

Introduction to Quantum Groups

Lecture 6 Hopf algebras

gft.itp.ac.ru/mbe/sht/quantum_groups.html

Hopf algebra

$(A, m, i, \Delta, \varepsilon, S)$

$\Delta: A \rightarrow A \otimes A$, $m: A \otimes A \rightarrow A$, $i: \mathbb{C} \rightarrow A$, $\varepsilon: A \rightarrow \mathbb{C}$, $S: A \rightarrow A$

The diagram shows five string diagrams. From left to right: 1) Two strands enter a single point, which then splits into two strands. 2) Two strands enter a single point, which then splits into two strands. 3) A single strand enters from the left and ends at a point. 4) A single strand enters from the left and ends at a point. 5) A single strand enters a circle labeled 'S' from the left, and exits from the right.

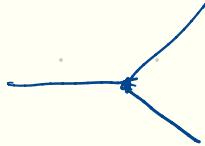
$$\text{Assosiat.} = \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array}$$

$$\text{Coassosiat.} = \begin{array}{c} \diagup \\ \diagdown \end{array} = \begin{array}{c} \diagdown \\ \diagup \end{array}$$

$$\text{unit} = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ \bullet \end{array}, \quad \text{counit} = \begin{array}{c} \diagup \quad \diagdown \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \end{array}$$

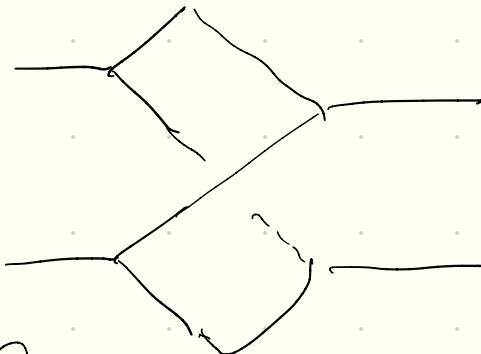
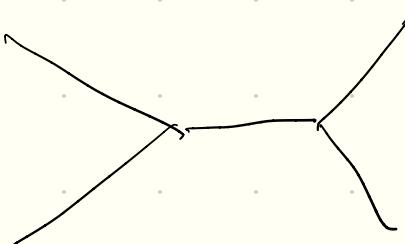
Hopf algebra

$$\Delta: A \rightarrow A \otimes A, \quad m: A \otimes A \rightarrow A, \quad i: \mathbb{C} \rightarrow A, \quad \epsilon: A \rightarrow \mathbb{C}, \quad S: A \rightarrow A$$

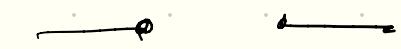
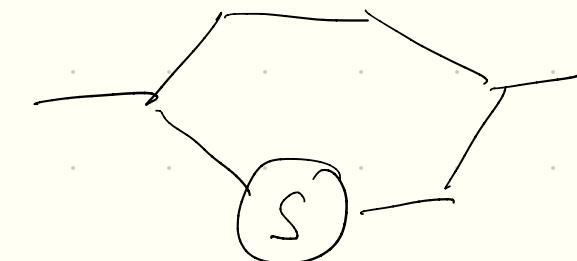
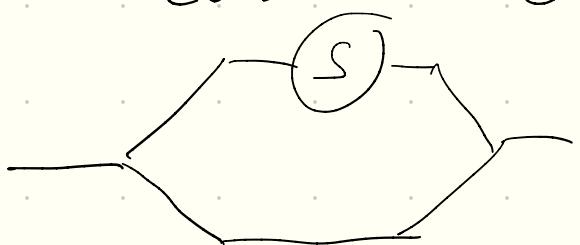


- Δ is homomorphism

$$\Delta(a \cdot b) = \Delta(a)\Delta(b)$$



- Relation on S



$$i\epsilon(a)$$

$U_h(SL_2)$

- $E, H, F :$ $[H, E] = 2E$, $[H, F] = -2F$

$$[E, F] = \frac{e^{\frac{\hbar H}{\hbar}} - e^{-\frac{\hbar H}{\hbar}}}{e^{\frac{\hbar}{\hbar}} - e^{-\frac{\hbar}{\hbar}}}$$

- $\Delta(E) = E \otimes e^{\frac{\hbar H}{\hbar}} + 1 \otimes E$, $\Delta(F) = F \otimes 1 + e^{-\frac{\hbar H}{\hbar}} \otimes F$
 $\Delta H = H \otimes 1 + 1 \otimes H'$

- ϵ, ξ, S are determined by m, Δ

$$\gamma = -$$

$$\epsilon_1 \Delta(a) = a$$

$\epsilon = \epsilon \otimes 1$

$$a = 1, \quad 1 = \epsilon_1(\Delta(1)) = \epsilon(1 \otimes 1) = \epsilon(1) 1 \quad \epsilon(1) = 1$$

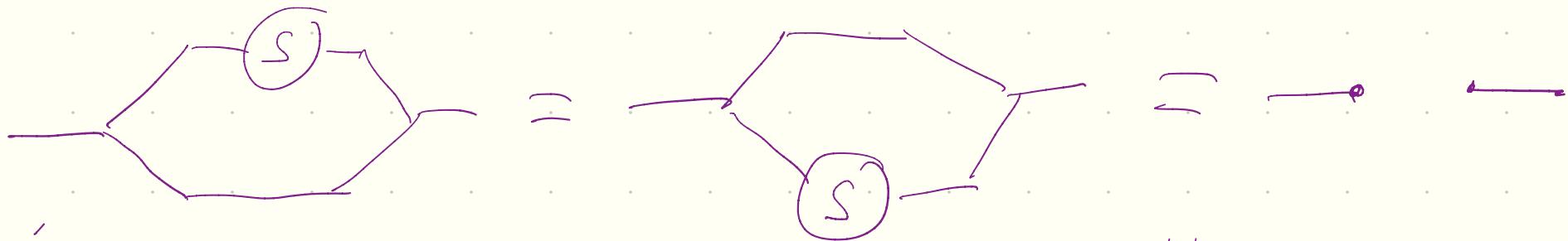
$$a = H, \quad H = \epsilon_1(\Delta(H)) = \epsilon(H) 1 + H \quad \epsilon(H) = 0$$

$$a = E, \quad E = \epsilon_1(\Delta(E)) = \epsilon(E) e^{\frac{\hbar H}{\hbar}} + E \quad \epsilon(E) = 0$$

$$\epsilon_2(\Delta(E)) = \epsilon(e^{\frac{\hbar H}{\hbar}}) E + \epsilon(E) \quad \epsilon(F) = 0$$

$U_h(SL_2)$

- $[H, E] = 2E, [H, F] = -2F, [E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}$
- $\Delta(E) = E \otimes e^{\hbar H} + 1 \otimes E, \Delta(F) = F \otimes 1 + e^{-\hbar H} \otimes F$
 $\Delta H = H \otimes 1 + 1 \otimes H$



$$0 = i\varepsilon(E) = m(S \otimes 1) \Delta E = m(S \otimes 1) (e \otimes e^{\hbar H} + 1 \otimes E) =$$

$$0 = i\varepsilon(H) = m(S \otimes 1) (H \otimes 1 + 1 \otimes H) = S(H) + H$$

- $S(E) = -E e^{-\hbar H}, \quad S(F) = -e^{\hbar H} F, \quad S(H) = -H$

S^2

- $S(E) = -E e^{-\hbar H}, \quad S(F) = -e^{\hbar H} F, \quad S(H) = -H.$

$$S^2(E) = -S(e^{-\hbar H})S(E) = e^{\hbar H} E e^{-\hbar H} \quad \text{conjugation by } e^{\hbar H}.$$

- Rem If A is comm or cocomm then $S^2 = \text{id}$

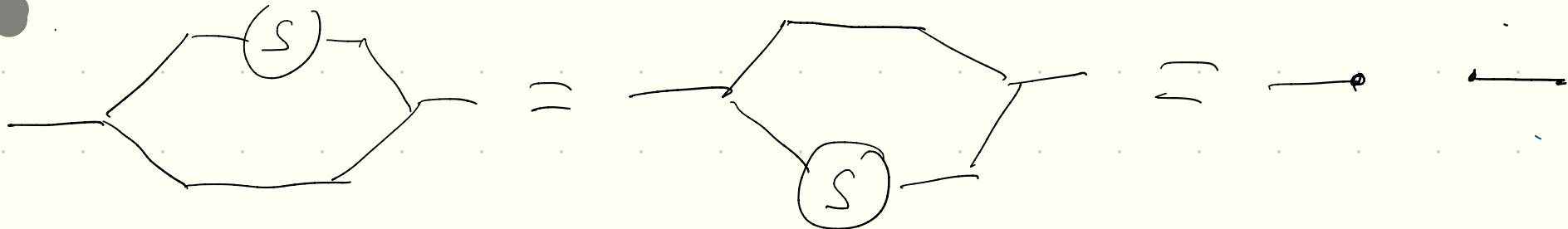
- In our case S^2 is conjugation by $e^{\hbar H}$
Sufficient to check on generators E, H, F due to

- Problem S is antihom of algebra and coalgebra
Hint Let $(\Delta \otimes \text{id})\Delta(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}, \quad (\Delta \otimes \text{id})\Delta(b) = \sum b_{(1)} \otimes b_{(2)} \otimes b_{(3)}$

Then $S(ab) = \sum S(a_{(1)}b)a_{(2)}S(a_{(3)})$
 $= \sum S(a_{(1)}b_{(1)})a_{(2)}b_{(2)}S(b_{(3)})S(a_{(3)}) = S(b)S(a)$

Remarks

- Usually we assume S is bijective.
- Def $f, g : A \rightarrow A$ Convolution $f \cdot g = m(f \circ g) \Delta$
 $\Delta(a) = a_{(1)} \otimes a_{(2)}$ $f \cdot g(a) = f(a_{(1)}) g(a_{(2)})$
ie : $A \rightarrow A$ — unit of the convolution product



Defining property of $S \Leftrightarrow S \cdot id = ie \rightarrow$
 S is determined by Δ, m .

sl_2

representations

- $[h, e] = 2e, [h, f] = -2f, [e, f] = h$

- $L_e = \langle v_e, v_{e-2}, v_{e-4}, \dots, v_{-e} \rangle$

$$e v_e = 0, \quad h v_e = e v_m.$$

Let $v_{e-2k} = f^k v_e$

$$f v_m = v_{m-2}, \quad h v_m = m v_m,$$

$$e v_m = \left(\frac{e-m}{2}\right) \left(\frac{e+m+2}{2}\right) v_{m+2}$$

- Another normalization

$$\tilde{v}_{e-2k} = \frac{1}{k!} f^k v_e$$

$$f \tilde{v}_m = \left(\frac{e-m+2}{2}\right) \tilde{v}_{m-2}, \quad h \tilde{v}_m = m \tilde{v}_m,$$

$$e \tilde{v}_m = \left(\frac{e+m+2}{2}\right) \tilde{v}_{m+2}$$

$U_{\hbar}(\mathfrak{sl}_2)$

representations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}$$

$L_e = \langle v_e, v_{e-2}, v_{e-4}, \dots, v_e \rangle$
 $E v_e = 0, \quad H v_e = e^{\hbar} v_e$ Let $v_{e-k} = F^k v_e$

Problem a) Find formulas for action of E, H, F
 b) Define basis \tilde{v}_m

Example $L_1 = \mathbb{C}^2 = \langle v_1, v_{-1} \rangle$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

same formulas
since for $H = \pm 1$

$$\frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}} = H$$

Representations of Hopf algebras

$$\Delta: A \rightarrow A \otimes A \quad | \quad \epsilon: A \rightarrow \mathbb{C}$$

$$V_1, V_2 \in \text{Mod}_A \quad \rightsquigarrow \quad V_1 \otimes V_2 \in \text{Mod}_A \quad | \quad \mathbb{C} \in \text{Mod}_A$$

Rem. Tensor product of group reps or Lie algebra reps is consequence of \exists of Δ

$$S: A \rightarrow A$$

$$V \in \text{Mod}_A$$

$$V^* \quad f_{V^*}(a) = f_V(S(a))^*$$

$${}^*V \quad f_{{}^*V}(a) = f_V(S^{-1}(a))^*$$

$${}^*(V^*) = V = ({}^*V)^*, \quad (V^*)^* \neq V \quad \text{but could be } \simeq$$

Rem If S^2 is conjugation by u , then ${}^*V = V^*$

Properties of Mod_A

$$\begin{array}{c} \text{Diagram showing two ways to contract three strands into one, resulting in equality.} \\ = \end{array}$$

monoidal category

$$(V_1 \otimes V_2) \otimes V_3 = V_1 \otimes (V_2 \otimes V_3)$$

$$\begin{array}{c} \text{Diagram showing a pentagon with a circle labeled 'S' inside, followed by an equals sign and a single horizontal line.} \\ = \end{array}$$

Rigid monoidal category

$$\exists \quad V^* \otimes V \rightarrow \mathbb{C} \quad \mathbb{C} \rightarrow V \otimes V^*$$

● Problem a) Show \exists of maps above.

b) Show that $(v \otimes w)^* = w^* \otimes v^*$ Hint: use problem above

Tensor product for SL_2

$$L_1 = \mathbb{C}^2$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad V_1 \quad V_0$$

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$L_1 \otimes L_1 \quad V_1 \otimes V_1 \quad V_1 \otimes V_{-1} \quad V_{-1} \otimes V_1 \quad V_1 \otimes V_{-1}$$

$$H \mapsto \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad E \mapsto \begin{pmatrix} 0 & 1 & e^{\frac{h}{2}} & 0 \\ 0 & 0 & 0 & e^{-\frac{h}{2}} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad F \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ e^{-\frac{h}{2}} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & e^{\frac{h}{2}} & 0 \end{pmatrix}$$

$$L_1 \otimes L_1 = L_2 \oplus L_0 \quad L_0 = \langle e^{\frac{h}{2}} V_1 \otimes V_{-1} - V_1 \otimes V_1 \rangle$$

$$\Delta E = E \otimes e^{\frac{h}{2} H} + 1 \otimes E \quad , \quad \Delta F = F \otimes 1 + e^{-\frac{h}{2} H} \otimes F$$

$$\Delta \quad \text{vs} \quad \Delta^{op} = P \circ \Delta \quad P = G_{12}$$

$$V_1 \quad V_2 \quad V_1 \otimes V_2$$

$$\Delta: A \rightarrow A \otimes A$$

$$\Delta^{op}: A \rightarrow A \otimes A$$

In our Example

$$L_1 \otimes L_1 = L_2 \oplus L_0$$

$$L_0 = \langle e^t V_1 \otimes V_1 - V_1 \otimes V_1 \rangle$$

$$L_1 \otimes_{\Delta^{op}} L_1 = L_2 \oplus L_0$$

$$L_0 = \langle V_1 \otimes V_1 - e^t V_1 \otimes V_1 \rangle$$

$$R: \quad L_1 \otimes_{\Delta} L_1 \downarrow \quad L_1 \otimes_{\Delta^{op}} L_1$$

$$R = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & 1 & e^t - e^{-t} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^t \end{pmatrix}$$

R matrix

- $R: V_1 \otimes V_2 \rightarrow V_1 \otimes V_2$

- $\tilde{R} = PR: V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$

$$\begin{pmatrix} e^{\frac{h}{2}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & e^{\frac{h}{2}} - e^{-\frac{h}{2}} & 0 \\ 0 & 0 & 0 & e^{\frac{h}{2}} \end{pmatrix} \quad \left(\tilde{R} - e^{\frac{h}{2}} \right) \left(\tilde{R} + e^{-\frac{h}{2}} \right) = 0$$

Hecke relation.

- Problem*
 - Show directly that $L_1 \otimes L_e = L_{e+1} \oplus L_{e-1}$, $e \geq 1$
 - Show that $L_{e_1} \otimes L_{e_2} = \bigoplus_{\substack{|e_1 - e_2| \leq l \leq e_1 + e_2 \\ l + e_1, l + e_2 - \text{even}}} L_l$

References

- Chari, Pressley A guide to quantum groups
Sec. 4.1
- Etingof, Gelaki, Nikshych, Ostrik Tensor categories
Ch 2,4,5 (very partially)