Introduction to Quantum Groups

Lecture 12
Lusztig's braid group

qft.itp.ac.ru/mbersht/quantum_groups.html

$\mathcal{U}_\hbar(\mathfrak{g})$

- Generators E_i, H_i, F_i $\quad K_i = e^{\hbar H_i}, \; q = e^{\hbar}$
- Relations $[H_i, H_j] = 0$ $\quad K_i E_j = q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i$

  q-Serre E, $\quad$ q-Serre F

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{e^{\hbar} - e^{-\hbar}}$$

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$$\Delta E_i = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i, \quad \Delta K_i = K_i \otimes K_i$$

- $S(E_i) = -E_i K_i^{-1} \quad S(F_i) = -K_i F_i \quad S(K_i) = K_i^{-1}$

Braid group

- <u>Def</u> Braid group $B = B_\Delta$ generated by T_i $i \in I$ with relations $\underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}}$

$m_{ij} = 2$ $a_{ij} = 0$ $m_{ij} = 3$ $a_{ij} a_{ji} = 1$

$m_{ij} = 4$ $a_{ij} a_{ji} = 2$ $m_{ij} = 6$ $a_{ij} a_{ji} = 3$

- <u>Rk</u> $\exists$ homomorphism $B \to W$ $T_i \mapsto s_i$

- <u>Fact (Matsumoto)</u> Let $w \in W$ $w = s_{i_1} s_{i_2} \cdots s_{i_\ell} = s_{i_1'} s_{i_2'} \cdots s_{i_\ell'}$ two reduced (shortest $\parallel$) expressions. Then $\underline{i} \to \underline{i}'$ only using Braid relations

<u>Corollary</u> $\forall w \in W$ $\exists !$ element $T_w = T_{i_1} \cdots T_{i_\ell} \in B$

- <u>Ex</u> $\mathfrak{g} = \mathfrak{sl}_3$ $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ $T_{w_0} = T_1 T_2 T_1 = T_2 T_1 T_2$

Action of T_i on $U_q(\mathfrak{g})$

- $E_i^{(r)} = E_i^r / [r]_q!$ $\quad F_i^{(r)} = F_i^r / [r]_q!$ $\quad [r]_q = \dfrac{q^r - q^{-r}}{q - q^{-1}}$, $[r]_q! = \prod_{j=1}^{r} [j]_q$

- <u>Def</u> $\quad T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i,$

$$T_i(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{-r} E_i^{(-a_{ij}-r)} E_j E_i^{(r)}$$

$$T_i(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{r} F_i^{(r)} F_j F_i^{(-a_{ij}-r)} \qquad T_i(K_j) = K_j K_i^{-a_{ij}}$$

- Inverse elements are defined by similar formulas

$T_i^{-1}(E_i) = -K_i^{-1} F_i \quad T_i^{-1}(F_i) = -E_i K_i \quad T_i^{-1}(K_j) = K_j K_i^{-a_{ij}}$

$$T_i^{-1}(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{-r} E_i^{(r)} E_j E_i^{(-a_{ij}-r)} \qquad T_i^{-1}(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{r} F_i^{(-a_{ij}-r)} F_j F_i^{(r)}$$

- Replace $q \to q^{-1}$, $K_i \to K_i^{-1}$ $\longrightarrow$ another Braid group

Remarks

- Rk $U_q(\mathfrak{g}) = \bigoplus_{\lambda \in Q} U_q(\mathfrak{g})_\lambda$ Q - root lattice

$wt(E_i) = \alpha_i$, $wt(F_i) = -\alpha_i$, $wt(K_i) = 0$

T_i reflects weights i.e $T_i : U_q(\mathfrak{g})_\lambda \to U_q(\mathfrak{g})_{s_i(\lambda)}$

T_i acts as reflection on Cartan $T_i(K_j) = K_j K_i^{a_{ij}}$ — c.f $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$

- Rk For $q=1$ $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ $W \circlearrowright \mathfrak{h}$, $W \circlearrowright \Delta$

Problem No natural action W on $\mathfrak{g}$. $W = N(H)/H$, W cannot be embedded into G

Solution (Tits) $\exists \tilde{W}$ central extension of W

$e \to (\mathbb{Z}/2\mathbb{Z})^{\mathrm{rk}\,\mathfrak{g}} \to \tilde{W} \to W \to e$ $\tilde{W} \subset G$, $\exists B \twoheadrightarrow \tilde{W}$

Ex $G = SL_2$ $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

Main theorems

- Th T_i is automorphism of $U_q(\mathfrak{sl})$

- Th $\{T_i, T_i^{-1}, i \in I\}$ satisfies Braid group relations

q-commutators

- $\forall t$ denote $[X,Y]_t = XY - tYX$
- $X \in U_q(\mathfrak{g})_\lambda$, $Y \in U_q(\mathfrak{g})_\mu$. Denote $ad_{q,X}Y = XY - q^{(\lambda,\mu)}YX$
- $ad^2_{q,E_1}(E_2) = ad_{q,E_1}(\underbrace{E_1E_2 - q^{-1}E_2E_1}_{=[E_1,E_2]_{q^{-1}}}) = E_1^2E_2 - (q+q^{-1})E_1E_2E_1 + E_2E_1^2$

  $(a_{12}a_{21} = 1$ — q-Serre

  $ad^2_{q^{-1},F_1}(F_2) = ad_{q^{-1},F_1}(F_1F_2 - qF_2F_1) = F_1^2F_2 - (q+q^{-1})F_1F_2F_1 + F_2F_1^2$

- $T_1(E_2) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{-r} E_i^{(-a_{ij}-r)} E_j E_i^{(r)} \Big|_{a_{ij}\cdot a_{ji}=1} = -E_1E_2 + q^{-1}E_2E_1 = [-E_1, E_2]_{q^{-1}} = ad_{q,-E_1}E_2$

  $T_1(F_2) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{r} F_i^{(r)} F_j F_i^{(-a_{ij}-r)} \Big|_{a_{ij}a_{ji}=1} = -F_2F_1 + qF_1F_2 = [-F_2, F_1]_q = ad_{q^{-1},-F_1}F_2$

- In general q-Serre $= ad^{-a_{ij}-1}_{q_i,E_i}E_j$ $\quad T_i(E_j) = ad^{-a_{ij}}_{q_i,-E_i}E_j$

Some checks

- $[E_i, F_j] = \delta_{i,j} \dfrac{K_i - K_i^{-1}}{q - q^{-1}}$

$$[T_i(E_i), T_i(F_i)] = [-F_i K_i, -K_i^{-1} E_i] = -[E_i, F_i] = \frac{K_i^{-1} - K_i}{q - q^{-1}} = T_i\left(\frac{K_i - K_i^{-1}}{q - q^{-1}}\right)$$

$$[T_i(E_i), T_i(F_j)]\Big|_{a_{ij}=-1} = [-F_i K_i, -q[F_i, F_j]_{q^{-1}}] = q^2 [F_i, [F_i, F_j]_{q^{-1}}]_{q^{-1}} K_i = 0 = T_i(0)$$

- <u>Problem</u> Check $[T_i(E_j), T_i(F_j)] = T_i([E_j, F_j])$ for $a_{ij} = -1$

- <u>Cumbersome check</u> Let $a_{21} a_{12} = a_{23} a_{32} = 1$

$$[T_2(E_1), T_2(E_3)] = [[E_2, E_1]_{q^{-1}}, [E_2, E_3]_{q^{-1}}] = [E_2, [E_1, [E_2, E_3]_{q^{-1}}]_q]_{q^{-2}} + q[\overbrace{[E_2, [E_2, E_3]_{q^{-1}}]_{q^{-1}}}^{=0}, E_1]_{q^2} =$$

$$= [E_2, [[E_1, E_2]_q, E_3]_{q^{-1}} + q[E_2, \underbrace{[E_1, E_3]}_{=0}]]_{q^{-1}}]_{q^{-2}} = [E_2, [[E_1, E_2]_q, E_3]_{q^{-1}}]_{q^2} = \longrightarrow [T_2(E_1), T_2(E_3)] = 0$$

$$\underbrace{[[E_2, [E_1, E_2]_q]_{q^{-1}}, E_3]_{q^2}}_{=0} + q^{-1}[[E_1, E_2]_q, [E_2, E_3]_{q^{-1}}] = -[[E_2, E_1]_{q^{-1}}, [E_2, E_3]_{q^{-1}}]$$

We used $[X, [Y, Z]_u]_v = [[X, Y]_t, Z]_{uv/t} + t[Y, [X, Z]_{v/t}]_{u/t}$ $\quad [[X, Y]_u, Z]_v = [X, [Y, Z]_t]_{uv/t} + t[[X, Z]_{v/t}, Y]_{u/t}$ $\quad \forall t$

Adjoint action

- **Def** Let A be Hopf a Hopf algebra $\mathrm{ad}_x y = x_{(1)} y S(x_{(2)})$
- **Ex** $A = U(\mathfrak{g})$, $\Delta(x) = x\otimes 1 + 1\otimes x$, $S(x) = -x$ $\mathrm{ad}_x Y = xY - Yx$
- **Ex** $A = \mathbb{C}[G]$, $\Delta(g) = g\otimes g$, $S(g) = g^{-1}$ $\mathrm{ad}_g Y = gYg^{-1}$
- **Ex** $A = U_q(\mathfrak{b}_-)$, $\Delta F_i = F_i \otimes 1 + K_i^{-1}\otimes F_i$, $S(K_i) = K_i^{-1}$, $S(F_i) = -K_i F_i$

$$\mathrm{ad}_{F_i} Y = F_i Y - K_i^{-1} Y K_i F_i = F_i Y - q^{-(\alpha_i,\, \mathrm{wt}\, Y)} Y F_i = \mathrm{ad}_{q^{-1}, F_i} Y$$

- **Problem** For $U_q(\mathfrak{g})^{\mathrm{coop}}$ (ie Δ^{op} coproduct) find $S(E_i)$
  Show that $\mathrm{ad}_{\Delta^{op} E_i} Y = \mathrm{ad}_{q, E_i} Y$
- **Conclusion** q-Serre and T_i are defined in terms of ad and $\mathrm{ad}_{\Delta^{op}}$

Convex order

- Let $w_0 \in W$ be the longest element. For sl_n $w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & & 1 \end{pmatrix}$
- $w_0 = s_{i_1} s_{i_2} \ldots s_{i_N}$ - reduced expression

$\beta_1 = \alpha_{i_1}$, $\beta_2 = s_{i_1}\alpha_{i_2}$, $\beta_3 = s_{i_1} s_{i_2} \alpha_{i_3}$, $\ldots$, $\beta_N = s_{i_1} s_{i_2} \cdots s_{i_{N-1}}(\alpha_{i_N})$

- $\forall\ w \in W$, $\ell(w) =$ length of reduced expression $= |\{\alpha \in \Delta_+ | w(\alpha) \in \Delta_-\}|$

$\ell(s_{i_1}) = 1$ $\quad \ell(s_{i_2} s_{i_1}) = 2$ $\quad \ell(s_{i_3} s_{i_2} s_{i_1}) = 3$ $\quad \ell(w_0^{-1}) = N = |\Delta_+|$

$s_{i_1}(\beta_1) < 0$ $\quad s_{i_2} s_{i_1}(\{\beta_1, \beta_2\}) < 0$ $\quad s_{i_3} s_{i_2} s_{i_1}(\{\beta_1, \beta_2, \beta_3\}) < 0$ $\quad w_0^{-1}(\beta_i) < 0\ \forall i$

Hence $\beta_1, \ldots, \beta_N$ all positive roots in some order

- The order is convex i.e. if $\alpha, \alpha', \alpha'' \in \Delta_+$

$\alpha = \alpha' + \alpha''$, $\alpha = \beta_i$, $\alpha' = \beta_{i'}$, $\alpha'' = \beta_{i''}$, $i' < i'' \Rightarrow i' < i < i''$

Cartan-Weyl elements

- Denote $E_{\beta_1}=E_{i_1}$, $E_{\beta_2}=T_{i_1}(E_{\beta_2})$, ... $E_{\beta_\ell}=T_{i_1}T_{i_2}\dots T_{i_{\ell-1}}(E_{i_\ell})$, ...
$F_{\beta_1}=F_{i_1}$, $F_{\beta_2}=T_{i_1}(F_{\beta_2})$, ... $F_{\beta_\ell}=T_{i_1}T_{i_2}\dots T_{i_{\ell-1}}(F_{i_\ell})$, ...

RK $w(E_{\beta_i})=\beta_i$, $wt(F_{\beta_i})=-\beta_i$. Elements $E_{\beta_1},\dots,E_{\beta_N},H_1,\dots,H_r,F_{\beta_1},\dots F_{\beta_N}$ analog of basis in $\mathfrak{g}\subset\mathcal{U}(\mathfrak{g})$

- Example $\mathfrak{g}=sl_3$. Two reduced expressions

$W_0=S_1S_2S_1$ $\beta_1=\alpha_1$, $\beta_2=\alpha_1+\alpha_2$, $\beta_3=S_1S_2(\alpha_1)=\alpha_2$

$E_{\beta_1}=E_1$, $E_{\beta_2}=[E_1,E_2]_{q^{-1}}$ $E_{\beta_3}=T_1T_2E_1=T_1([-E_2,E_1]_{q^{-1}})=[[E_1,E_2]_{q^{-1}},-F_1K_1]_{q^{-1}}=$

$=q^{-1}F_1K_1E_1E_2-q^{-2}F_1K_1E_2E_1-E_1E_2F_1K_1+q^{-1}E_2E_1F_1K_1$

$=qK_1[F_1,E_1]E_2+q^{-1}E_2[E_1,F_1]K_1=\frac{q(1-K_1^2)}{q-q^{-1}}E_2+E_2\frac{q^{-1}(K_1^2-1)}{q-q^{-1}}=E_2$

Basis depends on reduced expr.

$W_0=S_2S_1S_2$ $\beta_1=\alpha_2$, $\beta_2=\alpha_1+\alpha_2$, $\beta_3=\alpha_1$

$E_{\beta_1}=E_2$, $E_{\beta_2}=[-E_2,E_1]_{q^{-1}}$, $E_{\beta_3}=E_1$

Dependence on $\bar{i}$

- <u>Problem</u> ⓐ If $a_{i_k i_{k+1}}=0$, then reversing i_k and i_{k+1} we get reduced expr $\bar{i}'$ Cartan-Weyl elements are unchanged (but reordered $E'_{\beta_k}=E_{\beta_{k+1}}$, $E'_{\beta_{k+1}}=E_{\beta_k}$)

ⓑ If $i_k=i_{k+2}$, $a_{i_{k+1}, i_k}=a_{i_{k+2} i_{k+1}}=-1$. Then $\beta_{k+1}=\beta_k+\beta_{k+2}$ $\quad E_{\beta_{k+1}}=[-E_{\beta_k}, E_{\beta_{k+2}}]_{q^{-1}}$
Replacing $i_k i_{k+1} i_k \to i_{k+1} i_k i_{k+1}$ $\quad \bar{i}\to\bar{i}'$, Cartan-Weyl elements $\{E'_\beta\}$ differs from $\{E_\beta\}$ only by $E'_{\beta_{k+1}}$ and $E_{\beta_{k+1}}$

ⓒ If $\beta_k=\alpha_i$ - simple root then $E_{\beta_k}=E_i$

- <u>Hint</u> ⓐⓑ — apply $T_{i_k}^{-1}\dots T_{i_1}^{-1}$ and reduce to rk=2.
ⓒ Use ⓐⓑ and fact that if $\alpha_i=\beta_1 \Rightarrow E_{\beta_1}=E_i$

Lemma

- **Problem** Relate e_{ij}^- generators in RTT realization and Cartan-Weyl basis.

- **Lemma** $\forall k$ $E_{\beta_k} \in U_q(n_+)$

- **Proof** Induction by $ht(\beta)$. For $ht(\beta)=1$, $\beta = \alpha_i$, $E_\beta = E_i$ by ©

- If $ht(\beta)>1 \Rightarrow \beta = \alpha+\gamma$, $\alpha, \gamma \in \Delta_+$, α - simple

- $\exists \bar{i}, i'$ such that $\alpha_{i_1} = \alpha$, $\alpha_{i'_N} = \alpha$.

Matsumoto Theorem $\rightarrow \bar{i}$ to i' using Braid relations
Hence α goes through $\beta \rightarrow$ by Ⓑ $E_\beta \in U_q(n_+)$

PBW Theorem

- <u>Th</u> a) Elements $E_{\beta_1}^{a_1} E_{\beta_2}^{a_2} \dots E_{\beta_N}^{a_N}$ form a basis in $U_q(n_+)$

б) Elements $E_{\beta_1}^{a_1} \dots E_{\beta_N}^{a_N} H_1^{b_1} \dots H_r^{b_r} F_{\beta_1}^{c_1} \dots F_{\beta_N}^{c_N}$ form a basis in $U_\hbar(\mathfrak{g})$

- Sufficient to prove a).
  Sufficient to prove linear independendance

- Prove that monomials $E_{\beta_1}^{a_1} \dots E_{\beta_k}^{a_k}$ by induction on k

$$T_{i_1}^{-1}\left(E_{\beta_1}^{a_1} E_{\beta_2}^{a_2} \dots E_{\beta_k}^{a_k}\right) = (-F_{i_1} K_{i_1})^{a_1} \otimes (E'_{\beta_1})^{a_1} \dots (E'_{\beta_{k-1}})^{a_k} \subset U_q(\mathfrak{b}^-) \otimes U_q(n^+) = U_q(\mathfrak{g})$$

where $\bar{\iota} = (i_2, i_3, \dots, i_N, w_0(i_1))$

References

- Chari, Pressley *A guide to quantum groups*
  Sec. 8.1

- Tingley *Elementary construction of Lusztig's canonical basis*