Introduction to Quantum Groups

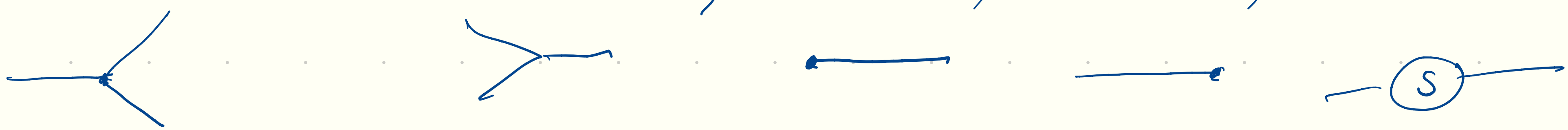
Lecture 6
Hopf algebras

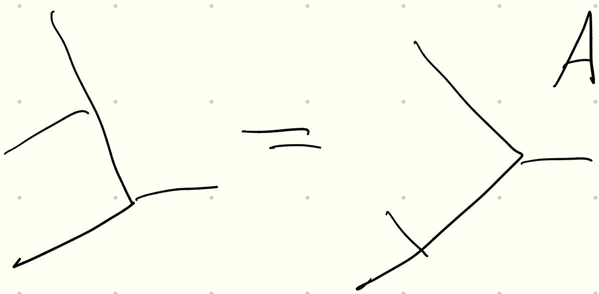
qft.itp.ac.ru/mbersht/quantum_groups.html

Hopf algebra

- $(A, m, i, \Delta, \varepsilon, S)$

- $\Delta : A \to A \otimes A$, $m: A \otimes A \to A$, $i: \mathbb{C} \to A$, $\varepsilon: A \to \mathbb{C}$, $S: A \to A$



- 

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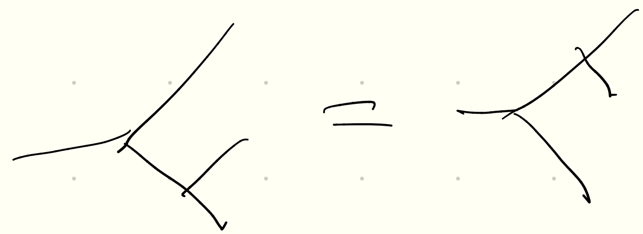

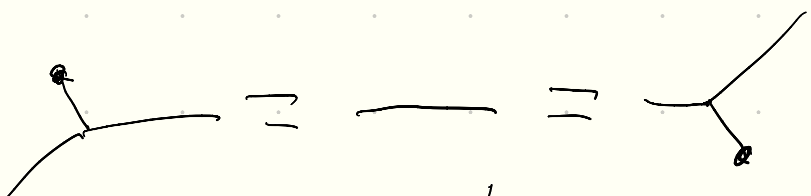

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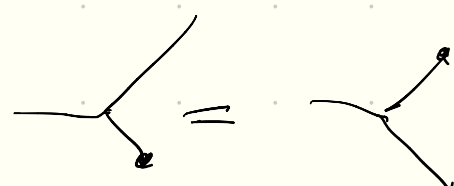

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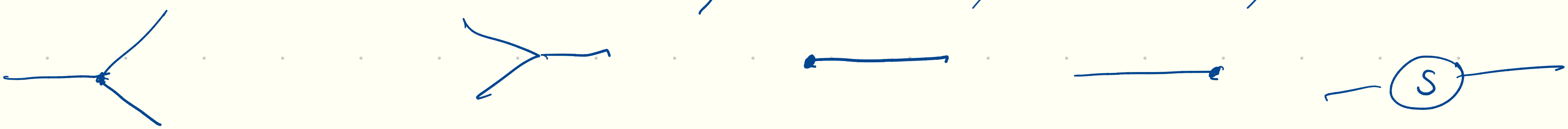
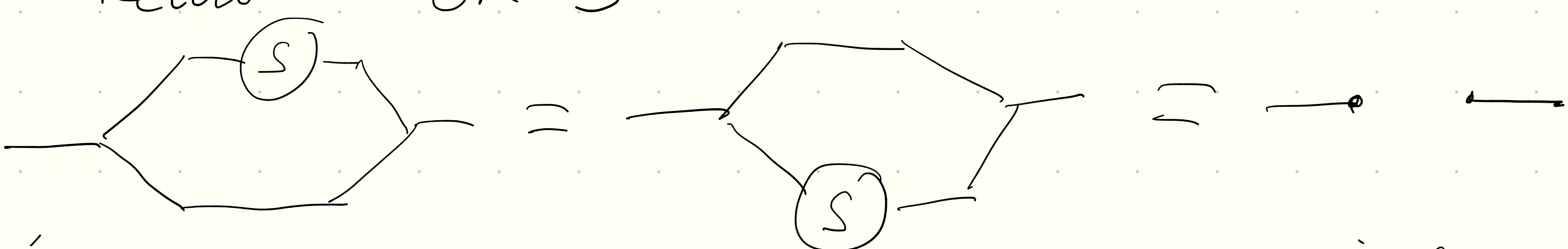
Hopf algebra

- $\Delta : A \to A \otimes A$, $m : A \otimes A \to A$, $i : \mathbb{C} \to A$, $\varepsilon : A \to \mathbb{C}$, $S : A \to A$



- Δ is homomorphism $\Delta(a \cdot b) = \Delta(a)\Delta(b)$

- Relation on S



$i\,\varepsilon(a)$

$U_\hbar(sl_2)$

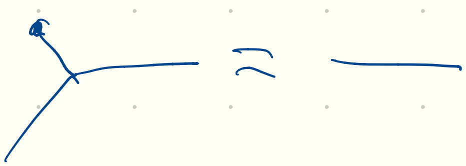
- E, H, F : $[H,E]=2E$, $[H,F]=-2F$

$$[E,F] = \frac{e^{\hbar H}-e^{-\hbar H}}{e^{\hbar}-e^{-\hbar}}$$

- $\Delta(E) = E\otimes e^{\hbar H} + 1\otimes E$, $\Delta(F) = F\otimes 1 + e^{-\hbar H}\otimes F$

$$\Delta H = H\otimes 1 + 1\otimes H$$

- i, ε, S are determined by m, Δ



$\varepsilon_1\Delta(a) = a$

$\varepsilon_1 = \varepsilon\otimes 1$

$a=1$, $1=\varepsilon_1(\Delta(1)) = \varepsilon(1\otimes 1) = \varepsilon(1)1$ $\qquad \varepsilon(1)=1$

$a=H$, $H=\varepsilon_1(\Delta(H)) = \varepsilon(H)1 + H$ $\qquad \varepsilon(H)=0$

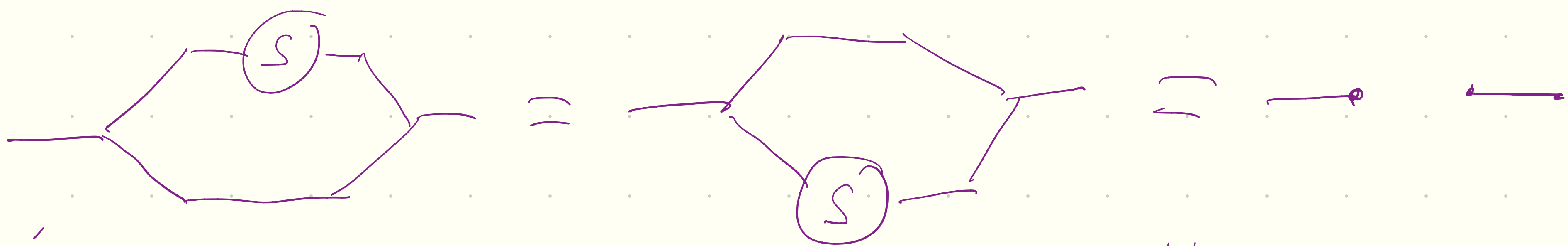
$a=E$, $E = \varepsilon_1(\Delta(E)) = \varepsilon(E)e^{\hbar H}+E$ $\qquad \varepsilon(E)=0$

$\quad \| \; \varepsilon_2(\Delta(E)) = \varepsilon(e^{\hbar H})E + \varepsilon(E)$ $\qquad \varepsilon(F)=0$

$U_\hbar(sl_2)$

- $[H,E]=2E$, $[H,F]=-2F$, $[E,F]=\dfrac{e^{\hbar H}-e^{-\hbar H}}{e^{\hbar}-e^{-\hbar}}$
- $\Delta(E)=E\otimes e^{\hbar H}+1\otimes E$, $\Delta(F)=F\otimes 1+e^{-\hbar H}\otimes F$

  $\Delta H = H\otimes 1+1\otimes H$

- 

$$0=i\varepsilon(E)=m(S\otimes 1)\Delta E=m(S\otimes 1)(E\otimes e^{\hbar H}+1\otimes E)=$$
$$=S(E)e^{\hbar H}+E$$

$$0=i\varepsilon(H)=m(S\otimes 1)(H\otimes 1+1\otimes H)=S(H)+H$$

- $S(E)=-Ee^{-\hbar H}$, $S(F)=-e^{\hbar H}F$, $S(H)=-H$

S^2

- $S(E) = -E e^{-\hbar H}$, $S(F) = -e^{\hbar H} F$, $S(H) = -H$.

$S^2(E) = -S(e^{-\hbar H})S(E) = e^{\hbar H} E e^{-\hbar H}$ — conjugation by $e^{\hbar H}$.

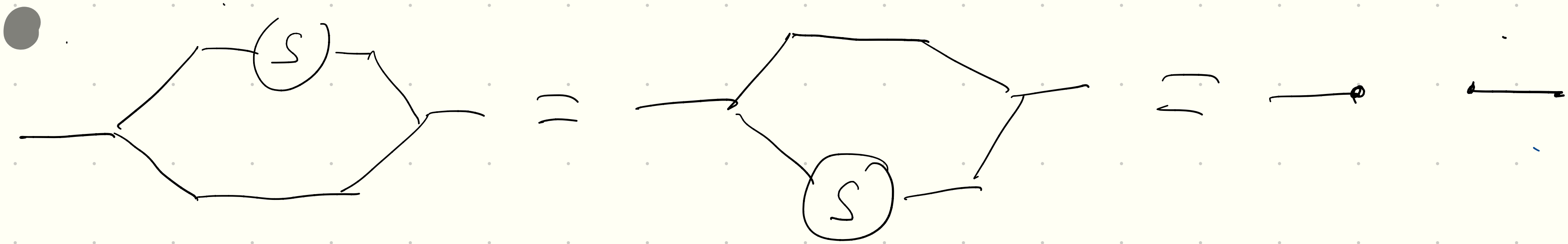
- <u>Rem</u> If A is comm. or cocomm. then $S^2 = id$

- In our case. S^2 is conjugation by $e^{\hbar H}$
Suffient to check on generators E, H, F due to

- <u>Problem</u> S is antihom of algebra and coalgebra
<u>Hint</u> Let $(\Delta \otimes id)\Delta(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$, $(\Delta \otimes id)\Delta(b) = \sum b_{(1)} \otimes b_{(2)} \otimes b_{(3)}$

Then $S(ab) = \sum S(a_{(1)} b) a_{(2)} S(a_{(3)})$
$= \sum S(a_{(1)} b_{(1)}) a_{(2)} b_{(2)} S(b_{(3)}) S(a_{(3)}) = S(b)S(a)$

Remarks

- Usually we assume S is bijective.
- <u>Def</u> $f, g : A \to A$ Convolution $f \bullet g = m(f \otimes g)\Delta$
  $\Delta(a) = a_{(1)} \otimes a_{(2)}$ $f \bullet g(a) = f(a_{(1)})g(a_{(2)})$
  $i\varepsilon : A \to A$ – unit of the convolution product
- 

Defining property of $S \iff S \cdot id = i\varepsilon \longrightarrow$
S is determined by Δ, m.

$\mathfrak{sl}_2$ representations

- $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$

- $L_\ell = \langle v_\ell, v_{\ell-2}, v_{\ell-4}, \ldots, v_{-\ell} \rangle$

  $e v_\ell = 0$, $h v_\ell = \ell v_m$. Let $v_{\ell-2k} = f^k v_\ell$

  $f v_m = v_{m-2}$, $h v_m = m v_m$, $e v_m = \left(\frac{\ell-m}{2}\right)\left(\frac{\ell+m+2}{2}\right) v_{m+2}$

- Another normalization $\tilde{v}_{\ell-2k} = \frac{1}{k!} f^k v_\ell$.

  $f \tilde{v}_m = \left(\frac{\ell-m+2}{2}\right)\tilde{v}_{m-2}$, $h \tilde{v}_m = m \tilde{v}_m$, $e \tilde{v}_m = \left(\frac{\ell+m+2}{2}\right)\tilde{v}_{m+2}$

$U_\hbar(sl_2)$ representations

- $[H,E]=2E$, $[H,F]=-2F$, $[E,F]=\dfrac{e^{\hbar H}-e^{-\hbar H}}{e^{\hbar}-e^{-\hbar}}$

- $L_\ell = \langle v_\ell, v_{\ell-2}, v_{\ell-4}, \dots, v_{-\ell}\rangle$
  $Ev_\ell=0$, $Hv_\ell=\ell v_\ell$ Let $v_{\ell-k}=F^k v_\ell$

<u>Probem</u> a) Find formulas for action of E,H,F
b) Define basis $\tilde{v}_m$

- <u>Example</u> $L_1=\mathbb{C}^2=\langle v_1, v_{-1}\rangle$

$H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $E=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $F=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ — same formulas since for $H=\pm1$

$$\frac{e^{\hbar H}-e^{-\hbar H}}{e^{\hbar}-e^{-\hbar}}=H.$$

Representations of Hopf algebras

- $\Delta : A \to A \otimes A$ | $\varepsilon : A \to \mathbb{C}$

  $V_1, V_2 \in Mod_A \rightsquigarrow V_1 \otimes V_2 \in Mod_A$ | $\mathbb{C} \in Mod_A$

- <u>Rem</u> Tensor product of group reps or Lie algebra reps is consequence of $\exists$ of Δ

- $S : A \to A$

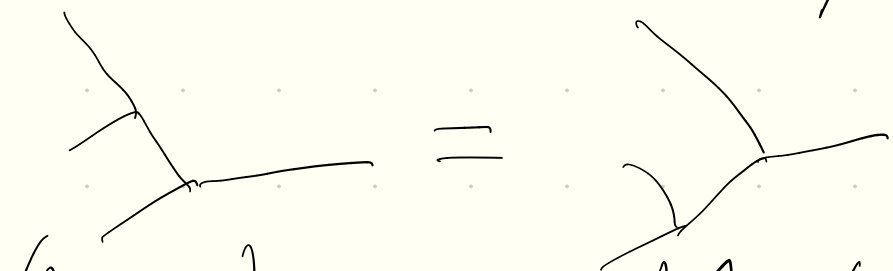
  $V \in Mod_A$ $\quad V^*$ $\quad \rho_{V^*}(a) = \rho_V(S(a))^*$

  $\quad {}^*V$ $\quad \rho_{{}^*V}(a) = \rho_V(S^{-1}(a))^*$

$^*(V^*) = V = ({}^*V)^*$; $\quad (V^*)^* \neq V$ but could be $\simeq$

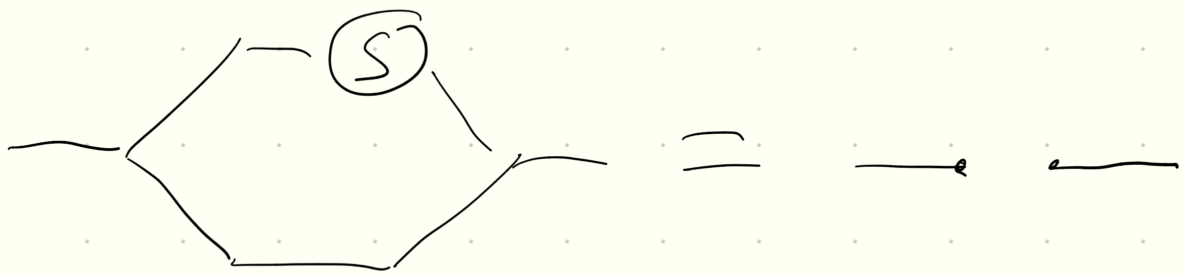
- <u>Rem</u> If S^2 is conjugation by u, then ${}^*V \simeq V^*$

Properties of Mod_A

- 

  monoidal category

  $(V_1 \otimes V_2) \otimes V_3 = V_1 \otimes (V_2 \otimes V_3)$

- 

  rigid monoidal category

$\exists \quad V^* \otimes V \to \mathbb{C} \qquad \mathbb{C} \to V \otimes V^*$

- Problem a) Show $\exists$ of maps above.

  b) Show that $(V \otimes W)^* = W^* \otimes V^*$ Hint: use problem above

Tensor product for sl_2

- $L_1 = \mathbb{C}^2 \qquad v_1 \qquad v_0$

$$H \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- $L_1 \otimes L_1 \qquad v_1 \otimes v_1 \quad v_1 \otimes v_{-1} \quad v_{-1} \otimes v_1 \quad v_{-1} \otimes v_{-1}$

$$H \mapsto \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad E \mapsto \begin{pmatrix} 0 & 1 & e^{\hbar} & 0 \\ 0 & 0 & 0 & e^{-\hbar} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ e^{-\hbar} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & e^{\hbar} & 0 \end{pmatrix}$$

$$L_1 \otimes L_1 = L_2 \oplus L_0 \qquad L_0 = \langle e^{\hbar} v_1 \otimes v_{-1} - v_{-1} \otimes v_1 \rangle$$

$$\Delta E = E \otimes e^{\hbar H} + 1 \otimes E, \quad \Delta F = F \otimes 1 + e^{-\hbar H} \otimes F$$

$$\Delta \quad \text{vs} \quad \Delta^{op} = P \circ \Delta$$

$$P = \sigma_{12}$$

- $V_1, \ V_2 \qquad V_1 \otimes V_2$

$\Delta : A \to A \otimes A \qquad \Big| \qquad \Delta^{op} : A \to A \otimes A$

- In our Example

$L_1 \otimes_{\Delta} L_1 = L_2 \oplus L_0 \qquad \Big| \qquad L_1 \otimes_{\Delta^{op}} L_1 = L_2 \oplus L_0$

$L_0 = \langle e^{\hbar} v_1 \otimes v_{-1} - v_{-1} \otimes v_1 \rangle \qquad \Big| \qquad L_0 = \langle v_1 \otimes v_{-1} - e^{\hbar} v_{-1} \otimes v_1 \rangle$

- $R : \quad L_1 \otimes_{\Delta} L_1 \to L_1 \otimes_{\Delta^{op}} L_1$

$$R = \begin{pmatrix} e^{\hbar} & 0 & 0 & 0 \\ 0 & 1 & e^{\hbar} - e^{-\hbar} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\hbar} \end{pmatrix}$$

R matrix

- $R: V_1 \otimes_\Delta V_2 \longrightarrow V_1 \otimes_{\Delta^{op}} V_2$
- $\tilde{R} = PR \;:\; V_1 \otimes_\Delta V_2 \longrightarrow V_2 \otimes_\Delta V_1$

$$\begin{pmatrix} e^{\hbar} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & e^{\hbar} - e^{-\hbar} & 0 \\ 0 & 0 & 0 & e^{\hbar} \end{pmatrix} \qquad \underbrace{(\tilde{R} - e^{\hbar})(\tilde{R} + e^{-\hbar}) = 0}_{\text{Hecke relation.}}$$

- Prolem* a) Show directly that $L_1 \otimes L_\ell = L_{\ell+1} \oplus L_{\ell-1}$, $\ell \geq 1$

  b) Show that $L_{\ell_1} \otimes L_{\ell_2} = \bigoplus\limits_{\substack{|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2 \\ \ell + \ell_1 + \ell_2 - \text{even}}} L_\ell$

References

- Chari, Pressley *A guide to quantum groups*
  Sec. 4.1

- Etingof, Gelaki, Nikshych, Ostrik *Tensor categories*
  Ch 2,4,5 (very partially)