Introduction to Quantum Groups

Lecture 5

Quantum groups and algebras. Example sl_2.

qft.itp.ac.ru/mbersht/quantum_groups.html

$\mathbb{C}[G]$

- Quantize G $\rightsquigarrow$ Quantize $\mathbb{C}[G]$
- G : group $\quad G \times G \to G$, $(g,h) \mapsto gh$ $\qquad G \to G$, $g \mapsto g^{-1}$
- $\Delta : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$ $\left(= \mathbb{C}[G \times G]\right)$

$f \mapsto \big((g,h) \mapsto f(gh) \big)$

antipode $S : \mathbb{C}[G] \to \mathbb{C}[G]$

$m : \mathbb{C}[G] \otimes \mathbb{C}[G] \to \mathbb{C}[G]$, product

$i : \mathbb{C} \to \mathbb{C}[G]$, unit $1 \mapsto 1$

$\varepsilon : \mathbb{C}[G] \to \mathbb{C}$ counit $f \mapsto f(e)$

$(\mathbb{C}[G], m, i, \Delta, \varepsilon, S)$ — is Hopf algebra

Hopf algebras

- $(A, m, i, \Delta, \varepsilon, S)$

associativity

$$\begin{tikzcd} & A\otimes A \arrow[dr, "m"] & \\ A\otimes A\otimes A \arrow[ur, "id\otimes m"] \arrow[dr, "m\otimes id"'] & & A \\ & A\otimes A \arrow[ur, "m"'] & \end{tikzcd}$$

coassociativity

$$\begin{tikzcd} & A\otimes A \arrow[dr, "\Delta\otimes id"] & \\ A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & A\otimes A\otimes A \\ & A\otimes A \arrow[ur, "id\otimes\Delta"'] & \end{tikzcd}$$

Δ is homomorphism

$$\Delta(ab) = \Delta(a)\Delta(b)$$

$$\begin{tikzcd} A\otimes A \arrow[r, mapsto, "\Delta\otimes\Delta"'] \arrow[d, "m"'] & A\otimes A\otimes A\otimes A \arrow[d, "m_{13}\otimes m_{24}"] \\ A \arrow[r, "\Delta"] & A\otimes A \end{tikzcd}$$

Axioms with i, ε, S . . .

Quantum $\mathbb{C}[G]$

- $\mathbb{C}[G] = (A_0, m_0, i_0, \Delta_0, \varepsilon_0, S_0)$
- Quantization : $(A, m, i, \Delta, \varepsilon, S)$

$A = A_0[[\hbar]]$, $m, \Delta, \ldots$ $\hbar$ - linear

$m \equiv m_0$, $\Delta = \Delta_0, \ldots$ mod $\hbar$

i.e. $m(\hbar a, b) = \hbar\, m(a, b)$

$\Delta(\hbar a) = \hbar\, \Delta(a)$

Quantum $\mathbb{C}[G]$

$A = A_0[[\hbar]]$
as vector space

- For short $m_0(a,b) = ab$ $\quad \Delta_0(a) = a_{(1)} \otimes a_{(2)}$

$m(a,b) = a * b = ab + \hbar m_1(a,b) + o(\hbar),$ $\quad m_1(a,b) - m_1(b,a) = \{a,b\},$

- $\Delta(a*b - b*a) = \hbar\, \Delta(\{a,b\}) + o(\hbar)$

$$\Delta(a) * \Delta(b) - \Delta(b) * \Delta(a) = (a_{(1)} * b_{(1)}) \otimes (a_{(2)} * b_{(2)}) - (b_{(1)} * a_{(1)}) \otimes (b_{(2)} * a_{(2)}) + o(\hbar)$$
$$= \hbar\big(m_1(a_{(1)}, b_{(1)}) \otimes a_{(2)} b_{(2)} + a_{(1)} b_{(1)} \otimes m_1(a_{(2)}, b_{(2)}) - m_1(b_{(1)}, a_{(1)}) \otimes b_{(2)} a_{(2)}$$
$$- b_{(1)} a_{(1)} \otimes m_1(b_{(2)}, a_{(2)})\big) + o(\hbar) = \hbar\big(\{a_1, b_1\} \otimes a_2 b_2 + a_{(1)} b_{(1)} \otimes \{a_{(2)}, b_{(2)}\}\big) + o(\hbar)$$

- Hence $\quad \Delta(\{a,b\}) = \{a_1, b_1\} \otimes a_2 b_2 + a_{(1)} b_{(1)} \otimes \{a_{(2)}, b_{(2)}\}$

Quantum $\mathbb{C}[G]$

- $\Delta\{a, b\} = \{a_1, b_1\} \otimes a_2 b_2 + a_{(1)} b_{(1)} \otimes \{a_{(2)}, b_{(2)}\}$

- Compare to P-L condition

$$\{\varphi, \psi\}(gh) = \{\varphi, \psi\}(gh)\Big|_{h\text{-fixed}} + \{\varphi, \psi\}(gh)\Big|_{g\text{-fixed}}$$

If $\varphi(gh) = \varphi_{(1)}(g)\varphi_{(2)}(h)$, $\psi(gh) = \psi_{(1)}(g)\psi_{(2)}(h)$

$$\{\varphi, \psi\}(gh) = \{\varphi_{(1)}, \psi_{(1)}\}(g)\, \varphi_{(2)}\psi_{(2)}(h) + \varphi_{(1)}\psi_{(1)}(g)\{\varphi_{(2)}, \psi_{(2)}\}(h)$$

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$\Delta : \mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$ map of Poisson algebras

$G \times G \to G$ is Poisson map. G is P-L group

Trivial Example

- G, $\Pi = 0$ $\qquad \mathbb{C}[G] \rightsquigarrow \mathbb{C}[G]$

- $G^* = \mathfrak{g}^*$ $\qquad \mathbb{C}[G^*] = S(\mathfrak{g}) \rightsquigarrow \mathcal{U}(\mathfrak{g}_\hbar)$

$\hbar$ rescaled comm

or Rees algebra

$U(\mathfrak{g})$ — Hopf Algebra

- $U(\mathfrak{g})$, $\Delta(a) = a\otimes 1 + 1\otimes a$, $\varepsilon(a)=0$, $S(a)=-a$. $a\in\mathfrak{g}$
- Dual to $\mathbb{C}[G]$: $U(\mathfrak{g})\otimes\mathbb{C}[G]\to\mathbb{C}$

$a_1\dots a_n\in\mathfrak{g}$ $\quad a_1\dots a_n\otimes f \longmapsto \frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_n} f(e^{t_1a_1}\cdots e^{t_na_n})\Big|_{t_1=\dots=t_n=0}$

right invar. diff. oper

$$(X, f_1f_2) = (\Delta X, f_1\otimes f_2)$$

$$(X_1X_2, f) = (X_1\otimes X_2, \Delta(f))$$

- $\mathbb{C}[G]$ – comm. $\longleftrightarrow$ $U(\mathfrak{g})$ cocomm.

$\Delta: U(\mathfrak{g}) \to S^2 U(\mathfrak{g})$

Quantum Algebras

- $U(\mathfrak{g}) = (A_0, m_0, i_0, \Delta_0, \varepsilon_0, S_0)$

  Quantization : $(A, m, i, \Delta, \varepsilon, S)$

  $A = A_0[[\hbar]]$, $m, \Delta, \ldots$ $\hbar$ - linear

  $m \equiv m_0$, $\Delta = \Delta_0, \ldots$ mod $\hbar$

- <u>Rem</u> If $\mathfrak{g}$ is s/s $\Rightarrow$ no deform $U(\mathfrak{g})$ as algebra

- <u>Rem</u> cocomm. deformations of $U(\mathfrak{g}) \longleftrightarrow$ deformations $\mathfrak{g}$ as Lie algebra

Quantization of Lie Bialgebra

- $U_\hbar(\mathfrak{g})$ quantization of $U(\mathfrak{g})$

$\forall a \in \mathfrak{g}$ $\quad \delta(a) = \dfrac{\Delta(a) - \Delta^{op}(a)}{\hbar} \mod \hbar$

- Th $(\mathfrak{g}, \delta)$ bialgebra Lie $\qquad \delta : \mathfrak{g} \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$

Proof $\delta(a) = \sum B_i \otimes C_i$.

We want $B_i \in \mathfrak{g} \iff \Delta_0(B_i) = B_i \otimes 1 + 1 \otimes B_i$

$\sum \Delta_0 B_i \otimes C_i = \frac{1}{\hbar}\left(\Delta \otimes 1 (\Delta - \sigma_{12}\Delta) a\right) \mod \hbar =$

$= \frac{1}{\hbar}\left((1\otimes\Delta - \sigma_{23}\, 1\otimes\Delta)\Delta a + \sigma_{23}(\Delta\otimes 1 - \sigma_{12}\,\Delta\otimes 1)\Delta a\right) \mod \hbar$

$= (1 \otimes \delta)\, \Delta_0 a + \sigma_{23}(\delta \otimes 1)\, \Delta_0 a = \sum 1 \otimes B_i \otimes C_i + B_i \otimes 1 \otimes C_i$

Hence $\delta : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$

- Problem* Prove Cocycle + coJacobi for δ.

Summary

- Deformotion of $\mathbb{C}[G]$ $\longrightarrow$ P-L group G.

- Deformation of $\mathcal{U}(\mathfrak{g})$ $\longrightarrow$ Lie bialgebra $\mathfrak{g}$

- Th (Etingof-Kazhdan) $\exists$ canonical quantizatio of $\forall$ Lie bialgebra

sl_2

- $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$
- $r=e\wedge f$

$\delta(e)=e\wedge h$, $\delta(h)=0$, $\delta(f)=f\wedge h$

- b_+, b_- – sub bialgebras

$$U_\hbar(b_+)$$

- $[H, E] = 2E, \quad \Delta H = H \otimes 1 + 1 \otimes H.$
- $\Delta E = E \otimes g_2(H) + g_1(H) \otimes E, \quad g_i(H) = 1 + O(H)$
- $(\Delta \otimes 1)(\Delta(E)) = E \otimes g_2(H) \otimes g_2(H) + g_1(H) \otimes E \otimes g_2(H) + \Delta g_1(H) \otimes E$

$$\|$$

$$(1 \otimes \Delta)(\Delta(E)) = E \otimes \Delta(g_2(H)) + g_1(H) \otimes E \otimes g_2(H) + g_1(H) \otimes g_1(H) \otimes E$$

Hence $\quad \Delta g_1(H) = g_1(H) \otimes g_1(H), \quad \Delta g_2(H) = g_2(H) \otimes g_2(H)$

$$U_\hbar(b_+)$$

- **Problem** $g(h) = 1 + \hbar(\ldots) \in U(h)[[\hbar]]$ is group like (i.e. $\Delta g = g \otimes g$) then $g(H) = \exp(\alpha \hbar H)$
  $\alpha \in \mathbb{C}[[\hbar]]$

- Hence $\Delta(E) = E \otimes e^{\hbar \alpha_2 H} + e^{\hbar \alpha_1 H} \otimes E$

- $\Delta(E) - \Delta^{21}(E) = \hbar\, \delta(E) + o(\hbar)$ Hence
  $(\alpha_2 - \alpha_1)(E \otimes H - H \otimes E) = (E \otimes H - H \otimes E)$
  $\alpha_2 - \alpha_1 = 1$

- Rescaling $\tilde{E} = e^{\hbar \alpha H} E \rightsquigarrow \Delta(\tilde{E}) = \Delta(e^{\hbar \alpha H}) \Delta(E)$
  $= \tilde{E} \otimes e^{\hbar(\alpha + \alpha_2) H} + e^{\hbar(\alpha + \alpha_1) H} \otimes E$

- We can take
  $$\Delta(E) = E \otimes e^{\hbar H} + 1 \otimes E$$

$$U_\hbar(sl_2)$$

- $U_\hbar(\mathfrak{b}_-)$ $\quad H, F, \quad [H,F]=-2F$

$$\Delta H = H\otimes 1 + 1\otimes H \quad \Delta(F) = F\otimes 1 + e^{-\hbar H}\otimes F$$

- $U_\hbar(sl_2)$ — generated by E, H, F,
  Know: $\Delta(E)$, $\Delta(H)$, $\Delta(F)$, $[H,E]$, $[H,F]$

- Problem Show that relation $[E,F] = \dfrac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}$
  agrees with Δ.

Hint Use relations $E\varphi(H) = \varphi(H-2)E$, $F\varphi(H) = \varphi(H+2)F$

- We found $U_\hbar(sl_2)$

$\mathcal{U}_\hbar(sl_2)$ as algebra.

- $c = fe + \frac{h(h+2)}{4} = ef + \frac{h(h-2)}{4} = \frac{ef+fe}{2} + \frac{h^2}{4}$ — generates $Z(\mathcal{U}(sl_2))$

- Problem $\exists$ homomorphism $\mathcal{U}_\hbar(sl_2) \to \mathcal{U}(sl_2)[[\hbar]]$

$E \mapsto e, \quad H \mapsto h, \quad F \mapsto \varphi(h,c)f$

Hint $[E,F] \mapsto e\varphi(h,c)f - \varphi(h,c)fe =$
$= \varphi(h-2,c)\,ef - \varphi(h,c)fe = \varphi(h-2)\left(c - \frac{h(h-2)}{4}\right) - \varphi(h,c)\left(c - \frac{h(h+2)}{4}\right)$

$\varphi(h,c) = \psi(h,c)\left(c - \frac{h(h+2)}{4}\right)^{-1}(e^{\hbar} - e^{-\hbar})$

$\psi(h-2,c) - \psi(h,c) = (e^{\hbar h} - e^{-\hbar h})$.

- $\exists$ inverse map

$$U_q(sl_2)$$

- Let $q = e^{\hbar}$, $K^{1/2} = q^{\hbar H/2}$
- $KE = q^2 EK$, $\quad$ (since $\varphi(H)E = E\varphi(H+2)$)

  $KF = q^{-2} FK^{-1}$

  $[E,F] = \dfrac{K - K^{-1}}{q - q^{-1}}$

- $\Delta K = K \otimes K$ $\quad$ (since K - group-like)

  $\Delta E = E \otimes K + 1 \otimes E$

  $\Delta F = F \otimes 1 + K^{-1} \otimes F$

$U_q(sl_2)$ — another generators

- $KE = q^2 EK, \quad KF = q^{-2} FK, \quad [E,F] = \dfrac{K-K^{-1}}{q-q^{-1}}$

$\Delta K = K \otimes K \qquad \Delta E = E \otimes K + 1 \otimes E, \quad \Delta F = F \otimes 1 + K^{-1} \otimes F$

- $A = K^{1/2} \qquad B = (q-q^{-1}) K^{-1/2} E \qquad C = (q-q^{-1}) F K^{1/2}$

$[A,B] = (q^2-1) BA, \quad [A,C] = (q^{-2}-1) CA$

$[B,C] = (q-q^{-1})^2 (K^{-1/2} EFK^{1/2} - FE) = (q-q^{-1}) \underline{(A^2 - A^{-2})}$

deformation of comm. product

- $\Delta A = A \otimes A$

$\Delta B = (q-q^{-1})(\Delta K^{-1/2}) \Delta E = B \otimes A + A^{-1} \otimes B$

$\Delta C = (q-q^{-1}) \Delta F \Delta K^{1/2} = C \otimes A + A^{-1} \otimes C$

$$\mathbb{C}[G^*]$$

- $G^* = B_+ \times B_- / H = \left\{ \begin{pmatrix} A^{-1} & B \\ 0 & A \end{pmatrix}, \begin{pmatrix} A & 0 \\ C & A^{-1} \end{pmatrix} \right\}$

  $\mathbb{C}[G^*] = \mathbb{C}[A, A^{-1}, B, C]$

- $\begin{pmatrix} A_1^{-1} & B_1 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} A_2^{-1} & B_2 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} A_1^{-1} A_2^{-1} & A_1^{-1} B_2 + B_1 A_2 \\ 0 & A_1 A_2 \end{pmatrix}$

  $\begin{pmatrix} A_1 & 0 \\ C_1 & A_1^{-1} \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ C_2 & A_2^{-1} \end{pmatrix} = \begin{pmatrix} A_1 A_2 & 0 \\ C_1 A_2 + A_1^{-1} C_2 & A_1^{-1} A_2^{-1} \end{pmatrix}$

  $\Delta A = A \otimes A, \quad \Delta B = B \otimes A + A^{-1} \otimes B, \quad \Delta C = C \otimes A + A^{-1} \otimes C.$

- $U_q(\mathfrak{sl}_2)$ -- deformation of $\mathbb{C}[G^*]$

  ------ $U(\mathfrak{sl}_2)$

References

- Chari, Pressley, *A guide to quantum groups*, Ch. 6
- Etingof, Schiffmann, *Lectures on quantum groups*, Ch. 9