Introduction to Quantum Groups

Lecture 13

Factorization of the universal R matrix

qft.itp.ac.ru/mbersht/quantum_groups.html

Quantum Weyl group

- $U_q(sl_2)$ <u>define</u> $t \in (U_q(sl_2)^\circ)^* = \mathbb{C}[SL_2]_q^*$
  $\forall \ell \geq 0$ L_ℓ – $\ell+1$ dim rep $U_q(sl_2)$ $L_\ell = \bigoplus_{\substack{-\ell \leq m \leq \ell \\ 2 \mid m+\ell}} L_\ell[m]$

$$t\big|_{L_\ell[m]} = \sum_{\substack{a-b+c=m, \\ a,b,c\geq 0}} (-1)^b q^{ac-b} F^{(a)} E^{(b)} F^{(c)} \qquad E^{(a)} = \frac{E^a}{[a]_q!}$$

- <u>Rk</u> t is well defined. $t : L_\ell[m] \to L_\ell[-m]$
- <u>Rk</u> For $q=1$ $\sum (-1)^b \frac{F^a}{a!}\frac{E^b}{b!}\frac{F^c}{c!} = e^F e^{-E} e^F = \begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix} = \begin{pmatrix}0&-1\\1&0\end{pmatrix}$

i.e. classically $t \in SL_2$, moreover t – reflection
$t \in N(H) < SL_2$
G acts on $\forall$ f.d. rep of $\mathfrak{g} \to q$-deformation acts on L_ℓ

Computation of t

- <u>Problem</u> (a) For $v \in L_\ell[m]$ $\quad E^{(a)}F^{(b)}v = \sum_{t\geq 0} F^{(b-t)}E^{(a-t)}\begin{bmatrix} m-b+a \\ t \end{bmatrix} v$

(b) $v_\ell \in L_\ell[\ell]$ highest weight vector. Let $\tilde{v}_m = F^{\left(\frac{\ell-m}{2}\right)} v_\ell \in L_\ell[m]$

$$t\,\tilde{v}_m = (-1)^{\frac{\ell-m}{2}} q^{-\frac{\ell+m+2}{2}\frac{\ell-m}{2}}\, \tilde{v}_{-m}.$$

(c) Show $tFv = -EKtv$, $\quad tKv = K^{-1}tv$, $\quad tEv = -K^{-1}Ftv$

- <u>Hint</u> (b) $\sum (-1)^b q^{ac-b} F^{(a)}E^{(b)}F^{(c)}F^{\left(\frac{\ell-m}{2}\right)} v_\ell = \left|\text{using } F^{(d_1)}F^{(d_2)} = \begin{bmatrix} d_1+d_2 \\ d_1 \end{bmatrix} F^{(d_1+d_2)} \text{ and (a)}\right|$

$$= \sum (-1)^b q^{ac-b} \begin{bmatrix} c+\frac{\ell-m}{2} \\ c \end{bmatrix}_q \begin{bmatrix} \frac{\ell+m}{2}+b-c \\ b \end{bmatrix}_q \begin{bmatrix} \frac{\ell+m}{2} \\ a \end{bmatrix} F^{\left(\frac{\ell+m}{2}\right)} v_\ell =$$

$$= \sum_a (-1)^{m+a} q^{-a-\frac{\ell+m+2}{2}\frac{\ell-m}{2}} \begin{bmatrix} \frac{\ell+m}{2} \\ a \end{bmatrix} \begin{bmatrix} a-1 \\ \frac{\ell+m}{2} \end{bmatrix} F^{\left(\frac{\ell+m}{2}\right)} v_\ell$$

$\left|\begin{array}{l}\text{using } 0 \leq a \leq \ell \\ \text{only } a=0 \text{ remains}\end{array}\right|$ $= (-1)^m q^{-\frac{\ell+m+2}{2}\frac{\ell-m}{2}} F^{\left(\frac{\ell+m}{2}\right)} v_\ell$

using $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^k \begin{bmatrix} -n+k-1 \\ k \end{bmatrix}$

$\sum_j \begin{bmatrix} n \\ j \end{bmatrix}\begin{bmatrix} m \\ k-j \end{bmatrix} q^{j(n+m)-nk} = \begin{bmatrix} n+m \\ k \end{bmatrix}$

follows from

$(x+y)^{n+m} = (x+y)^n (x+y)^m$

$xy = q\,yx$

Corollaries

- $U_q(sl_2)^\circ$ has basis $c^{m}_{\ell\, m'}$-matrix elements $\tilde{v}_m \to \tilde{v}_{m'} \in L_\ell$

  $t \in (U_q(sl_2)^\circ)^*$, $\quad t(c^{m}_{\ell, m'}) = (-1)^{\frac{\ell-m}{2}} q^{-\frac{\ell+m+2}{2}\frac{\ell-m}{2}} \delta_{m+m', 0}$

- Th $\quad t^{-1}xt = T(x)$ where T is generator of Lusztig's braid group

- In other words

  $t^{-1}Et = -FK \qquad t^{-1}Kt = K^{-1} \qquad t^{-1}Ft = -K^{-1}E$

  in L_ℓ $\forall \ell \in \mathbb{Z}_{\geq 0}$

Coproduct

- <u>Problem</u> Let $\bar{R}=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(q-q^{-1})^n}{[n]_q!}E^n\otimes F^n$ Prove $\bar{R}^{-1}=\sum\frac{(-1)^n q^{-\binom{n}{2}}(q-q^{-1})^n}{[n]_q}E^n\otimes F^n$

- <u>Problem*</u> $\Delta(t)=\bar{R}^{-1}t\otimes t$

Equivalently $\forall v_1\in L_{e_1},\ v_2\in L_{e_2}$ $\quad t(v_1\otimes v_2)=\bar{R}^{-1}(tv_1\otimes tv_2)$

- <u>Hint</u> Possible plan of the computation

a) If $\Delta(t)v_1\otimes v_2=\bar{R}^{-1}(t\otimes t)v_1\otimes v_2\Rightarrow\Delta(t)\Delta(E)v_1\otimes v_2=\bar{R}^{-1}(t\otimes t)\Delta(E)v_1\otimes v_2$

b) Hence sufficient to check for v_2, $Fv_1=0$

c) For $Fv_1=0\Rightarrow Etv_1=0\Rightarrow R^{-1}t\otimes t\,v_1\otimes v_2=t\otimes t\,v_1\otimes v_2$

$$\Delta t\,v_1\otimes v_2=\sum(-1)^b q^{\#}\Delta(F)^{(a)}\Delta(E)^{(b)}(v_1\otimes F^{(c)}v_2)=$$

$$=\sum(-1)^{b_1+b_2}q^{\#}F^{(a_1)}E^{(b_1)}v_1\otimes F^{(a_2)}E^{(b_2)}F^{(c_2)}v_2=\sum(-1)^{b_1+b_2}\begin{bmatrix}e_1-b_1+a_1\\a_1\end{bmatrix}E^{(b_1-a_1)}v_1\otimes F^{(a_2)}E^{(b_2)}F^{(c_2)}v_2$$

$$=\sum(-1)^{b_2+g}q^{\#}\Big(\sum_{a_1}\begin{bmatrix}e_1-g\\a_1\end{bmatrix}(-1)^{a_1}q^{\#}E^{(g)}v_1\otimes F^{(a_2)}E^{(b_2)}F^{(c_2)}v_2$$

$$=(-1)^n q^{\#}E^{(e_1)}v_1\otimes\sum(-1)^{b_2}q^{\#}F^{(a_2)}E^{(b_2)}F^{(c_2)}v_2=tv_1\otimes tv_2$$

Universal R matrix

- **Lem** $\overline{R}\Delta(x)\overline{R}^{-1} = (T^{-1}\otimes T^{-1})\Delta(T(x))$

Pf $\Delta(T^{-1}(x)) = \Delta(txt^{-1}) = \overline{R}^{-1}t\otimes t\Delta(x)t^{-1}\otimes t^{-1}\overline{R} = \overline{R}^{-1}T^{-1}\otimes T^{-1}(\Delta(x))\overline{R}$

- **Th** Let $R_H = e^{\hbar H\otimes H/2}$, $R = R_H\overline{R}$. Then $R\Delta(x)R^{-1} = \Delta^{op}(x)$ universal R matrix

Pf $R\Delta(E)R^{-1} = R_H T^{-1}\otimes T^{-1}(\Delta(T(E)))R_H^{-1} = R_H T^{-1}\otimes T^{-1}(\Delta(-FK))R_H^{-1}$

$= R_H T^{-1}\otimes T^{-1}(-FK\otimes K + 1\otimes FK)R_H^{-1} = q^{H\otimes H/2}(E\otimes K^{-1} + 1\otimes E)q^{-H\otimes H/2}$

$= E\otimes 1 + K\otimes E = \Delta^{op}(E)$ (We used $q^{\frac{H\otimes H}{2}}E\otimes 1 = E\otimes K q^{\frac{H\otimes H}{2}}$)

- **Problem** $t^2 = e^{\hbar H^2/2}u$, where u - central (relate u to central elements from Lect 7)

General case

- $U_q(\mathfrak{g})$ $E_i, K_i^{\pm 1}, F_i$ $i \in I$ $\rightsquigarrow t_i \in (U_q(\mathfrak{g})^\circ)^*$
- $\forall \lambda \in P^+, \exists L_{\lambda,q}$ f.d. $U_q(\mathfrak{g})$-mod

w.r.t $E_i, K_i^{\pm 1}, F_i$ $L_{\lambda,q} = \bigoplus L_{\ell^{(s)}}$ $U_q(\mathfrak{sl}_2)$-mod $t_i\big|_{L_{\ell^{(s)}}} = t$

$$t_i\big|_{L_\ell[m]} = \sum_{a-b+c=m,\ a,b,c\geq 0} (-1)^b q^{ac-b} F_i^{(a)} E_i^{(b)} F_i^{(c)}$$

t_i is reflection, $v \in L_{\lambda,q}[\mu] \rightarrow t_i v \in L_{\lambda,q}[s_i(\mu)]$

- RK Morally $t_i \in G$, moreover $t_i \in N(H)$

Quantum Weyl group

Main properties

- **Th** a) $T_i(x) = t_i^{-1} x t_i \quad \forall i \in I, \ x \in U_q(\mathfrak{g})$
  б) $\underbrace{t_i t_j \cdots}_{m_{ij}} = \underbrace{t_j t_i \cdots}_{m_{ij}}$ Braid relations

For proof – $\mathrm{rk}\,\mathfrak{g} = 1, 2$.

<u>Corollary</u> T_i – automorphisms, give braid group action on $U_q(\mathfrak{g})$

- <u>Th</u> $\Delta(t_i) = \bar{R}_i^{-1}\, t_i \otimes t_i$, where $\bar{R}_i = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (q-q^{-1})^n}{[n]_q!} E_i^n \otimes F_i^n$

Follows from $\mathrm{rk}\,\mathfrak{g} = 1$ above.

<u>Cor</u> $\bar{R}_i \Delta(x) \bar{R}_i^{-1} = (T_i^{-1} \otimes T_i^{-1}) \Delta(T_i(x))$

<u>Rk</u> t_i is not group-like, $\bar{R}$ measures it.
Classically $r|_H = 0$, but $r|_{WH} \neq 0$. $r|_{s_i} \sim E_i \wedge F_i$

Cartan - Weyl elements

- Fix reduced decomp $w_0 = s_{i_1} \dots s_{i_N}$

$E_{\gamma_1} = T_{i_N}^{-1} \cdots T_{i_2}^{-1}(E_{i_1}), \dots\dots, E_{\gamma_{N-1}} = T_{i_N}^{-1}(E_{i_{N-1}}), \quad E_{\gamma_N} = E_{i_N}$

$F_{\gamma_1} = T_{i_N}^{-1} \cdots T_{i_2}^{-1}(F_{i_1}), \dots\dots, F_{\gamma_{N-1}} = T_{i_N}^{-1}(F_{i_{N-1}}) \quad F_{\gamma_N} = F_{i_N}$

Similar to elements from Lect 12.
Properties follows from rank=2 computations and Braid group

- $\overline{R}_{\gamma_k} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} (q-q^{-1})^n}{[n]_q!} E_{\gamma_k}^n \otimes F_{\gamma_k}^n = T_{i_N}^{-1} \cdots T_{i_{k+1}}^{-1}(\overline{R}_{i_k})$

- $\overline{R}_H = e^{\hbar \sum H_j \otimes H^j}$ $\quad H_j, H^j$ – dual bases in $\mathfrak{h}$

Universal R matrix

- <u>Th</u> $R\Delta(x)R^{-1} = \Delta^{op}(x)$, for $R = R_H R_{\gamma_1} \dots R_{\gamma_N}$

<u>Pf</u> $R\Delta(x)R^{-1} = \bar{R}_H \bar{R}_{\gamma_1} \cdots \bar{R}_{\gamma_N} \Delta(x) \bar{R}_{\gamma_N}^{-1} \cdots \bar{R}_{\gamma_1}^{-1} \bar{R}_H^{-1}$

$$= \bar{R}_H \bar{R}_{\gamma_1} \cdots \bar{R}_{\gamma_{N-1}} (T_{i_N}^{-1} \otimes T_{i_N}^{-1})(\Delta(T_{i_N}(E_i))) \bar{R}_{\gamma_{N-1}}^{-1} \cdots \bar{R}_{\gamma_1}^{-1} \bar{R}_H^{-1}$$

$$= \bar{R}_H (T_{i_N}^{-1} \otimes T_{i_N}^{-1}) \Big[(T_{i_N} \otimes T_{i_N})(\bar{R}_{\gamma_1}) \cdots \overbrace{(T_{i_N} \otimes T_{i_N})(\bar{R}_{\gamma_{N-1}})}^{= \bar{R}_{i_{N-1}}} \Delta(T_{i_N}(E_i)) \overbrace{(T_{i_N} \otimes T_{i_N})(\bar{R}_{\gamma_{N-1}}^{-1})}^{\bar{R}_{i_{N-1}}^{-1} \,=} \cdots \Big] \bar{R}_H^{-1}$$

$$= \bar{R}_H (T_{i_N}^{-1} \cdots T_{i_1}^{-1} \otimes T_{i_N}^{-1} \cdots T_{i_1}^{-1}) [\Delta(T_{i_1} \cdots T_{i_N}(E_i))] \bar{R}_H^{-1} = R_H (T_{w_0}^{-1} \otimes T_{w_0}^{-1}) [\Delta(T_{w_0}(E_i))] R_H^{-1}$$

Using $w_0 = s_{i'} w_0 s_i$, where $w_0(\alpha_i) = -\alpha_{i'}$, $T_{w_0 s_i}(E_i) = E_{i'}$, $T_{w_0}(E_i) = -F_{i'}K_{i'}$

$$= R_H (T_{w_0}^{-1} \otimes T_{w_0}^{-1}) [\Delta(-F_{i'}K_{i'})] R_H^{-1} = R_H (T_{w_0}^{-1} \otimes T_{w_0}^{-1}) [-F_{i'}K_{i'} \otimes K_{i'} + 1 \otimes F_{i'}K_{i'}) R_H^{-1}$$

$$= R_H (E_i \otimes K_i^{-1} + 1 \otimes E_i) R_H^{-1} = E_i \otimes 1 + K_i \otimes E_i = \Delta^{op}(E_i)$$

Remark and Example

using

$$\boxed{\overline{R}_i \Delta(x) \overline{R}_i^{-1} = (T_i^{-1} \otimes T_i^{-1}) \Delta(T_i(x))}$$

- **Rk** $\Delta E_{\gamma_k} = \Delta(T_{i_N}^{-1} \cdots T_{i_{k+1}}^{-1}(E_{i_k}))$

$$= R_{i_N}^{-1} (T_{i_N}^{-1} \otimes T_{i_N}^{-1}) [\Delta(T_{i_{N-1}}^{-1} \cdots T_{i_{k+1}}^{-1}(E_{i_k}))] R_{i_N}$$

$$= R_{\gamma_N}^{-1} \cdots R_{\gamma_{k+1}}^{-1} (T_{i_N}^{-1} \cdots T_{i_{k+1}}^{-1} \otimes T_{i_N}^{-1} \cdots T_{i_{k+1}}^{-1}) [E_{i_k} \otimes K_{i_k} + 1 \otimes K_{i_k}] R_{\gamma_{k+1}} \cdots R_{\gamma_N}$$

$$= \prod_{e>k} R_{\gamma_k}^{-1} (E_{\gamma_k} \otimes K_{\gamma_k} + 1 \otimes E_{\gamma_k}) \prod_{e>k} R_{\gamma_k}$$

- **Problem** For $\mathfrak{g} = \mathfrak{sl}_n$ compute $(\rho_{\mathbb{C}^n} \otimes \rho_{\mathbb{C}^n}) R$ (at least for $n=3$)

Hint In this rep $E_i \mapsto E_{i,i+1}$, $F_i \mapsto E_{i+1,i}$ and similarly for all E_{γ_k}

References

- Chari, Pressley *A guide to quantum groups*
  Sec. 8.1
- Lusztig *Introduction to quantum groups* Ch 5, 37, 39
- Khoroshkin Tolstoy *Universal R-matrix for Quantized (Super) algebras*