Affine Quantum Groups

Lecture 1

Quantum groups (Quick Review)

Definition $U_q(\mathfrak{g})$

- Φ - root system $\Pi = \{\alpha_1, \dots, \alpha_r\}$ - simple roots

Usually assume $(\alpha,\alpha) = 2$, $\forall \alpha \in \Pi$

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_j)}$$ — Cartan matrix

- **Def** $U_q(\mathfrak{g})$ is generated by

E_i, F_i, K_i, K_i^{-1} for $\alpha_i \in \Pi$ $\qquad q_i = q^{(\alpha_i, \alpha_i)/2}$

quadratic relations

- $K_i K_i^{-1} = K_i^{-1} K_i = 1$, $\quad K_i K_j = K_j K_i$
- $K_i E_j K_i^{-1} = q^{a_{ij}} E_j$, $\quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j$ $\qquad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$
- $\sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0$, $\quad \sum_{k=0}^{1-a_{ij}} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0$

Serre relations

- coproduct $\quad \Delta: U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$

$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta K_i = K_i \otimes K_i, \quad \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i$

counit $\quad \varepsilon: U_q(\mathfrak{g}) \to \mathbb{C}$

$\varepsilon(E_i) = 0 \qquad \varepsilon(K_i) = 1 \qquad \varepsilon(F_i) = 0$

antipod $\quad S: U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$

$S(E_i) = -E_i K_i^{-1} \qquad S(K_i) = K_i^{-1} \qquad S(F_i) = -K_i F_i$

- For $\lambda = \sum n_i \alpha_i$ let $K_\lambda = \prod K_i^{n_i}$.

- RK For $X \in U_q(\mathfrak{g})_\lambda$, $Y \in U_q(\mathfrak{g})_\mu$ $\quad \mathrm{ad}_{q,X} Y = XY - q^{(\lambda,\mu)} YX$

Serre relations $\quad \mathrm{ad}_{q,E_i}^{1-a_{ij}} E_j = 0, \quad \mathrm{ad}_{q,F_i}^{1-a_{ij}} F_j = 0$

- $U_q(\mathfrak{g}) = U^- \otimes U^0 \otimes U^+$ $\qquad U^{\geq} = U^0 \otimes U^+$

generated by F_i $\quad$ — $K_i^{\pm 1}$ $\quad$ — $E_i^{\pm 1}$ $\qquad U^{\leq} = U^- \otimes U^0$

Universal R-matrix

- **Th** $\exists!$ Pairing $\mathcal{U}^{\leq} \otimes \mathcal{U}^{\geq} \to \mathbb{C}$ s.t.

$$(x, yy') = (\Delta(x), y' \otimes y), \quad (xx', y) = (x \otimes x', \Delta(y))$$

$$(K_i, K_j) = q^{(\alpha_i, \alpha_j)}, \quad (F_i, K_j) = (K_i, E_j) = 0 \quad (F_i, E_j) = \delta_{ij} \frac{1}{q_i - q_i^{-1}}$$

- **Th** $R = e^{\hbar \sum H_i \otimes H^i} \sum x_a \otimes y^a = \bar{R}_H \bar{R}$ satisfies

  - $R\,\Delta(x)\,R^{-1} = \Delta^{op}(x) \quad \forall x \in \mathcal{U}_q(\mathfrak{g})$
  - $(\Delta \otimes Id) R = R_{13} R_{23} \quad (Id \otimes \Delta) R = R_{13} R_{12}$

($q = e^{\hbar}$, $K_i = e^{\hbar H_i}$
H_i, H^i – dual bases in $\mathfrak{h}$, w.r.to Killing form
$x_a \in \mathcal{U}^+$, $y^a \in \mathcal{U}^-$ dual bases w.r.t. pairing)

- **Ex** $\mathfrak{g} = Sl_2$ $\quad (F^n, E^n) = \dfrac{[n]_q!}{q^{\binom{n}{2}} (q - q^{-1})^n}$

$$R = e^{\frac{1}{2}\hbar\, H \otimes H} \sum q^{\binom{n}{2}} \frac{(q - q^{-1})^n}{[n]_q!} E^n \otimes F^n = \bar{R}_H \bar{R}$$

- For V, W - reps of $\mathcal{U}_q(\mathfrak{g})$

$R_{V,W} = \rho_V \otimes \rho_W(R) : V \otimes_\Delta W \longrightarrow V \otimes_{\Delta^{op}} W$

$\widetilde{R}_{V,W} = P R_{V,W} : V \otimes_\Delta W \longrightarrow W \otimes_\Delta V$

intertwiners

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$
$$\widetilde{R}_{23} \widetilde{R}_{12} \widetilde{R}_{23} = \widetilde{R}_{12} \widetilde{R}_{23} \widetilde{R}_{12}$$

Yang-Baxter

- Problem (urgent) For $\mathfrak{g} = \mathfrak{sl}_3$, $V = \mathbb{C}^3$

ⓐ Compute $\overline{R}$ in 1 & 2 order by E and F using pairing.

ⓑ Compute $R_{V,V}$ and R_{V,V^*}

ⓒ Find eigenvalues $\widetilde{R}_{V,V}$

Hint E_i, F_i acts on V, V^* as for $q=1$

ⓐ is sufficient for ⓑ & ⓒ. Convenient to write answer in matrix units $\sum \# E_{ab} \otimes E_{cd}$

Braid group

- **Def** Braid group B_φ is generated by

$T_i, \quad \forall \alpha_i \in \Pi \quad$ s.t $\quad \underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}}$

$m_{ij} = 2 \quad a_{ij} = 0, \quad m_{ij} = 3 \quad a_{ij} a_{ji} = 1$

- **Rk** $\exists$ homomorphism $B_\varphi \twoheadrightarrow W_\varphi \quad T_i \mapsto s_i$

- **Def** $T_i(E_i) = -F_i K_i, \; T_i(F_i) = -K_i^{-1} E_i$ ~ like antipode

Lusztig

$T_i(K_j) = K_j K_i^{-a_{ij}}$ — as reflection

$$T_i(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} E_i^{(-a_{ij}-r)} E_j E_i^{(r)}$$

$$T_i(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{r} F_i^{(r)} F_j F_i^{(-a_{ij}-r)}$$

$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$

$[k]_q! = \prod_{j=1}^{k} [j]_q$

$E_i^{(k)} = \frac{E_i^k}{[k]_{q_i}!}$

- **RK** $T_i(E_j) = \frac{1}{[-a_{ij}]_{q_i}!} ad_{q_i, E_i}^{-a_{ij}} E_j = ad_{\Delta^{op}, E_i^{(-a_{ij})}} E_j$
- **Th** (Lusztig)

ⓐ T_i is automorphism of $U_q(\mathfrak{g})$

ⓑ $T_1, \dots, T_r$ satisfy B_Φ relations.

- Fix reduced decomposition $w_0 = s_{i_1} \cdot \ldots \cdot s_{i_N}$ — longest element in W_Φ

$\beta_1 = s_{i_N} \cdot \ldots \cdot s_{i_2}(\alpha_{i_1}), \dots, \beta_{N-1} = s_{i_N}(\alpha_{i_{N-1}}), \beta_N = \alpha_{i_N}$

$\beta_1, \dots, \beta_N$ — all positive roots in some convex order

- Define

$E_{\beta_1} = T_{i_N}^{-1} \cdot \ldots \cdot T_{i_2}^{-1}(E_{i_1}), \dots, E_{\beta_{N-1}} = T_{i_N}^{-1}(E_{i_{N-1}}), E_{\beta_N} = E_{i_N}$

$F_{\beta_1} = T_{i_N}^{1} \cdot \ldots \cdot T_{i_2}^{-1}(F_{i_1}), \dots, F_{\beta_{N-1}} = T_{i_N}^{-1}(F_{i_{N-1}}), F_{\beta_N} = F_{i_N}$

- **Th** (PBW) a) Elements $E_{\beta_1}^{a_1}\cdots E_{\beta_N}^{a_N}$ form a basis in U^+

b) Elements $F_{\beta_1}^{b_1}\cdots F_{\beta_N}^{b_N} K_\lambda E_{\beta_1}^{a_1}\cdots E_{\beta_N}^{a_N}$ form a basis in U

- useful formulas for T_i^{-1}

$$T_i^{-1}(E_i) = -K_i^{-1}F_i, \quad T_i^{-1}(F_i) = -E_iK_i, \quad T_i^{-1}(K_j) = K_jK_i^{-a_{ij}}$$

$$T_i^{-1}(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} E_i^{(r)} E_j E_i^{(-a_{ij}-r)} \qquad T_i^{-1}(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{r} F_i^{(-a_{ij}-r)} F_j F_i^{(r)}$$

- $\bar{R}_{\beta_k} = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \dfrac{(q-q^{-1})^n}{[n]_q!} E_{\beta_k}^n \otimes F_{\beta_k}^n$

- **Th** (Levendorskii - Soibelman, Kirillov-Reshetikhin Khoroshkin-Tolstoy, Rosso) $R = \bar{R}_H \bar{R}_{\beta_1}\cdots \bar{R}_{\beta_N}$

- <u>Problem</u> For $\mathfrak{g} = \mathfrak{sl}_3$ compute $\bar{R}$ in $1 \& 2$ order using this formula.

Representations

- For any homomorphism $\sigma : Q \to \{\pm 1\}$
  - Automorphism σ $\quad E_i \mapsto \sigma(\alpha_i) E_i \quad K_i \mapsto \sigma(\alpha_i) K_i, \quad F_i \mapsto F_i$
  - Representation $\mathbb{C}_\sigma$ $\quad E_i, F_i \mapsto 0 \quad K_i \mapsto \sigma(\alpha_i)$
- V – f.d rep. $\quad V = \bigoplus_{\lambda, \sigma} V_{\lambda, \sigma} \quad K_i v = \sigma(\alpha_i) q^{(\lambda, \alpha_i)} v \quad v \in V_{\lambda, \sigma}$

  $V^\sigma = \bigoplus_\lambda V_{\lambda, \sigma} \qquad V = \bigoplus V^\sigma$
- Category $\mathrm{Rep}^{\text{type I}}_{U_q \mathfrak{g}}$ equivalent to $\mathrm{Rep}_{\mathfrak{g}}$

  For any σ. $\mathrm{Rep}^{\text{type I}}_{U_q \mathfrak{g}}$ equivalent to $\mathrm{Rep}^{\text{type } \sigma}_{U_q \mathfrak{g}}$
- <u>Rem</u> Only type I for $U_\hbar(\mathfrak{g})$ – $\mathbb{C}[[\hbar]]$ algebra with generators E_i, F_i, H_i, $q = e^\hbar$, $K_i = e^{\hbar H_i}$

Desription of the center

- For f.d. module V let $f_V : U_q(\mathfrak{g}) \to \mathbb{C}$ — quantum trace

$$f_V(x) = Tr_V(x K_{2\rho})$$

- **Lem** $f_V(xy) = f_V(y\, S^2(x))$

Pf $Tr(xy K_{2\rho}) = Tr(y\, K_{2\rho} x K_{-2\rho} K_{2\rho}) = Tr(y\, S^2(x)\, K_{2\rho})$ $\square$

- **Th** (Drinfeld–Reshetikhin)

a) $C_V = (id \otimes f_V)(R_{21} R_{12})$ – central element

b) $V \to C_V$ homomorphism $K(U_q(\mathfrak{g})\text{-mod}) \to Z(U_q(\mathfrak{g}))$

- **Problem** Find central elements for $\mathfrak{sl}_3$, V, V^* (classical limit)

Rem C_V belong to extension of $U_q(\mathfrak{g})$, by K_λ, $\lambda \in P$ weight lattice

RTT realization

- $\mathcal{U}(R)$ ~ Hopf algebra with generators

$e_{ij}^+, \quad e_{ji}^- \qquad 1 \le i \le j \le n$

$$L^+ = \begin{pmatrix} e_{11}^+ & e_{12}^+ & \cdots & e_{1n}^+ \\ 0 & \ddots & & \vdots \\ & \ddots & \ddots & \\ & & 0 & e_{nn}^+ \end{pmatrix} \qquad L^- = \begin{pmatrix} e_{11}^- & & 0 & \\ e_{21}^- & \ddots & & \\ & & \ddots & 0 \\ & & e_{nn-1}^- & e_{nn}^- \end{pmatrix}$$

with relations $\quad e_{ii}^+ e_{ii}^- = 1$

$$R L_1^+ L_2^+ = L_2^+ L_1^+ R, \qquad R L_1^- L_2^- = L_2^- L_1^- R, \qquad R L_1^+ L_2^- = L_2^- L_1^+ R$$

Coproduct, antipode

$$\Delta(L^\pm) = L^\pm \otimes L^\pm \qquad S(L^\pm) = (L^\pm)^{-1}$$

- Here $L_1 = L \otimes 1$, $\quad L_2 = 1 \otimes L$, $\qquad \Delta e_{ij}^\pm = \sum_k e_{ik}^\pm \otimes e_{kj}^\pm$

$$R = q \sum E_{ii} \otimes E_{ii} + \sum_{i \ne j} E_{ii} \otimes E_{jj} + \sum_{i<j} E_{ij} \otimes E_{ji}$$

- $\underline{RK}$ $n=2$ $R=\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$

- $\underline{Th}$ (Ding-Frenkel) Hopf algebras $\mathcal{U}(R)$ and $\mathcal{U}_q(\mathfrak{gl}_n)^{coop}$ are isomorphic $(m \mapsto m, \ \Delta \to \Delta^{op})$

- Here $\mathcal{U}_q(\mathfrak{gl}_n)$ generated by $E_1, \dots, E_{n-1}, K_1^{\pm 1}, \dots K_n^{\pm 1}, F_1, \dots, F_{n-1}$

Different from sl_n relations

- $[E_i, F_j] = \delta_{ij} \dfrac{K_i K_{i+1}^{-1} - K_i K_{i+1}}{q-q^{-1}}$
- $K_i E_j = q^{\delta_{ij}} q^{-\delta_{i\,j+1}} E_j K_i$
- $K_i F_j = q^{-\delta_{ij}} q^{\delta_{i\,j+1}} F_j K_i$

- $\Delta E_i = E_i \otimes K_i K_{i+1}^{-1} + 1 \otimes E_i$
- $\Delta F_i = F_i \otimes 1 + K_i^{-1} K_{i+1} \otimes F_i$

- Homomorphism $\mathcal{U}(\mathfrak{gl}_n) \to \mathcal{U}(R)$

$$L^+ = \begin{pmatrix} K_1 & 0 & & 0 \\ 0 & \ddots & & \\ & & \ddots & 0 \\ 0 & & 0 & K_n \end{pmatrix} \begin{pmatrix} 1 & (q-q^{-1})F_1 & & * \\ & 1 & & \\ & & \ddots & (q-q^{-1})F_{n-1} \\ & & & 1 \end{pmatrix} = (\rho_V \otimes id) R$$

$$L^- = \begin{pmatrix} 1 & 0 & & \\ (q^{-1}-q)E_1 & 1 & & \\ & & \ddots & \\ * & & (q^{-1}-q)E_{n-1} & 1 \end{pmatrix} \begin{pmatrix} K_1^{-1} & & 0 \\ & \ddots & \\ 0 & & K_n^{-1} \end{pmatrix} = (id \otimes \rho) R^{-1}$$

- Relations surjectivity injectivity

- Problem a) For sl_3 relate elements of L^+, L^- to Cartan-Weyl basis.

b)* The same for sl_n

Hints ① Let

$$L^+ = \begin{pmatrix} K_1 & 0 & & 0 \\ 0 & K_2 & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & K_n \end{pmatrix} \begin{pmatrix} 1 & (q-q^{-1})f_{12} & (q-q^{-1})f_{13} & \cdots & (q-q^{-1})f_{1n} \\ 0 & 1 & (q-q^{-1})f_{23} & & \\ 0 & 0 & 1 & & \\ \cdots & & & & \\ 0 & & & & 1 \end{pmatrix}$$

we have $i<j<k$ $[e^+_{ij}, e^+_{jk}] = -(q-q^{-1}) e^+_{jj} e^+_{ik}$

$$f_{ik} = [f_{jk}, f_{ij}]_{q^{-1}} = -q^{-1}[f_{ij}, f_{jk}]_q$$

② For $w_0 = s_{i_1} s_{i_2} \cdot \ldots \cdot s_{i_N}$

ⓐ If $a_{i_k i_{k+1}} = 0$ then reversing i_k, i_{k+1} get $\bar{i}'$ with same Cartan-Weyl elements

(reordered $F_{\beta'_k} = F_{\beta_{k+1}}$, $F_{\beta'_{k+1}} = F_{\beta_k}$)

ⓑ If $i_k = i_{k+2}$ $a_{i_k i_{k+1}} = a_{i_{k+1} i_{k+2}} = -1$ then

$$F_{\beta_{k+1}} = -[F_{\beta_k}, F_{\beta_{k+2}}]_q \qquad \beta_{k+1} = \beta_k + \beta_{k+2}$$

$i' \rightsquigarrow i'_k = i'_{k+2} = i_{k+1}$, $i'_{k+1} = i_k = i_{k+2}$, $F_{\beta'_k} = F_{\beta_{k+2}}$, $F_{\beta'_{k+2}} = F_{\beta_k}$

difference $F_{\beta'_{k+1}} = -[F_{\beta_{k+2}}, F_{\beta_k}]_q \neq F_{\beta_{k+1}}$

ⓒ If $\beta_k = \alpha_i$ - simple root $F_{\beta_k} = F_i$

Drinfeld Double

- Pairing $\langle\cdot,\cdot\rangle : A^- \otimes A^+ \to \mathbb{C}$

Drinfeld Double $A = A^- \otimes A^+$ $\quad \Delta a = a_{(1)} \otimes a_{(2)}$

$$\langle a_{(1)}, b_{(1)} \rangle a_{(2)} * b_{(2)} = b_{(1)} * a_{(1)} \langle a_{(2)}, b_{(2)} \rangle$$

- Define $\langle L_1^+, L_2^- \rangle = R_{12}$ $\qquad \langle \ell^+_{ij}, \ell^-_{i'j'} \rangle = R^{jj'}_{ii'}$

Relation $\quad R_{12} L_1^+ L_2^- = L_2^- L_1^+ R_{12}$

- <u>Problem</u> $\langle R_{12} L_1^+ L_2^+ - L_2^+ L_1^+ R, \ \dots \rangle = 0$