Introduction to Quantum Groups

Lecture 9
RTT realization

qft.itp.ac.ru/mbersht/quantum_groups.html

$$\mathbb{C}[GL_n]$$

- $\mathbb{C}[Mat_n] = \mathbb{C}[t_{ij}]$ $\quad 1 \le i,j \le n.$

  $\mathbb{C}[Mat_n] \to \mathbb{C}[Mat_n] \otimes \mathbb{C}[Mat_n]$ – (dual to $M_1 M_2$)

  $\Delta t_{ij} = \sum_k t_{ik} \otimes t_{kj}$

  $\Delta T = T \otimes T$ $\quad T = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ t_{n1} & & t_{nn} \end{pmatrix} = \sum t_{ij} E_{ij}$

  $\varepsilon(t_{ij}) = \delta_{ij}$ (evaluation at E)

- $S: \mathbb{C}[GL_n] \to \mathbb{C}[GL_n]$ $\quad \mathbb{C}[GL_n] = \mathbb{C}[t_{ij}, \det^{-1}]$

  $T \mapsto T^{-1}$

R matrix

- $R = q \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i<j} E_{ij} \otimes E_{ji}$

- $n = 2$

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

$$R : \underset{\Delta}{\mathbb{C}^2 \otimes \mathbb{C}^2} \to \underset{\Delta^{op}}{\mathbb{C}^2 \otimes \mathbb{C}^2}$$

$$(\rho_{\mathbb{C}} \otimes \rho_{\mathbb{C}^2}) R_{univ} = R$$

- $n > 2$ $\quad R : \underset{\Delta}{\mathbb{C}^n \otimes \mathbb{C}^n} \to \underset{\Delta^{op}}{\mathbb{C}^n \otimes \mathbb{C}^n}$ $\quad \mathcal{U}_\hbar(sl_n)$

- <u>Lemma</u> $\quad R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$

RTT quantum group

- **Def** $U(R)$ – is assos. alg with unit generated by ℓ^+_{ij}, ℓ^-_{ji}, $1 \le i \le j \le n$.

- $$L^+ = \begin{pmatrix} \ell^+_{11} & \ell^+_{12} & \cdots & \\ & \ddots & & \ell^+_{n-1n} \\ 0 & & & \ell^+_{nn} \end{pmatrix}, \quad L^- = \begin{pmatrix} \ell^-_{11} & & 0 & \\ \ell^-_{21} & \ddots & & \\ \vdots & & \ddots & \\ & & \ell^-_{nn-1} & \ell^-_{nn} \end{pmatrix}$$

with relations $\ell^-_{ii}\ell^+_{ii} = \ell^+_{ii}\ell^-_{ii} = 1$.

$$R L^+_1 L^+_2 = L^+_2 L^+_1 R, \quad R L^-_1 L^-_2 = L^-_2 L^-_1 R, \quad R L^+_1 L^-_2 = L^-_2 L^+_1 R$$

$$L^\pm_1 = L^\pm \otimes 1, \quad L^\pm_2 = 1 \otimes L$$

- Coproduct $\Delta(L^\pm) = L^\pm \otimes L^\pm$, $S(L^\pm) = (L^\pm)^{-1}$

RTT quantum group

- $\underline{\text{Rk}}$ This is quantization of $\mathbb{C}[G^*] = \mathbb{C}[B_+ \times_H B_-]$
- $\underline{\text{Rk}}$ $(L^+)^{-1}$ is well defined.
- Explicitly $R^{kk'}_{ii'} L^+_{kj} L^+_{k'j'} = L^+_{i'k'} L^+_{ik} R^{jj'}_{kk'}$, $R^{ij}_{ji} = 0$. $j > i$

. . .

- $\underline{\text{Th}}$ Hopf algebras $\mathcal{U}(R)$ and $U_q(\mathfrak{gl}_n)^{coop}$ are isomorphic
  $(m \mapsto m, \ \Delta \mapsto \Delta^{op})$

- $\underline{\text{Rk}}$ $\mathcal{U}(R)$ has n^2 generators and $\underline{\text{quadratic}}$ rel.
  $\mathcal{U}(\mathfrak{gl}_n)$ has $3n-2$ generators and relations include Serre
  $E_1 \dots E_{n-1}$ $F_1, \dots F_{n-1}$ $K_1 \dots K_n$

$U_q(\mathfrak{sl}_n)$

- Generators $E_1, \dots, E_{n-1}, K_1^{\pm 1}, \dots, K_n^{\pm 1}, F_1, \dots, F_{n-1}$

- Relations

$$K_i K_j = K_j K_i, \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1$$

$$K_i E_j = q^{\delta_{ij}} q^{-\delta_{ij}+1} E_j \qquad K_i F_j K_i^{-1} = q^{-\delta_{ij}} q^{\delta_{ij}+1} F_j$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}}$$

$$E_i E_j = E_j E_i \qquad F_i F_j = F_j F_i \qquad |i-j| \leq 2$$

$$E_i^2 E_{i\pm 1} - (q + q^{-1}) E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 = 0$$

$$F_i^2 F_{i\pm 1} - (q + q^{-1}) F_i F_{i\pm 1} F_i + F_{i\pm 1} F_i^2 = 0$$

— q-serre

- $\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1} \qquad \Delta E_i = E_i \otimes K_i K_{i+1}^{-1} + 1 \otimes E_i \qquad \Delta F_i = F_i \otimes 1 + K_i^{-1} K_{i+1} \otimes F_i$

- $S(K_i^{\pm 1}) = K_i^{\mp 1} \qquad S(E_i) = -E_i K_i^{-1}, \qquad S(F_i) = -K_i F_i$

Homomorphism $U_q(\mathfrak{sl}_n) \to U(R)^{coop}$

- $L^+ = \begin{pmatrix} K_1 & 0 & & 0 \\ 0 & \ddots & & \\ & & \ddots & 0 \\ 0 & & 0 & K_n \end{pmatrix} \begin{pmatrix} 1 & (q-q^{-1})F_1 & & * \\ 0 & \ddots & \ddots & \\ & & \ddots & (q-q^{-1})F_{n-1} \\ 0 & & 0 & 1 \end{pmatrix}$ $\quad L^- = \begin{pmatrix} 1 & 0 & \cdot & 0 \\ (q^{-1}-q)E_1 & \ddots & & \\ & \ddots & \ddots & 0 \\ * & & (q^{-1}-q)E_{n-1} & 1 \end{pmatrix} \begin{pmatrix} K_1^{-1} & 0 & & 0 \\ 0 & \ddots & & \\ & & \ddots & 0 \\ 0 & & 0 & K_n^{-1} \end{pmatrix}$

- Coproduct

$e^+_{ii} = K_i$, $e^-_{ii} = K_i^{-1}$, $e^+_{i\,i+1} = (q-q^{-1})K_iF_i$, $e^-_{i+1,i} = (q^{-1}-q)E_iK_i^{-1}$

$$\Delta^{op} e^+_{i,i+1} = e^+_{i\,i+1} \otimes e^+_{ii} + e^+_{i+1\,i+1} \otimes e^+_{i\,i+1}$$

$$\Delta K_iF_i = K_iF_i \otimes K_i + K_{i+1} \otimes K_iF_i$$

$$\Delta^{op} e^-_{i\,i-1} = e^-_{i\,i-1} \otimes e^-_{ii} + e^-_{i-1\,i-1} \otimes e^-_{i\,i-1}$$

$$\Delta E_iK_i^{-1} = E_iK_i^{-1} \otimes K_{i+1}^{-1} + K_i^{-1} \otimes E_iK_i^{-1}$$

Homomorphism $U_q(\mathfrak{sl}_n) \to \mathcal{U}(R)^{coop}$

- $$L^+ = \begin{pmatrix} K_1 & 0 & & 0 \\ 0 & \ddots & & \\ & & \ddots & 0 \\ 0 & & 0 & K_n \end{pmatrix} \begin{pmatrix} 1 & (q-q^{-1})F_1 & & * \\ 0 & \ddots & \ddots & \\ & & \ddots & (q-q^{-1})F_{n-1} \\ 0 & & 0 & 1 \end{pmatrix} \qquad L^- = \begin{pmatrix} 1 & 0 & \cdot & 0 \\ (q^{-1}-q)E_1 & \ddots & & \\ & & \ddots & 0 \\ * & & (q^{-1}-q)E_{n-1} & 1 \end{pmatrix} \begin{pmatrix} K_1^{-1} & 0 & & 0 \\ 0 & \ddots & & \\ & & \ddots & 0 \\ 0 & & 0 & K_n^{-1} \end{pmatrix}$$

- <u>Problem</u> a) Check directly quadratic relations on E, F, K from RTT b)* Serre relations

- <u>Problem</u> Show that $\mathcal{U}(R)$ is generated by $\ell^+_{ii}, \ell^+_{i\,i+1}, \ell^-_{ii}, \ell^-_{i\,i-1}$

Surjectivity

From universal R matrix

- $\mathcal{R} \in U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n)$ – universal R matrix
- $\rho: U(\mathfrak{gl}_n) \to Mat_n$ n-dim representation,
  $(\rho \otimes \rho)\mathcal{R} = R$
- $L^+ := (\rho \otimes id)\mathcal{R}$, $L^- := (id \otimes \rho)\mathcal{R}^{-1}$
- Since $\mathcal{R} \in U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-) \longrightarrow L^+$ upper triang in $\mathfrak{b}^-$
  L^- lower triang in $\mathfrak{b}^+$
- Problem Find $(\rho \otimes id)\mathcal{R}$, $(id \otimes \rho)\mathcal{R}^{-1}$ for $\mathfrak{g} = sl_2$

From universal R matrix

- $L^+ := (\rho\otimes id)R, \quad L^- := (id\otimes\rho)R^{-1}$

- $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \xrightarrow{\rho\otimes\rho\otimes id} R L^+_1 L^+_2 = L^+_2 L^+_1 R$

  $R_{23}R^{-1}_{12}R^{-1}_{13} = R^{-1}_{13}R^{-1}_{12}R_{23} \xrightarrow{\rho\otimes id\otimes\rho} R L^-_1 L^-_2 = L^-_2 L^-_1 R$

  $R_{13}R_{12}R^{-1}_{23} = R^{-1}_{23}R_{12}R_{13} \xrightarrow{\rho\otimes id\otimes\rho} R L^+_1 L^-_2 = L^-_2 L^+_1 R$

- $(id\otimes\Delta)R = R_{13}R_{12} \xrightarrow{\rho\otimes id\otimes id} \Delta L^+ = L^+_2 \otimes L^+_1$

  $(\Delta\otimes id)R = R_{13}R_{23} \longrightarrow (\Delta\otimes id)R^{-1} = R^{-1}_{23}R^{-1}_{13} \xrightarrow{id\otimes id\otimes\rho}$

  $\Delta L^- = L^-_2 \otimes L^-_1$

End of the proof

- <u>Fact</u> $\forall \Lambda \in P^+$ $\exists L_{\Lambda,q}$ representation of $U_q(\mathfrak{gl}_n)$

- $\forall \Lambda$ $L^+_\Lambda = (\rho_{\mathbb{C}^n} \otimes \rho_{L_{\Lambda,q}}) R$, $L^-_\Lambda = (\rho_{L_{\Lambda,q}} \otimes \rho_{\mathbb{C}^n}) R^{-1}$ satisfy $U(R)$ relations

$$\begin{tikzcd} U_q(\mathfrak{gl}_n) \arrow[r] \arrow[rr, bend right] & U(R) \arrow[r, dashed] & \mathrm{Mat}_{L_{\Lambda,q}} \end{tikzcd}$$

$I = \mathrm{Ker}(U_q(\mathfrak{gl}_n) \to U(R)) \rightsquigarrow I \subset \bigcap \mathrm{Ker}_{L_{\Lambda,q}}$

- <u>Fact</u> $\bigcap \mathrm{Ker}\, \rho_{L_{\Lambda,q}} = 0$ (We know for $q=1$, algebra is not deformed)

$\downarrow$

$I = 0$

References

- Ding Frenkel Isomorphism of two realizations on quantum afinne algebra $U_q(\widehat{gl}(n))$