Introduction to Quantum Groups

Lecture 2

Poisson-Lie groups and Lie Bialgebras

qft.itp.ac.ru/mbersht/quantum_groups.html

More Poisson Geometry

- (M, Π_M), (N, Π_N) – Poisson manyfolds
- Def $(M \times N, \Pi_M + \Pi_N)$ – product of P. m.
- Def $\varphi : M \to N$ is Poisson map if $\varphi_* \Pi_M = \Pi_N$
- Def $M \subset N$ is Poisson submanifold if $\Pi_N|_M = \Pi_N$.

Ex. Symplectic leaves are Poisson submanifolds

Poisson-Lie groups

- **Def** G — Poisson-Lie group if
  - G — Lie group
  - G — Poisson manifold
  - $G \times G \to G$ is Poisson map
- More explicitly: $\forall \varphi, \psi \in C^\infty(G)$

$$\{\varphi, \psi\}(gh) = \{\varphi, \psi\}(gh)\Big|_{h\text{-fixed}} + \{\varphi, \psi\}(gh)\Big|_{g\text{-fixed}}$$

- **Rem** $i : G \to G$, $g \mapsto g^{-1}$, is not Poisson,

One can show: $\{\varphi \circ i, \psi \circ i\} = -\{\varphi, \psi\} \circ i$

Example <u>1</u>

- $\mathfrak{g}^*$, group w.r.t "+", P.b. linear Poisson bracket $\{x_i, x_j\} = \sum c_{ij}^k x_k$

- $\mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ $\quad x_i = x_i' + x_i''$, $\quad x_j = x_j' + x_j''$

$$\sum c_{ij}^k x_k = \{x_i, x_j\} = \{x_i' + x_i'', x_j' + x_j''\} = \sum c_{ij}^k x_k' + \sum c_{ij}^k x_k''$$

$$\{\varphi, \psi\}(gh) = \{\varphi, \psi\}(gh)\Big|_{h\text{-fixed}} + \{\varphi, \psi\}(gh)\Big|_{g\text{-fixed}}$$

Example 2

- G – matrix group

$$\{L_1 \overset{\otimes}{,} L_2\} = [r, L_1 \otimes L_2] \qquad r \in Mat \otimes Mat$$

- L – matrix of coordinate functions

$$\left(e.g \quad L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$$

- $\{L_1 \overset{\otimes}{,} L_2\}$ – matrix

consist of brackets of cordinate functions

$$\left(e.g. \quad \{L_1 \overset{\otimes}{,} L_2\} = \begin{pmatrix} \{a,a\} & \{a,b\} & \cdots \\ \{a,c\} & \{a,d\} & \cdots \\ \{c,a\} & \{c,b\} & \cdots \\ \{c,c\} & \{c,d\} & \cdots \end{pmatrix} \right)$$

Example 2 (continuation)

- $\{L_1 \overset{\otimes}{,} L_2\} = [r, L_1 \otimes L_2]$

- r – <u>classical r-matrix</u>.

$\{\cdot,\cdot\}$ should satisfy skew symm + Jacobi ident
$\Longrightarrow$ conditions on r

- $G \times G \to G$ $\quad L_1 = L_1' L_1''$, $\quad L_2 = L_2' L_2''$

$$[r, L_1' L_1'' \otimes L_2' L_2''] = [r, L_1' \otimes L_2'](L_1'' \otimes L_2'') + $$
$$+ L_1' \otimes L_2' [r, L_1'' \otimes L_2'']$$

$$\{\varphi, \psi\}(gh) = \{\varphi, \psi\}(gh)\Big|_{h\text{-fixed}} + \{\varphi, \psi\}(gh)\Big|_{g\text{-fixed}}$$

Problems

- **Def** G is Poisson-Lie group. H is P-L subrgoup if H is subgroup and Poisson submanifold

- **Problem** $H \subset G$ Poisson Lie subgroup. Show that $C^\infty(G)^H$ is Poisson subalgebra

Rem Hence G/H is Poisson manifold.

- **Problem** $G = GL(2)$, $r = \frac{1}{4} h \otimes h + e \otimes f$

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Find brackets of a, b, c, d.

Check skew-comm. Check Poisson-Lie property

In terms of Π

- $\{\varphi,\psi\}(gh) = \{\varphi,\psi\}(gh)\Big|_{h\text{-fixed}} + \{\varphi,\psi\}(gh)\Big|_{g\text{-fixed}}$

- In terms of Π:
$$\Pi(gh) = (\rho_h)_* \Pi(g) + (\lambda_g)_* \Pi(h)$$

- Here
$\rho_h : G \to G \quad x \mapsto xh$ right multiplication
$\lambda_h : G \to G \quad x \mapsto hx$ left multiplication

- <u>Rem</u> If $g = h = e$, $\quad \Pi(e) = \Pi(e) + \Pi(e)$

Hence $\Pi(e) = 0$, P-L group is not symplectic

Lie algebra on $\mathfrak{g}^*$

- $I = \{\varphi \in C^\infty(G) \mid \varphi(e) = 0\}$.

$$\varphi, \psi \in C^\infty(G) \implies \{\varphi, \psi\} \in I \text{ since } \Pi(e) = 0$$
$$\varphi \in I^2,\ \psi \in C^\infty(G) \implies \{\varphi, \psi\} \in I^2$$

- $\delta^* = \{\cdot, \cdot\} : \quad \Lambda^2 I/I^2 \longrightarrow I/I^2$

$$I/I^2 = T_e^* G = \mathfrak{g}^* : \quad \delta^* : \Lambda^2 \mathfrak{g}^* \to \mathfrak{g}^*$$

$$d\varphi \wedge d\psi \longmapsto d\{\varphi, \psi\}$$

We get Lie alg. str. on $\mathfrak{g}^*$

Lie Bialgebras

- $\delta^*: \Lambda^2 \mathfrak{g}^* \to \mathfrak{g}^*$, dual: $\mathfrak{g} \to \Lambda^2 \mathfrak{g}$
- Another way to define δ: $G \to \Lambda^2 \mathfrak{g}$ $g \mapsto (\lambda_g)_* \Pi(g^{-1})$
  differential $\delta: \mathfrak{g} \mapsto \Lambda^2 \mathfrak{g}$
- Jacobi for α, $\{$ $\Rightarrow$ Jacobi for δ^* $\Rightarrow$ CoJacobi for δ
- <u>Prop</u>: δ satisfies "cocycle condition

$$\delta([a,b]) = ad_a(\delta(b)) - ad_b(\delta(a)),$$

ad: action of $\mathfrak{g}$ on $\Lambda^2 \mathfrak{g}$

- Proof: see in References

Lie bialgebra

- **Def** A Lie bialgebra $(\mathfrak{g}, [\cdot,\cdot], \delta)$

$\delta : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$,
  - δ satisfy co Jacobi relation
  - δ satisfy cocycle condition

- **Th** ⓐ If G is P-L group then $\mathfrak{g}$ is Lie bialgebra. ⓑ If $\mathfrak{g}$ is Lie bialgebra then $\exists!$ connected, simple connected P-L group G s. t. $\operatorname{Lie} G = \mathfrak{g}$

- **Proof** ⓐ – above, ⓑ – see in References

- **Rem** Notion of Lie bialgebra is self-dual

Manin triples

- **Def** Manin triple is $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$
  - $\mathfrak{g}$ is Lie algebra with nondegenerate symm, invariant form $(\cdot, \cdot)$
  - $\mathfrak{g}, \mathfrak{g}_-$ — Lie subalgebras
  - $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector space (not as Lie algebra)
  - $\mathfrak{g}_+, \mathfrak{g}_-$ — are isotropic for $(\cdot, \cdot)$

Example 1

- $\mathfrak{g}$ – simple Lie algebra $\mathfrak{g} = \underbrace{\mathfrak{n}_- \oplus \mathfrak{h}}_{\mathfrak{b}_-} \oplus \mathfrak{n}_+$, with $\overbrace{\mathfrak{h} \oplus \mathfrak{n}_+}^{\mathfrak{b}_+}$

$$\mathfrak{g}_1 = \mathfrak{g} \oplus \mathfrak{h} \qquad \mathfrak{g}_+ = \left\{ (a,b) \,\middle|\, \begin{array}{l} a \in \mathfrak{b}_+,\ b \in \mathfrak{h}, \\ pr_+ a = b \end{array} \right\}$$

$$\mathfrak{g}_- = \left\{ (a,b) \,\middle|\, \begin{array}{l} a \in \mathfrak{b}_-,\ b \in \mathfrak{h} \\ pr_- a = -b \end{array} \right\}$$

$$\big((a_1, b_1), (a_2, b_2) \big) = (a_1, a_2) - (b_1, b_2) \quad \text{form on } \mathfrak{g}_1$$

Examples 2, 3

- G – Lie group with trivial Poisson bracket

$$\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*, \quad \mathfrak{g}_+ = \mathfrak{g}, \quad \mathfrak{g}_- = \mathfrak{g}^*$$

commutator of $\mathfrak{g}, \mathfrak{g}^*$: $[a, \alpha] = ad^*_a \alpha$
$a \in \mathfrak{g}, \ \alpha \in \mathfrak{g}^*$

- $\mathfrak{g}$ simple Lie algebra

$$\mathfrak{g} = \mathfrak{g}[t^{\pm 1}], \quad \mathfrak{g}_+ = \mathfrak{g}[t], \quad \mathfrak{g}_- = \mathfrak{g}[t^{-1}]t^{-1}$$

$$(f, g) = \operatorname{Res}_{t=0} f g \, dt$$

Example 4

- $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}$, $\quad \mathfrak{g}_{+} = \{(a,a) \mid a \in \mathfrak{g}\} \simeq \mathfrak{g}$

$$\mathfrak{g}_{-} = \left\{(a,b) \;\middle|\; \begin{array}{l} a \in \mathfrak{b}_+,\ b \in \mathfrak{b}_- \\ pr\, a + pr\, b = 0 \end{array} \right\} \simeq \frac{\mathfrak{b}_+ \oplus \mathfrak{b}_-}{\mathfrak{h}}$$

$$((a_1,b_1),(a_2,b_2)) = (a_1,a_2) - (b_1,b_2)$$ form on $\mathfrak{g}$

Manin triples $\longleftrightarrow$ Lie bialgebras

• **Problem** For $\forall$ fin. dim $\mathfrak{g}$, $\exists$ bijection

$$\left\{\begin{matrix}\text{Lie bialgebra str}\\ \text{on } \mathfrak{g}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Manin triples}\\ \text{with } \mathfrak{g}_+ = \mathfrak{g}\end{matrix}\right\}$$

Hint If $\mathfrak{g}$ is Lie bialg. $\Rightarrow \exists$ str of Lie algebra on $\mathfrak{q} = \mathfrak{g} \oplus \mathfrak{g}^*$, namely

$$[a_i, \alpha^j] = \sum c_{ik}^j a_j - \sum \tilde{c}_i^{ej} \alpha_e$$

$\{a_i\}$ – basis in $\mathfrak{g}$, $\{\alpha^i\}$ – basis on $\mathfrak{g}^*$, $c, \tilde{c}$ – structure constants in $\mathfrak{g}$, $\mathfrak{g}^*$

• **Problem** Find Lie bialgebra structure (i.e. δ) for Examples 1, 4 and $\mathfrak{g} = sl_2$.

References

- Etingof, Schiffmann, *Lectures on quantum groups* Ch 2
- Chari, Pressley *A guide to quantum groups* Sec. 1.2, 1.3