Introduction to Quantum Groups

Lecture 4

Classical r-matrices

qft.itp.ac.ru/mbersht/quantum_groups.html

Coboundary Lie Bialgebras

- Lie bialgebra: $(\mathfrak{g}, \delta)$ $\quad \delta : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$
  - CoJacobi
  - cocycle $\quad \delta([a,b]) = ad_a \delta(b) - ad_b \delta(a)$
- Def δ is coboundary if $\exists\, r \in \Lambda^2 \mathfrak{g}$ s.t.

$$\delta(a) = ad_a r \qquad (= [a \otimes 1 + 1 \otimes a, r])$$

- Rem If $\mathfrak{g}$ is semisimple then $H^1(\mathfrak{g}, \#) = 0 \Rightarrow \exists\, r$

Coboundary vs CoJacobi

- **Th** $\mathfrak{g}$ - Lie alg, $r \in \mathfrak{g}\otimes\mathfrak{g}$, $\delta : \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$, $\delta(a) = ad_a r$

ⓐ δ maps to $\Lambda^2\mathfrak{g} \iff r_{12} + r_{21} \in (\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$

ⓑ δ satisfies CoJacoby $\iff$

$$[[r,r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$$

$r = \sum x_i \otimes y_i$, $r_{12} = \sum x_i \otimes y_i \otimes 1$, $r_{13} = \sum x_i \otimes 1 \otimes y_i$.

- **Problem** Prove a)

CYBE

- Assume $\exists$ non-degen, invar scalar prod. on $\mathfrak{g}$

$\exists$ canonical invariant $\Lambda^3\mathfrak{g} \to \mathbb{C}$

$$a_1 \wedge a_2 \wedge a_3 \mapsto ([a_1, a_2], a_3)$$

Invariance: $\forall b\ a_1 a_2 a_3$ $\quad [b, a_1] \wedge a_2 \wedge a_3 + a_1 \wedge [b, a_2] \wedge a_3 + a_1 \wedge a_2 \wedge [b, a_3]$

$$([[b, a_1], a_2], a_3) + ([a_1, [b, a_2]], a_3) + ([a_1, a_2], [b, a_3]) = 0$$

- In other terms $[a_i, a_j] = \sum C_{ij}^k a_k \to C^{ijk} \in (\Lambda^3\mathfrak{g})^{\mathfrak{g}}$

- MCYBE $\quad [[r, r]] = \varepsilon C$

CYBE $\quad [[r, r]] = 0$

CYBE vs MCYBE

- $r = r^S + r^A$, $r^S \in S^2\mathfrak{g}$, $r^A \in \Lambda^2\mathfrak{g}$. $r^S \in S^2\mathfrak{g}^{\mathfrak{g}}$.

Natural: $r^S = \alpha\Omega = \sum a_i \otimes a^i \in S^2\mathfrak{g}$ — $\mathrm{Id} \in \mathfrak{g}^* \otimes \mathfrak{g}$ — tensor Casimir

- Problem a) $\delta_r = \delta_{r^A}$, where $\delta_r(a) = ad_a r$
  b) $[[r,r]] = [[r^A, r^A]] + \alpha^2 C$

- Hence $\begin{pmatrix} r^A \in \Lambda^2\mathfrak{g} \\ \text{MCYBE} \end{pmatrix} \longleftrightarrow \begin{pmatrix} r \in \mathfrak{g} \otimes \mathfrak{g} \\ \text{CYBE} \end{pmatrix}$

- Rem $\mathfrak{g}$ - simple. Then $(S^2\mathfrak{g})^{\mathfrak{g}} = \langle \Omega \rangle$, $(\Lambda^3\mathfrak{g})^{\mathfrak{g}} = \langle C \rangle$

- $r \rightsquigarrow \hat{r} : \mathfrak{g} \to \mathfrak{g}$ (for $r \in \Lambda^2\mathfrak{g}$)

MCYBE: $[\hat{r}a, \hat{r}b] - \hat{r}[\hat{r}a, b] - \hat{r}[a, \hat{r}b] = \varepsilon [a,b]$

Example $\mathfrak{g} = sl_2$

- $\Lambda^3 \mathfrak{g} = \mathbb{C}$ — triv rep $\quad sl_2 = \langle e, h, f \rangle$
- $r \in \Lambda^2 \mathfrak{g} \qquad [[r,r]] \in \Lambda^3 \mathfrak{g} = (\Lambda^3 \mathfrak{g})^{\mathfrak{g}}$

$\Lambda^2 \mathfrak{g}$ — 3 dim rep sl_2 $\simeq$ adjoint

different $r \longleftrightarrow$ adjoint orbits for sl_2

- $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$

(i) $\sim \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \qquad r = \lambda\, e \wedge f$

(ii) $\sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad r = e \wedge h$

(iii) $\sim \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad r = 0$

$\Lambda^2 \mathfrak{g} = \langle e \wedge h, e \wedge f, f \wedge h \rangle$

Example $\mathfrak{g} = sl_2$

- $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ (i) $\sim \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ $r = \lambda\, e \wedge f$

(ii) $\sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $r = e \wedge h$

$\Lambda^2 \mathfrak{g} = \langle e \wedge h,\ e \wedge f,\ f \wedge h \rangle$ (iii) $\sim \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $r = 0$

- CYBE $\Lambda^2 \mathfrak{g} \to \Lambda^3 \mathfrak{g} = \mathbb{C}$ (ii), (iii)

$[[r,r]] :=$ def :

- (i) $\delta(e) = \lambda\, e \wedge h$, $\delta(h) = 0$, $\delta(f) = \lambda\, f \wedge h$

(ii) $\delta(e) = 0$, $\delta(h) = 2 e \wedge h$, $\delta(f) = 2 e \wedge f$

(iii) $\delta = 0$

Classical double

- $(\mathfrak{g}, \delta)$ – bialg. Lie.
- $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ – Manin triple

$\mathfrak{g}_+ = \mathfrak{g}$, $\mathfrak{g}_- = \mathfrak{g}^*$, $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ – Lie alg.

- <u>Prop</u> $\exists\, r \in \Lambda^2 \mathfrak{g}$ satisfy MCYBE $\quad r \longleftrightarrow Id \in \mathfrak{g} \otimes \mathfrak{g}^*$,
- If $\mathfrak{g} = \langle a_i \rangle$ $\mathfrak{g}^* = \langle a^i \rangle$ – dual bases

then $r = \sum a_i \wedge a^i$

- Another way: $r = \sum a_i \otimes a^i \in \mathfrak{g} \otimes \mathfrak{g}^* \subset \mathfrak{g} \otimes \mathfrak{g}$,

r is <u>not</u> antisymm, but satisfies CYBE

Classical double

- Prop $\exists\, r \in \Lambda^2 \mathfrak{g}$, $r \longleftrightarrow Id \in \mathfrak{g} \otimes \mathfrak{g}^*$, satisfy MCYBE

- Proof $r = \sum a_i \otimes a^i - \sum a^i \otimes a_i$

$\hat{r} = P_+ - P_-$, where $P_\pm$ - projectors on $\mathfrak{g}_\pm$

$a = a_+ + a_-$, $b = b_+ + b_-$

$$[\hat{r}a, \hat{r}b] - \hat{r}[\hat{r}a, b] - \hat{r}[a, \hat{r}b] = [a_+ - a_-, b_+ - b_-] -$$

$$- \hat{r}\left([a_+ - a_-, b_+ + b_-] + [a_+ + a_-, b_+ - b_-]\right) =$$

$$= -[a_+, b_+] - [a_-, b_-] - [a_+, b_-] - [a_-, b_+] = -[a, b]$$

Standard structure for simple $\mathfrak{g}$

- $(\mathfrak{g}_r, \mathfrak{g}_{r+}, \mathfrak{g}_{r-}) = (\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{b}_+, \mathfrak{b}_-)$

  $\mathfrak{b}_+ = (a, pr_+(a))$, $\mathfrak{b}_- = (a, -pr_-(a))$

$((x_1, y_1), (x_2, y_2)) = (x_1, x_2) - (y_1, y_2)$ $\quad pr_\pm : \mathfrak{b}_\pm \to \mathfrak{h}$

- $\exists\, r \in \Lambda^2(\mathfrak{g} \oplus \mathfrak{h})$ $\qquad r = \sum_{\alpha \in \Delta_+} e_\alpha \wedge e_{-\alpha} + \sum h_i \wedge h^i$

  since $\mathfrak{g}_r$ – double

- $0 \oplus \mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{h}$ – Lie ideal

  $\delta(0 \oplus \mathfrak{h}) = 0$

Hence $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{h} / \oplus \mathfrak{h}$ has structure of quotient Lie bialgebra

Standard structure for simple $\mathfrak{g}$

- $r \in \Lambda^2(\mathfrak{g} \oplus \mathfrak{h}) \rightsquigarrow r = \sum e_\alpha \otimes e_{-\alpha} \in \Lambda^2 \mathfrak{g}$

standard ~~coboundary~~ structure on $\mathfrak{g}$

- <u>Problem</u> a) Find $\delta(h_i)$, $\delta(e_\alpha)$, $\delta(e_{-\alpha})$, α - simple

b) Find Lie algebra $\mathfrak{g}^*$

c) Show that $r = \sum h_i \otimes h_i + 2 \sum_{\alpha \in \Delta_+} e_\alpha \otimes e_{-\alpha}$ defines the same δ and satisfies CYBE

r-matrix str for P-L groups

- $(\mathfrak{g}, \delta)$ – coboundary Lie Bialgebra
  $\delta(a) = ad_a r$, G – corr. connected group Π – ?
- Define $\Pi_e(g) = r - (Ad_{g^{-1}})_* r$

We have: $\Pi_e(e) = 0$, $d\Pi_e(a) = ad_a r$

$$\Pi_e(gh) = \Pi_e(h) + (Ad_{h^{-1}})\Pi_e(g) \text{ – multiplicativity}$$

$$\Pi_e(gh) = r - (Ad_{h^{-1}g^{-1}})_*(r) \qquad (Ad_{h^{-1}})\Pi_e(g) = r - (Ad_{h^{-1}})_* r + (Ad_{h^{-1}})_* r - (Ad_{h^{-1}})_*(Ad_{g^{-1}})_* r$$

- $\Pi(g) = (\lambda_g)_* r - (\rho_g)_* r = (\lambda_g)_* \Pi_e(g)$

Sklyanin bracket

Sklyanin bracket for matrix groups

- $\Pi(g) = (\lambda_g)_* r - (\rho_g)_* r$

- Let G - matrix group.
  $r = \sum r_{i_1 j_1 i_2 j_2} \partial_{i_1 j_1} \wedge \partial_{i_2 j_2}$

$$(\lambda_g)_* r = \sum r_{i_3 j_1 i_4 j_2} \, g_{i_1 i_3} \, g_{i_2 i_4} \, \partial_{i_1 j_1} \wedge \partial_{i_2 j_2}$$

$$(\rho_g)_* r = \sum r_{i_1 j_3 i_2 j_4} \, g_{j_3 j_1} \, g_{j_4 j_2} \, \partial_{i_1 j_1} \wedge \partial_{i_2 j_2}$$

- $\{g_{i_1 j_1}, g_{i_2 j_2}\} = \sum \left(g_{i_1 i_3} \, g_{i_2 i_4} \, r_{i_3 j_1 i_4 j_2} - r_{i_1 j_3 i_2 j_4} \, g_{j_3 j_1} \, g_{j_4 j_2} \right)$

  In matrix terms $\{g \overset{\otimes}{,} g\} = [g \otimes g, r]$

Sklyanin bracket

- $\Pi(g) = (\lambda_g)_* r - (\rho_g)_* r$

We know:
- Π is multiplicative
- $d\Pi_e$ gives δ, $\delta(a) = ad_a r$

- Problem* Show that Π defines Poisson structure

- Hint (based on Lu-Weinstein)

Def $K \in \Lambda^k TG$ is <u>multiplicative</u> if $K(gh) = (\lambda_g)_* K(h) + (\rho_h)_* K(g)$

- K is mult. $\Longleftrightarrow$ $K(e) = 0$, $\forall X, Y \in Vect(G)$ X - left inv, Y - right inv, $L_X L_Y K = 0$
- K is mult, $d_e K = 0 \Rightarrow K = 0$

- Π is mult $\Rightarrow$ Schouten bracket $[\Pi, \Pi]$ - is mult
- r satisfies MCYBE $\Rightarrow d_e[\Pi, \Pi] = 0$ $\longrightarrow$ $[\Pi, \Pi] = 0$

References

- Etingof, Schiffmann, *Lectures on quantum groups* Ch 3
- Chari, Pressley *A guide to quantum groups*, Ch. 2
- Lu, Weinstein *Poisson-Lie groups, Dressing transformations and Bruhat decompositions*