Introduction to Quantum Groups

Lecture 11

Functions on quantum group SL_n

qft.itp.ac.ru/mbersht/quantum_groups.html

$\mathbb{C}[Mat_n]_q$

- **Def** $\mathbb{C}[Mat_n]_q$ - is an algebra gen by t_{ij} $1\le i,j\le n$

  $\tilde{R}\, T_1 T_2 = T_1 T_2 \tilde{R}$, where

  $$\tilde{R} = \sum q E_{ii}\otimes E_{ii} + \sum_{i\ne j} E_{ji}\otimes E_{ij} + (q-q^{-1})\sum_{i>j} E_{ii}\otimes E_{jj}$$

  $$\Delta t_{ij} = \sum_k t_{ik}\otimes t_{kj}$$

  $$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

- $$\sum_{k,k'} \tilde{R}^{kk'}_{ii'}\, t_{kj}\, t_{k'j'} = \sum_{k,k'} t_{ik}\, t_{i'k'}\, \tilde{R}^{jj'}_{kk'}$$

- More explicitly

  a) $t_{ij}\, t_{i'j} = q\, t_{i'j}\, t_{ij}$ $\quad i<i'$ $\qquad t_{ij}\, t_{ij'} = q\, t_{ij'}\, t_{ij}$ $\quad j<j'$

  b) $t_{ij}\, t_{i'j'} = t_{i'j'}\, t_{ij}$ $\quad i<i',\ j>j'$

  c) $t_{ij}\, t_{i'j'} = t_{i'j'}\, t_{ij} + (q-q^{-1})\, t_{ij'}\, t_{i'j}$ $\quad i<i',\ j<j'$

Remarks

- For $R = P\tilde{R}$ we have $R T_1 T_2 = T_2 T_1 R$
- In $q \to 1$ limit $\tilde{R} \to P$ $\quad P_{ij}^{i'j'} = \delta_{ij'}\delta_{i'j}$

$$P_{i'i}^{ii'} t_{ij} t_{i'j'} = t_{i'j'} t_{ij} P_{j'j}^{jj'}$$ Hence t_{ij} - commute

Quantization of $\{g \overset{\otimes}{,} g\} = [r, g \otimes g]$

$$\lim_{\hbar \to 0} \frac{g_{ij} * g_{i'j'} - g_{i'j'} * g_{ij}}{2\hbar} = \{g_{ij}, g_{i'j'}\} = r_{ii'}^{kk'} g_{kj} g_{k'j'} - g_{ik} g_{i'k'} r_{kk'}^{jj'}$$

- Transposition $t_{ij} \to t_{ji}$ is automorphism (since $\tilde{R} = \tilde{R}^t$)

Comodules $S_q^\cdot V$, $\Lambda_q^\cdot V$

- **Def** $S_q^\cdot V = \mathbb{C}\langle x_1, \dots, x_n \rangle / x_i x_j = q^{-1} x_j x_i \quad i<j$ (q deform of $S^\cdot V$)

As vector space $S_q V$ has basis $x_1^{a_1} \dots x_n^{a_n} \quad a_i \in \mathbb{Z}_{\geq 0}$

$$\Delta : S_q^\cdot V \to \mathbb{C}[Mat_n]_q \otimes S_q^\cdot V \qquad \Delta(x_i) = \sum t_{ij} \otimes x_j$$

- **Def** $\Lambda_q^\cdot V = \mathbb{C}\langle \xi_1, \dots, \xi_n \rangle / \xi_i \xi_j = -q \xi_j \xi_i \quad i<j$ (q deform of $\Lambda_q^\cdot V$)

As vector space $\Lambda_q V$ has basis $\xi_1^{a_1} \dots \xi_n^{a_n} \quad a_i \in \{0,1\}$

$$\Delta : \Lambda_q^\cdot V \to \mathbb{C}[Mat_n]_q \otimes \Lambda_q^\cdot V \qquad \Delta(\xi_i) = \sum t_{ij} \otimes \xi_j$$

- **Prop** For both $S_q^\cdot V$ and $\Lambda_q V$ Δ is homomorphism of algebras

Relations from $U_q(sl_2)$

- $$\tilde{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad \tilde{R}: \mathbb{C}^2\otimes\mathbb{C}^2 \to \mathbb{C}^2\otimes\mathbb{C}^2$$

$$\mathbb{C}^2\otimes\mathbb{C}^2 = S_q^2\mathbb{C}^2 \oplus \Lambda^2\mathbb{C}^2$$

- $\tilde{R} \sim \mathrm{diag}(q, q, q, -q^{-1})$ (Hecke relation $(\tilde{R}-q)(\tilde{R}+q^{-1})=0$)

$$S_q^2\mathbb{C}^2 = \mathrm{Ker}(\tilde{R}-q) \qquad \Lambda_q^2\mathbb{C}^2 = \mathrm{Ker}(\tilde{R}+q^{-1})$$

- For $U_q(sl_n)$ $\tilde{R}$ consist of Blocks (q) and $\begin{pmatrix} 0 & 1 \\ 1 & q-q^{-1} \end{pmatrix} \sim \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}$.

$$S_q^2\mathbb{C}^n = \mathrm{Ker}(\tilde{R}-q) \qquad \Lambda_q^2\mathbb{C}^n = \mathrm{Ker}(\tilde{R}+q^{-1})$$

Relations from $U_q(sl_2)$

- Bilinears of ε_i belong to $\Lambda_q^2\mathbb{C}^2$

Hence $(\tilde{R}+q^{-1})\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}\otimes\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}=0 \Rightarrow \begin{pmatrix} q+q^{-1} & & & \\ & q^{-1} & 1 & \\ & 1 & q & \\ & & & q+q^{-1}\end{pmatrix}\begin{pmatrix}\varepsilon_1\varepsilon_1\\ \varepsilon_1\varepsilon_2\\ \varepsilon_2\varepsilon_1\\ \varepsilon_2\varepsilon_2\end{pmatrix}=0$

$\Leftarrow$

$\varepsilon_1^2=\varepsilon_2^2=0 \quad \varepsilon_1\varepsilon_2=-q\varepsilon_2\varepsilon_1$

- Bilinears of x_i belong to $S_q^2\mathbb{C}^2$

Hence $(\tilde{R}-q)\begin{pmatrix}x_1\\ x_2\end{pmatrix}\otimes\begin{pmatrix}x_1\\ x_2\end{pmatrix}=0 \Rightarrow \begin{pmatrix} 0 & & & \\ & -q & 1 & \\ & 1 & -q^{-1} & \\ & & & 0\end{pmatrix}\begin{pmatrix}x_1x_1\\ x_1x_2\\ x_2x_1\\ x_2x_2\end{pmatrix}=0$

$\Leftarrow$

$x_1x_2=q^{-1}x_2x_1$

q-Minors

- $S_q^\bullet V = \bigoplus S_q^k V \qquad \Lambda_q^\bullet V = \bigoplus \Lambda_q^k V$

  $\Delta : S_q^k V \to \mathbb{C}[Mat_n]_q \otimes S_q^k V$

  $\Delta : \Lambda_q^k V \to \mathbb{C}[Mat_n]_q \otimes \Lambda_q^k V$

- For $I = \{1 \le i_1 < i_2 < \dots < i_r \le n\}$, $J = \{1 \le j_1 < j_2 < \dots < j_r \le n\}$ define t_I^J by

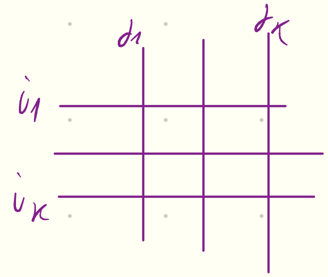
  $\Delta \xi_I = \sum t_I^J \otimes \xi_J$, where $\xi_I = \xi_{i_1} \cdots \xi_{i_k}$

- <u>Calculation</u> $\Delta(\xi_{i_1} \cdots \xi_{i_k}) = \Delta(\xi_{i_1}) \cdot \ldots \cdot \Delta(\xi_{i_k})$

$$= \Big(\sum_{e_1} t_{i_1 e_1} \otimes \xi_{e_1}\Big) \cdot \ldots \cdot \Big(\sum_{e_k} t_{i_k e_k} \otimes \xi_{e_k}\Big) = \sum_{j_1 < \dots < j_k} \Big(\sum_{\sigma \in S_k} (-q)^{-|\sigma|} t_{i_1 j_{\sigma(1)}} \cdots t_{i_k j_{\sigma(k)}}\Big) \otimes \xi_J$$

$$|\sigma| = \#\{i,j \mid i<j,\ \sigma(i) > \sigma(j)\}$$

minor



$$t_I^J = \sum_{\sigma \in S_k} (-q)^{-|\sigma|} t_{i_1 j_{\sigma(1)}} \cdot \ldots \cdot t_{i_k j_{\sigma(k)}} = \sum_{\sigma \in S_k} (-q)^{-|\sigma|} t_{i_{\sigma(1)} j_1} \cdot \ldots \cdot t_{i_{\sigma(k)} j_k}$$

Properties

- **Prop** $\Delta t_I^J = \sum_K t_I^K \otimes t_K^J$

Proof $(\Delta\otimes id)\Delta = (id\otimes\Delta)\Delta$

$\Delta t_I^J \otimes \xi_J = (\Delta\otimes id)(\Delta\xi_I) = (id\otimes\Delta)\Delta\xi_I = (id\otimes\Delta)(t_I^K\otimes\xi_K) = t_I^K\otimes t_K^J\otimes\xi_J$

- **Rem** For $|I|=|J|=k$, $\{t_I^J\}$ - matrix element on $\Lambda^k\mathbb{C}^n$

Coproduct $\Delta T = T\otimes T$

- $q\det = t_{1\dots n}^{1\dots n}$

Corol $\Delta\, q\det = q\det\otimes q\det$

Laplace expansion

- **Prop** ⓐ For given $J_1 \sqcup J_2 = J$

$$\mathrm{sgn}(J_1, J_2)\, t_I^J = \sum_{I_1 \sqcup I_2 = I} t_{I_1}^{J_1} t_{I_2}^{J_2}\, \mathrm{sgn}_q(I_1, I_2)$$

ⓑ For given $I_1 \sqcup I_2 = I$

$$\mathrm{sgn}_q(I_1, I_2)\, t_I^J = \sum_{J_1 \sqcup J_2 = J} t_{I_1}^{J_1} t_{I_2}^{J_2}\, \mathrm{sgn}_q(J_1, J_2)$$

where $\mathrm{sgn}_q(I, J) = \begin{cases} 0 & \text{if } I \cap J \neq 0 \\ (-q)^{-\#\{i \in I, j \in J \mid i > j\}} & \end{cases}$

- <u>Rk</u> For $|J_1| = 1$ or $|J_2| = 1$ a) is column expansion
For $|I_1| = 1$ or $|I_2| = 1$ b) is row expansion

Laplace expansion - proof

- (b) $\xi_{I_1}\xi_{I_2} = \xi_I \, sgn(I_1, I_2)$

$sgn_q(I, J) = \begin{cases} 0 & \text{if } I \cap J \neq 0 \\ (-q)^{-\#\{i \in I, j \in J \mid i > j\}} \end{cases}$

$\xi_i \xi_j = -q \xi_j \xi_i \quad i < j$

$$sgn(I_1, I_2) \sum t_I^J \otimes \xi_J = sgn(I_1, I_2) \Delta \xi_I = \Delta(\xi_{I_1} \xi_{I_2})$$

$$= \sum t_{I_1}^{J_1} \otimes \xi_{J_1} \sum t_{I_2}^{J_2} \otimes \xi_{J_2} = \sum sgn(J_1, J_2) \, t_{I_1}^{J_1} t_{I_2}^{J_2} \otimes \xi_J$$

- (b) $\Rightarrow$ (a) using automorphism $t_{ij} \mapsto t_{ji}$

$$t_I^J \mapsto t_J^I$$

- <u>Problem</u> [Plucker relations] For given $J = \{j_1 < \ldots < j_{r-1}\}$, $K = \{k_0 < \ldots < k_r\}$
$I = \{i_1 < \ldots < i_r\}$

$$\sum_{s=0}^{r} sgn(J, \{k_s\}) (-q)^{-s} \, t_{j_1 \ldots k_s \ldots j_{r-1}}^{i_1 \ldots\ldots i_r} \, t_{k_0 \ldots \hat{k}_s \ldots k_r}^{i_1 \ldots\ldots i_r} = 0$$

Matrix $T^\vee$, Hopf algebra structure

- $T^\vee = \sum E_{ij}\, t_{1\cdots\hat{i}\cdots n}^{1\cdots\hat{j}\cdots n}\,(-q)^{i-j}$
- Laplace $\Rightarrow$ $T^\vee T = \mathrm{qdet}\cdot \mathrm{Id}_n = T T^\vee$
- <u>Corol</u> $T\,\mathrm{qdet} = T T^\vee T = \mathrm{qdet}\cdot T$ Hence qdet is central
- <u>Def</u> $\mathbb{C}[GL_n]_q = \mathbb{C}[t_{ij}, \mathrm{qdet}^{-1}]$, $\mathbb{C}[SL_n]_q = \mathbb{C}[t_{ij}]/(\mathrm{qdet}-1)$
- <u>Lem</u> $S(T) = T^\vee \mathrm{qdet}^{-1}$ satisfies antipode properties

<u>Pf</u>

$\Leftrightarrow \mathrm{qdet}^{-1} T^\vee T = \mathrm{Id}_n$

- <u>RK</u> S antihomomorphism:

$$\tilde{R} T_1 T_2 = T_2 T_1 \tilde{R} \qquad \tilde{R} T_2^\vee T_1^\vee = T_1^\vee T_2^\vee \tilde{R}$$

Main Theorems

- <u>Th</u> $\mathbb{C}[Mat_n]_q = \langle t_{11}^{a_{11}} t_{12}^{a_{12}} \cdots t_{1n}^{a_{1n}} t_{21}^{a_{21}} \cdots t_{2n}^{a_{2n}} \cdots t_{nn}^{a_{nn}} \rangle$

<u>Idea of Pf.</u> Use diamond lemma for lexicographical order $t_{11} < t_{12} < \cdots < t_{1n} < t_{21} < \cdots < t_{2n} < t_{31} < \cdots < t_{nn}$

- <u>Th</u> $\mathbb{C}[SL_n]_q = U_q(sl_n)^\circ = \bigoplus_{\lambda \in P_+} L_{\lambda,q} \otimes L^*_{\lambda,q}$

<u>Idea of Pf</u> Construct homomorphism $\psi: U_q(sl_n)^\circ \to \mathbb{C}[SL_n]_q$ using $\mathbb{C}^n \in U_q(sl_n)$-mod. $\forall \lambda$ $L_{\lambda,q} \subset (\mathbb{C}^n)^{\otimes N}$ for some $N \Rightarrow \psi$ is surj.
Compare sizes (using Th above) $\Rightarrow \psi$ is inj. (c.f. SL_2 proof in prev. Lecture)

- <u>Problem</u> Center of $\mathbb{C}[SL_n]_q$ is generated by qdet.

<u>Hint</u> Show that if $Y \in \mathbb{C}[SL_n]_q$ is central, then its highest term w.r.t. lexicographical order above is $(t_{11} t_{22} \cdots t_{nn})^d$ for some $d \in \mathbb{Z}_{\geq 0}$

Primitive ideals, subalgebras

- **Def** $I\subset A$ is primitive if $I=\mathrm{Ann}\,M$, M - simple A-mod

<u>Th</u> Primitive ideals in $\mathbb{C}[G]_q \longleftrightarrow$ symplectic leaves in G for G - s/s

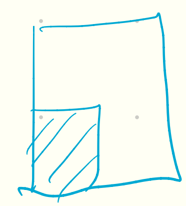
- For such questions one uses more general elements $\mathbb{C}[SL_n]_q$

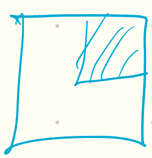
For $v\in L_{\lambda,q}[\lambda]$, $\ell\in L^*_{\lambda,q}[\mu]$, $t^{L_{\lambda,q}}_{\ell,v}\in U_q(\mathfrak{g})^*$ – corresp. matrix elements

We abbreviate $t^{L_{\lambda,q}}_{\ell,v}$ to $t^{\lambda}_{\mu,\lambda}$. Note $(L_{\lambda,q})^*\simeq L_{-w_0(\lambda),q}$

- <u>Example</u> For $\mathfrak{g}=\mathfrak{sl}_n$, ϖ_k - k-th fundamental weight

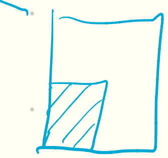
$t^{\varpi_k}_{-w(\varpi_k),\varpi_k}=t^{1\dots k}_{i_1\dots i_k}$, $t^{\varpi_k}_{-w(\varpi_k),w_0(\varpi_k)}=t^{n-k+1\dots n}_{i_1\dots i_k}$ where $i_s=w(s)$

In particular $t^{\varpi_k}_{-\varpi_k,\varpi_k}=t^{1\dots k}_{1\dots k}$, $t^{\varpi_k}_{-w_0(\varpi_k),\varpi_k}=t^{1\dots k}_{n-k+1\dots n}$ — 

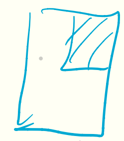
 — $t^{\varpi_k}_{-\varpi_k,w_0(\varpi_k)}=t^{n-k+1\dots n}_{1\dots k}$, $t^{\varpi_k}_{-w_0(\varpi_k),\varpi_k}=t^{n-k+1\dots n}_{n-k+1\dots n}$

Relations

Problem* (a) $t^{\Lambda}_{-w_0(\Lambda),\Lambda}\, t^{\Lambda'}_{-\mu,\lambda} = q^{(\Lambda,\lambda)-(w_0(\Lambda),\mu)}\, t^{\Lambda'}_{-\mu,\lambda}\, t^{\Lambda}_{-w_0(\Lambda),\Lambda}$



(b) $t^{\Lambda}_{-\Lambda,w_0(\Lambda)}\, t^{\Lambda'}_{-\mu,\lambda} = q^{(\Lambda,\mu)-(w_0(\Lambda),\lambda)}\, t^{\Lambda'}_{-\mu,\lambda}\, t^{\Lambda}_{-\Lambda,w_0(\Lambda)}$



(c) Elements $t^{\Lambda}_{-w_0(\Lambda),\Lambda}$, $t^{\Lambda'}_{-\Lambda',w_0(\Lambda')}$ form commutative algebra

Hint (a) Use $RTT = TTR$,

Universal R-matrix has form $R = \overline{R}_H \overline{R}$, where

$R_H = q^{H_j\otimes H^j}$, H_j, H^j - dual bases, $\overline{R}\in U_q(n^+)\otimes U_q(n^-)$ (see Lect 13, 14)

$R\, v_\Lambda\otimes v = q^{(\Lambda,\lambda)}\, v_\Lambda\otimes v$, for $v_\Lambda\in L_{\Lambda,q}[\Lambda]$, $v\in L_{\Lambda',q}[\lambda]$

$(e_{-w_0(\Lambda)}\otimes e, R\,-) = q^{(w_0(\Lambda),\mu)}(e_{-w_0(\Lambda)}\otimes e, -)$ for $e_{w_0(\Lambda)}\in (L^*_{\Lambda,q})[-w_0(\lambda)]$, $e\in L^*_{\Lambda,q}[-\mu]$

(b) Use $R_{21}^{-1}TT = TTR_{21}^{-1}$, $\quad R_{21}^{-1} = \overline{R}_{21}^{-1} R_H^{-1}$

"Triangular" decomposition

- Def A_+ subalgebra generated by $t^{\Lambda}_{\lambda\ \Lambda}$
  A_- — " — $t^{\Lambda}_{\lambda\ w_0(\Lambda)}$

- Th $A_+ \otimes A_- \to \mathbb{C}[G]_q$ is surj

- Problem d) A_+ is genenerated by $t^{1 \dots k}_{i_1 \dots i_k}$
  A_- — " — $t^{n-k+1 \dots n}_{i_1 \dots i_k}$

e) Commutative subalgebra ⓒ above is generated by $t^{n-k+1 \dots n}_{1 \dots k}$ $t^{1 \dots k}_{n-k+1 \dots n}$

Hint $\Lambda = \sum \ell_k \varpi_k \rightsquigarrow V_\Lambda \sim \bigotimes V_{\varpi_k}^{\otimes \ell_k}$

$$\begin{array}{ccc} V_\Lambda & \sim & \bigotimes V_{\varpi_k}^{\otimes \ell_k} \\ \cup & & \cup \\ L^q_\Lambda & \hookrightarrow & \bigotimes L_{\varpi_k}^{\otimes \ell_k} \end{array}$$

References

- Chari, Pressley *A guide to quantum groups*, Sec. 7.3
- Korogodski, Soibelman *Algebras of Functions on quantum group* Sec 3.1, 3.2
- Noumi Yamada Mimachi *Finite dimensional representations of the quantum group $GL_q(n,\mathbb{C})$ and zonal spherical functions on $U_q(n-1)\backslash U_q(n)$*
- Hodges Levasseur *Primitive Ideals of $\mathbb{C}[SL_3]_q$*