Introduction to Quantum Groups

Lecture 9

Drinfeld-Jimbo quantum groups

qft.itp.ac.ru/mbersht/quantum_groups.html

Simple Lie algebras

- For brevity $\mathfrak{g}$ is ADE

- $i \in I$. generators: h_i, e_i, f_i

$$A_{A_n} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & & 0 & \\ & -1 & \ddots & & \\ & 0 & & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}$$

- $A = (a_{ij})$ — Cartan matrix

$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j$

$[e_i, f_j] = \delta_{ij} h_i$

$(\mathrm{ad})_{e_i}^{1-a_{ij}} e_j = 0, \quad (\mathrm{ad})_{f_i}^{1-a_{ij}} f_j = 0$ — Serre relations

- $e_i^2 e_j - 2 e_i e_j e_i + e_j e_i^2 = 0 \qquad i - j \qquad a_{ij} = -1$

$e_i e_j - e_j e_i = 0 \qquad i \quad j \qquad a_{ij} = 0$

What we know

- $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}$, $\mathfrak{g}_+ = \mathfrak{g}$, $\mathfrak{g}_- = \mathfrak{b}_+ \oplus \mathfrak{b}_- / \mathfrak{h}$
  $\delta(e_i) = e_i \wedge h_i$, $\delta(f_i) = f_i \wedge h_i$, $\delta(h_i) = 0$

- $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{h}$ $\mathfrak{g}_+ = \mathfrak{b}_+$, $\mathfrak{g}_- = \mathfrak{b}_-$

- $\mathfrak{g} = D(\mathfrak{b}_+)/\mathfrak{h}$, here $D(\mathfrak{b}_+)$ – double of bialgebra.

- $U_\hbar(sl_2)$

$\Delta E = E \otimes e^{\hbar H} + 1 \otimes E$, $\Delta F = F \otimes 1 + e^{-\hbar H} \otimes F$, $\Delta H = H \otimes 1 + 1 \otimes H$

$$U_\hbar(\mathfrak{b}_+)$$

- $H_i, E_i,\quad i \in I$
- $[H_i, H_j] = 0, \quad [H_i, E_j] = a_{ij} E_j$

$$\Delta E_i = E_i \otimes e^{\hbar H_i} + 1 \otimes E_i, \quad \Delta H_i = H_i \otimes 1 + 1 \otimes H_i$$

- q-Serre relations $\displaystyle\sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q E_i^k E_j E_i^{1-a_{ij}-k} = 0$
- $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]_q!}{[k]_q!\,[n-k]_q!}$ $\quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q = \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q = \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q = 1$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = \frac{[2]_q!}{[1]_q!^2} = [2]_q = \frac{(q^2 - q^{-2})}{(q - q^{-1})} = q + q^{-1} \qquad q = e^{\hbar}$$

- $a_{ij} = 0 \qquad E_i E_j - E_j E_i = 0$

$a_{ij} = -1 \qquad E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0$

q- Serre relations

- Denote $K_i = e^{\hbar H_i}$. Sufficient $\mathfrak{g} = \mathfrak{sl}_3$. $K_1 E_2 = q^{-1} E_2 K_1$

- $$\Delta\left(E_1^2 E_2 - (q+q^{-1}) E_1 E_2 E_1 + E_2 E_1^2\right)$$
$$= (E_1 \otimes K_1 + 1 \otimes E_1)(E_1 \otimes K_1 + 1 \otimes E_1)(E_2 \otimes K_2 + 1 \otimes E_2)$$
$$- (q+q^{-1})(E_1 \otimes K_1 + 1 \otimes E_1)(E_2 \otimes K_2 + 1 \otimes E_2)(E_1 \otimes K_1 + 1 \otimes E_1)$$
$$+ (E_2 \otimes K_2 + 1 \otimes E_2)(E_1 \otimes K_1 + 1 \otimes E_1)(E_1 \otimes K_1 + 1 \otimes E_1)$$
$$= (q\text{-Serre}) \otimes K_1^2 K_2 + E_1^2 \otimes E_2 K_1^2 \left(q^{-2} - (q+q^{-1}) q^{-1} + 1\right)$$
$$+ E_1 E_2 \otimes E_1 K_1 K_2 \left((1+q^2) - (q+q^{-1}) q\right) + E_2 E_1 \otimes E_1 K_1 K_2 \left(-(q+q^{-1}) + q^{-1}(1+q^2)\right)$$
$$+ (E_2 \otimes E_1^2 K_2)\left(1 - (q+q^{-1}) q^{-1} + q^{-2}\right) + E_1 \otimes E_1 E_2 K_1 \left((q+q^{-1}) - (q+q^{-1})\right)$$
$$+ E_1 \otimes E_2 E_1 K_1 \left(-q(q+q^{-1}) + 1 + q^2\right) + 1 \otimes (q\text{-Serre}) = 0$$

- Remark. $ab = q\, ba$.
$$a^2 b - (q+q^{-1}) aba + b a^2 = 0, \qquad ab^2 - (q+q^{-1}) bab + b^2 a = 0$$

- <u>Remark</u> $\Delta(q\text{-Serre}) = q\text{-Serre} \otimes K_{q\text{-Serre}} + 1 \otimes q\text{-Serre}$ — "Lie type" element

$$\mathcal{U}_\hbar(\mathfrak{g})$$

- Generators E_i, H_i, F_i
- Relations $[H_i, H_j]=0$ $[H_i, E_j] = a_{ij} E_j$, $[H_i, F_j] = -a_{ij} F_j$

  q-Serre E_i, q-Serre F_j

$$[E_i, F_j] = \delta_{ij} \frac{e^{\hbar H_i} - e^{-\hbar H_i}}{e^{\hbar} - e^{-\hbar}}$$

($i \neq j$ ... no weight $\alpha_i - \alpha_j$
$i = j$ ... know from sl_2)

- $\Delta E_i = E_i \otimes e^{\hbar H_i} + 1 \otimes E_i$, $\Delta F_i = F_i \otimes 1 + e^{-\hbar H_i} \otimes F_i$, $\Delta H_i = H_i \otimes 1 + 1 \otimes H_i$

- Problem $[F_i, q\text{-Serre } E_i] = 0$
- Problem Find $H \in \mathfrak{h}$ s.t. $S^2(x) = e^{\hbar H} x e^{-\hbar H}$

Hopf pairing

- **Def** Given bialgebras A^-, A^+ a bialgebra pairing $(\cdot,\cdot): A^- \otimes A^+ \to \mathbb{C}$ s.t

$$(a \cdot a', b) = (a \otimes a', \Delta(b)) = (a, b_{(1)})(a', b_{(2)})$$
$$(a, b \cdot b') = (\Delta^{op}(a), b \otimes b') = (a_{(2)}, b)(a_{(1)}, b')$$

- Equivalenly $\quad (A^-)^{coop} \longrightarrow (A^+)^*$

$$a \longmapsto (a, \cdot)$$

Self duality of $\mathcal{U}_\hbar(\mathfrak{b})$

- <u>Th</u> (Drinfeld) $\exists!$ non degenerate pairing
  $\mathcal{U}_\hbar(\mathfrak{b}^-)\otimes\mathcal{U}_\hbar(\mathfrak{b}^+)\to\mathbb{C}$ s.t.

  - $(e^{\hbar H_i}, e^{\hbar H_j}) = e^{+\hbar a_{ij}}$
  - $(e^{\hbar H_i}, E_j) = (F_i, e^{\hbar H_j}) = 0$
  - $(F_i, E_j) = \delta_{i,j}\dfrac{1}{e^{\hbar}-e^{-\hbar}}$

- $(\cdot,\cdot)$ defined on generators $\longrightarrow$ uniqueness
  In particular $(E_i e^{\hbar H_j}, F_i) = \delta_{i,j}\dfrac{1}{e^{\hbar}-e^{-\hbar}}$

- <u>Rem.</u> Root lattice grading $\longrightarrow (F_i, E_j)\sim\delta_{i,j}$

- For $\hbar\to 0$ $\mathcal{U}(\mathfrak{b}^+)$ and $\mathcal{U}(\mathfrak{b}^-)$ are <u>not</u> dual as Hopf alg.
  But $\mathfrak{b}^+$ and $\mathfrak{b}^-$ are dual as bialgebras.

Drinfeld theorem, proof

- $U_\hbar(\mathfrak{b}^+) = U_\hbar(\mathfrak{h}) \otimes U_\hbar(\mathfrak{n}^+)$ as vector space (and algebra) (but not coalgebra)

$$U_\hbar(\mathfrak{b}^+) = \bigoplus_{\beta\in Q^+} U_\hbar(\mathfrak{b}^+)_\beta = \bigoplus_{\beta\in Q^+} U_\hbar(\mathfrak{h})\otimes U_\hbar(\mathfrak{n}^+)$$

Q_+ grading

- $\forall i$ $\quad \Psi_i \in U_\hbar(\mathfrak{h})^* \subset U(\mathfrak{b}^+)^*$: $\quad \Psi_i(e^{\hbar H_j}) = e^{\hbar a_{ij}}$ $\quad$ generators

$X_i \in U_\hbar(\mathfrak{b}^+)^*_{\alpha_i} \subset U(\mathfrak{b}^+)^*$: $\quad X_i(e^{\hbar H} E_i) = 1 \quad H \in \mathfrak{h}$

- $\Delta\Psi_i(e^{\hbar H_j}\otimes e^{\hbar H_k}) = \Psi_i(e^{\hbar(H_j+H_k)}) = e^{\hbar(a_{ij}+a_{ik})} \longrightarrow \Delta\Psi_i = \Psi_i\otimes\Psi_i$

$\Delta X_i(E_i\otimes e^{\hbar H_j} + e^{\hbar H_k}\otimes E_i) = X_i(e^{-\hbar a_{ji}} e^{\hbar H_j} E_i + e^{\hbar H_k} E_i)$
$= e^{-\hbar a_{ji}} + 1 \longrightarrow \Delta X_i = X_i\otimes\Psi_i^{-1} + 1\otimes X_i$

- $\Psi_j X_i(e^{\hbar H_k}E_i) = (\Psi_j\otimes X_i)(e^{\hbar H_k}E_i\otimes e^{\hbar(H_k+H_j)} + e^{\hbar H_k}\otimes e^{\hbar H_k}E_i) = e^{\hbar a_{kj}}$

$X_i\Psi_j(e^{\hbar H_k}E_i) = (X_i\otimes\Psi_j)(e^{\hbar H_k}E_i\otimes e^{\hbar(H_k+H_j)} + e^{\hbar H_k}\otimes e^{\hbar H_k}E_i) = e^{\hbar(a_{kj}+a_{ij})}$

$\Psi_i X_j = e^{-\hbar a_{ij}} X_j \Psi_i$ $\quad$ relations

Drinfeld theorem, proof

- $\Delta\psi_i = \psi_i \otimes \psi_i \quad \Delta X_i = X_i \otimes \psi_i^{-1} + 1 \otimes X_i \quad \psi_i X_j = e^{-\hbar a_{ij}} X_j \psi_i$
- Define $U_\hbar(\mathfrak{b}^-)^{coop} \to U_\hbar(\mathfrak{b}^+)^* \quad F_i \mapsto \frac{X_i}{e^{\hbar} - e^{-\hbar}}, \quad K_i \mapsto \psi_i$

  Uniquenes - trivial. Existence - relations

- $\mathfrak{g} = \mathfrak{sl}_3$. $(q\text{-Serre } X)(E_1^2 E_2) =$

$$= (X_1 \otimes X_1 \otimes X_2 - (q+q^{-1}) X_1 \otimes X_2 \otimes X_1 + X_2 \otimes X_1 \otimes X_1)[(\Delta \otimes id)\Delta E_1^2 E_2]$$

$$= (X_1 \otimes X_1 \otimes X_2 - (q+q^{-1}) X_1 \otimes X_2 \otimes X_1 + X_2 \otimes X_1 \otimes X_1)[(E_1 \otimes K_1 \otimes K_1 + 1 \otimes E_1 \otimes K_1 + 1 \otimes 1 \otimes E_1)^2 (E_2 \otimes K_2 \otimes K_2 + 1 \otimes E_2 \otimes K_2 + 1 \otimes 1 \otimes E_2)] = (X_1 \otimes X_1 \otimes X_2 - (q+q^{-1}) X_1 \otimes X_2 \otimes X_1 + X_2 \otimes X_1 \otimes X_1)$$

$$[(1+q^{-2}) E_1 \otimes K_1 E_1 \otimes K_1^2 E_2 + (q+q^{-1}) E_1 \otimes K_1 E_2 \otimes K_1 K_2 E_1 + (q^2+1) E_2 \otimes K_2 E_1 \otimes K_1 K_2 E_1]$$

$$= (1+q^{-2}) - (q+q^{-1})^2 + (1+q^2) = 0$$

- <u>Rem</u> For X_1, X_2 q-Serre is automatical.

Nondegeneracy

- For $\beta = \sum m_i \alpha_i$ $\quad K_\beta = e^{\hbar(\sum m_i H_i)}$

- $\underline{\text{Prop}}$ a) $\forall\, y \in U_\hbar(n^+)_\beta$, $\quad \Delta y = y \otimes K_\beta + 1 \otimes y + \sum_{0<\gamma<\beta} y_\gamma \; 1 \otimes K_\gamma$,
$y_\gamma \in U_\hbar(n^+)_\gamma \otimes U_\hbar(n^+)_{\beta-\gamma}$
b) $\forall\, y \in U_\hbar(n^+)_{\alpha_i+\alpha_j}$, $\quad \Delta(y) \neq y \otimes K_\beta + 1 \otimes y$ — For $\hbar = 0$ not true
c) $\Delta(q\text{-Serre}) = q\text{-Serre} \otimes K_{q\text{-Serre}} + 1 \otimes q\text{-Serre}$

- Proof b) $\quad \Delta E_i E_j = \ldots + \left(E_i \otimes K_i E_j + e^{\hbar a_{ij}} E_j \otimes K_j E_i\right)$
$a_{ij} = 0 \Rightarrow E_i E_j - E_j E_i = 0$
$a_{ij} = -1 \Rightarrow \Delta(E_i E_j - q E_j E_i) = \ldots + (1-q^2) E_i \otimes K_i E_j$

- Fact $\quad \forall\, y \in U(n^+)_\beta$, $\beta \neq \alpha_i \Rightarrow \Delta(y) \neq y \otimes K_\beta + 1 \otimes y$

- $\underline{\text{Problem}}$ Using fact show nondegeneracy $\quad U_\hbar(b) \otimes U_\hbar(b^+) \to \mathbb{C}$.

References

- Chari, Pressley *A guide to quantum groups*
  Sec. 4.2, 6.5

- Tanisaki *Killing forms, Harish-Chandra isomorphisms and universal R-matrices for quantum algebras*