Introduction to Quantum Groups

Lecture 7

Quantum R-matrices

qft.itp.ac.ru/mbersht/quantum_groups.html

Hopf algebras

- $(A, m, i, \Delta, \varepsilon, S)$
- Mod_A $\otimes$, $\mathbb{C}$, V^*, ${}^*V$
- <u>Want</u> $V \otimes W \simeq W \otimes V$ $\qquad \tilde{R}_{V,W} = P R_{V,W}$

  $V \underset{\Delta}{\otimes} W \simeq V \underset{\Delta^{op}}{\otimes} W$ $\qquad R_{V,W}$

  $\Delta^{op} = P\Delta$

- $$\begin{tikzcd}
(V_1 \otimes V_2) \otimes V_3 \arrow[rr, "\tilde{R}_{V_1 \otimes V_2, V_3}"] \arrow[dr, "Id \otimes \tilde{R}_{V_2, V_3}"'] & & V_3 \otimes (V_1 \otimes V_2) \\
& V_1 \otimes V_3 \otimes V_2 \arrow[ur, "\tilde{R}_{V_1, V_3} \otimes Id"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
V_1 \otimes V_2 \otimes V_3 \arrow[rr, "\tilde{R}_{V_1, V_2 \otimes V_3}"] \arrow[dr, "\tilde{R}_{V_1, V_2} \otimes Id"'] & & V_2 \otimes V_3 \otimes V_1 \\
& V_2 \otimes V_1 \otimes V_3 \arrow[ur, "Id \otimes \tilde{R}_{V_1, V_3}"'] &
\end{tikzcd}$$

<u>Want</u>

Quasitriangular structure

- **Def** Quasitriangular str on Hopf algebra A is invertible $R \in A \otimes A$ s.t
  - $R\Delta(x) = \Delta^{op}(x) R \quad \forall x \in A$
  - $(\Delta \otimes \mathrm{Id})(R) = R_{13} R_{23}$, $(\mathrm{Id} \otimes \Delta) R = R_{13} R_{12}$

- Notation $R = \sum a_i \otimes b_i \qquad R_{13} = \sum a_i \otimes 1 \otimes b_i$,
  $R_{23} = \sum 1 \otimes a_i \otimes b_i \qquad (\Delta \otimes \mathrm{Id}) R = \sum \Delta(a_i) \otimes b_i$

- R is called universal R-matrix

$\forall\ V, W \in \mathrm{Mod}_A \quad (\pi_V \otimes \pi_W) R : \ V \otimes_\Delta W \to V \otimes_{\Delta^{op}} W$ — intertwiner

$(\Delta \otimes \mathrm{Id}) \Delta(x) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \qquad V_1 \otimes V_2 \otimes V_3 \longrightarrow V_1 \otimes V_2 \otimes V_3$

$\pi_1(x_{(1)}) \otimes \pi_2(x_{(2)}) \otimes \pi_3(x_{(3)}) \qquad \pi_1(x_{(2)}) \otimes \pi_2(x_{(3)}) \otimes \pi_3(x_{(1)})$

Two intertwiners should equal..

QYBE

- **Th** $$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$

- **Rem** In terms of $\tilde{R}$ we have

Left side $\sigma_{12} \tilde{R}_{12} \sigma_{13} \tilde{R}_{13} \sigma_{23} \tilde{R}_{23} = \sigma_{12}\sigma_{13}\sigma_{23} \tilde{R}_{23} \tilde{R}_{12} \tilde{R}_{23}$

Right side $\sigma_{23} \tilde{R}_{23} \sigma_{13} \tilde{R}_{13} \sigma_{12} \tilde{R}_{12} = \sigma_{23}\sigma_{13}\sigma_{12} \tilde{R}_{12} \tilde{R}_{23} \tilde{R}_{12}$

$$\tilde{R}_{23} \tilde{R}_{12} \tilde{R}_{23} = \tilde{R}_{12} \tilde{R}_{23} \tilde{R}_{12} \quad \text{— braid relation}$$

- **Rem** In terms of reps

$$\begin{tikzcd}
 & V_1 \otimes V_3 \otimes V_2 \arrow[r] & V_3 \otimes V_1 \otimes V_2 \arrow[dr] & \\
V_1 \otimes V_2 \otimes V_3 \arrow[ur] \arrow[dr] & & & V_3 \otimes V_2 \otimes V_1 \\
 & V_2 \otimes V_1 \otimes V_3 \arrow[r] & V_2 \otimes V_3 \otimes V_1 \arrow[ur] &
\end{tikzcd}$$

QYBE

- **Th** $\quad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$

- **Proof** $\quad R = \sum a_i \otimes b_i = \sum c_i \otimes d_i$

$$R_{12}R_{13}R_{23} = R_{12}(\Delta \otimes Id_A)R = \left[\begin{array}{c} (a_i \otimes b_i \otimes 1)(\Delta(c_i) \otimes d_i) \\ \| \\ (\Delta^{op}(c_i)\otimes d_i)(a_i \otimes b_i \otimes 1) \end{array}\right]$$

$$= (\Delta^{op} \otimes Id_A)(R)R_{12} = \sigma_{12}\big((\Delta \otimes Id_A)(R)\big)R_{12} =$$

$$= \sigma_{12}(R_{13}R_{23})R_{12} = R_{23}R_{13}R_{12}.$$

- $R\,\Delta(x) = \Delta^{op}(x)\,R$
- $(\Delta \otimes Id)(R) = R_{13}R_{23}$
- $(id \otimes \Delta)R = R_{13}R_{12}.$

Quasitriangular QUE

- <u>Prop</u> $U_\hbar(\mathfrak{g})$ be QUE with $R = 1 + \hbar r \pmod{\hbar^2}$.
  Then $r \in \mathfrak{g} \otimes \mathfrak{g}$ and $\delta(x) = ad_x r$.

- Proof $\quad \delta(a) = \dfrac{\Delta(a) - \Delta^{op}(a)}{\hbar} \mod \hbar$

- $(\Delta \otimes Id)(R) = R_{13} R_{23}$

  $(\Delta_0 \otimes 1) r = r_{13} + r_{23} \qquad \Delta_0 : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$

  $\Delta_0(a_i) = a_i \otimes 1 + 1 \otimes a_i \implies a_i \in \mathfrak{g}$

  Similarly $b_i \in \mathfrak{g} \implies r \in \mathfrak{g} \otimes \mathfrak{g}$

- $R\Delta(x) = \Delta^{op}(x) R \quad \implies \delta(x) = \dfrac{\Delta(x) - \Delta^{op}(x)}{\hbar} = ad_x r$

- <u>Rem</u> QYBE $R \implies$ CYBE r

Unitary R

- $R\Delta(x) = \Delta^{op}(x) R \Rightarrow R_{21}\Delta^{op} = \Delta R_{21} \rightarrow$
  $\Rightarrow (R_{21})^{-1}\Delta = \Delta^{op}(R_{21})^{-1} \Rightarrow R_{21}^{-1}$ intertwines Δ and Δ^{op}
- Def R is unitary if $RR^{21} = Id$
- R unitary $\Rightarrow r + r^{21} = 0 \Rightarrow r \in \Lambda^2 \mathfrak{g}$
- Def R is triangular if R is quasitriangular and unitary

$$U_\hbar(sl_2)$$

- $[H,E]=2E$, $[H,F]=-2F$, $[E,F]=\dfrac{e^{\hbar H}-e^{-\hbar H}}{e^{\hbar}-e^{-\hbar}}$
- $\Delta E = E\otimes e^{\hbar H}+1\otimes E$, $\Delta F = F\otimes 1 + e^{-\hbar H}\otimes F$
- Th $\quad R = e^{\frac{1}{2}\hbar H\otimes H}\sum_{n=0}^{\infty} q^{\binom{n}{2}}\dfrac{(q-q^{-1})^n}{[n]_q!}(E^n\otimes F^n)$
- Notations $\quad q=e^{\hbar}$, $[n]_q=\dfrac{q^n-q^{-n}}{q-q^{-1}}$ $\quad [n]_q! = [n]_q[n-1]_q\cdots[1]_q$

  $\binom{n}{2}=\dfrac{n(n-1)}{2}$.

- Can be proven by direct computation.

More conceptual proof Drinfeld double later

$$U_\hbar(sl_2)$$

- **Problem** Show that $\Delta^{op}(E)R = R\Delta(E)$.

Hint We want:

$$(E\otimes 1 + e^{\hbar H}\otimes E)e^{\frac{1}{2}\hbar H\otimes H}\sum_{n=0}^{\infty} q^{\binom{n}{2}}\frac{(q-q^{-1})^n}{[n]_q!}(E^n\otimes F^n) -$$

$$e^{\frac{1}{2}\hbar H\otimes H}\sum_{n=0}^{\infty} q^{\binom{n}{2}}\frac{(q-q^{-1})^n}{[n]_q!}(E^n\otimes F^n)(E\otimes e^{\hbar H} + 1\otimes E) = 0$$

$$\sum(\cdots)E^{n+1}\otimes F^n$$

- $$(E\otimes 1)e^{\frac{1}{2}\hbar H\otimes H} = e^{\frac{1}{2}\hbar H\otimes H}E\otimes e^{-\hbar H}$$

- $$[E, F^{n+1}] = \frac{[n+1]_q}{q-q^{-1}}\left(e^{\hbar(H+n)} - e^{-\hbar(H+n)}\right)F^n$$

Example

- $L_1 = \mathbb{C}^2 \qquad v_1 \quad v_0$

$$H \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- $R \qquad L_1 \otimes L_1$

$$e^{\frac{1}{2}\hbar H\otimes H} \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} (E^n \otimes F^n) \rightsquigarrow e^{\frac{1}{2}\hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \left(1 + (q-q^{-1}) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) =$$

$$= q^{-\frac{1}{2}} \begin{pmatrix} q & & & \\ & 1 & & \\ & & 1 & \\ & & & q \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & q-q^{-1} & \\ & & 1 & \\ & & & 1 \end{pmatrix} = q^{-\frac{1}{2}} \begin{pmatrix} q & & & \\ & 1 & q-q^{-1} & \\ & & 1 & \\ & & & q \end{pmatrix}$$

R matrix and duals

- Duality: V^* $\quad \rho_{V^*}(a) = \rho_V(S(a))^*$

  $V^* \otimes V \to \mathbb{C}, \qquad \mathbb{C} \to V \otimes V^*$

- $\exists R \rightsquigarrow V \to V \otimes V^* \otimes V^{**} \xrightarrow{P R \otimes id} V^* \otimes V \otimes V^{**} \to V^{**}$

- In basis $V = \langle e_j \rangle$, $V^* = \langle e^j \rangle$, $V^{**} = \langle e_j \rangle$, $R = \sum a_i \otimes b_i$

$$e_k \mapsto \sum e_k \otimes e^j \otimes e_j \mapsto \sum b_i e^j \otimes a_i e_k \otimes e_j \mapsto \sum (S(b_i) a_i)^j_k \, e_j$$

- Define $u = \sum S(b_i) a_i$, intertwines

  V and V^{**} (or ${}^*V$ and V^*)

  a $\qquad S^2(a)$

S^2 — conjugation

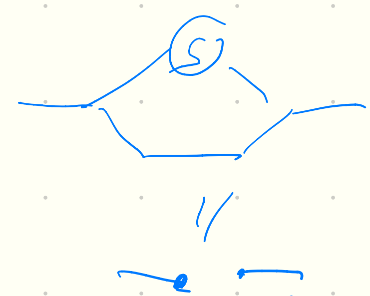
- Let $R=\sum a_i\otimes b_i$, $R^{-1}=\sum c_i\otimes d_i$. $u=\sum S(b_i)a_i$

- <u>Th</u> a) $\forall x\in A$, $S^2(x)=uxu^{-1}$ b) $u^{-1}=\sum S^{-1}(d_i)c_i$

- <u>Proof</u> a) $(\Delta\otimes id)\Delta(x)=x_{(1)}\otimes x_{(2)}\otimes x_{(3)}$

$$\sum a_i x_{(1)}\otimes b_i x_{(2)}\otimes x_3=\sum x_{(2)}a_i\otimes x_{(1)}b_i\otimes x_{(3)}$$

$$\sum S^2(x_{(3)})S(b_i x_{(2)})a_i x_{(1)}=\sum S^2(x_{(3)})S(x_{(1)}b_i)x_{(2)}a_i$$



$$\sum S(x_{(2)}S(x_{(3)}))S(b_i)a_i x_{(1)}=\sum S^2(x_{(3)})S(b_i)S(x_{(1)})x_{(2)}a_i$$

$$ux=S^2(x)u.$$

b) $\sum_j uS^{-1}(d_j)c_j=\sum_j S(d_j)uc_j=\sum S(b_id_j)a_ic_j=1$

Some central elements

- $R=\sum a_i\otimes b_i$, $R^{-1}=\sum c_i\otimes d_i$, $u=\sum S(b_i)a_i$, $u^{-1}=\sum S^{-1}(d_i)c_i$

- R_{21}^{-1} intertwines Δ and Δ^{op} $\longrightarrow$
$v=\sum S(c_i)d_i$, $vxv^{-1}=S^2(x)$ $\longrightarrow$
$uv^{-1}=u\,S(u)$ — central

cf. $\mathrm{Tr}\,R_{21}R_{12}$ in integr. models

- $A=U_\hbar(sl_2)$, S^2 is conj by $e^{\hbar H}$.
Hence $e^{-\hbar H}u$ is central

- Problem a) Show $C_\hbar=FE+\dfrac{e^{\hbar(H+1)}+e^{-\hbar(H+1)}}{(e^{\hbar}-e^{-\hbar})^2}$ is central
b) Find action of $C_\hbar$ and $e^{-\hbar H}u$ on L_m
c) $\varphi_\hbar^{-1}: U(sl_2)[[\hbar]]\to U_\hbar(sl_2)$ isomorphism.
$c=fe+\frac{h(h+2)}{4}$. Find $\varphi_\hbar^{-1}(c)$ and $\varphi_\hbar^{-1}(e^{\hbar c})$
on L_m, relate to elements above.

References

- Chari, Pressley A guide to quantum groups
  Sec. 4.2
- Drinfeld Almost cocommutative Hopf algebras