Affine Quantum Groups

Lecture 2

Affine algebras (Quick Review)

Loop algebras

- <u>Untwisted</u> affine Kac-Moody algebras

- $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ $\qquad \hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K \qquad \tilde{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$

- $X \in \mathfrak{g}$, notation $X \otimes t^n = X_n = X[n]$

- Relations $\quad [X_n, Y_m] = [X, Y]_{n+m} + n\,\delta_{n+m,0} \langle X, Y \rangle K.$

Killing form

$$[K, X_n] = 0$$

$$[d, K] = 0, \quad [d, X_n] = n X_n. \quad — \quad d = t\partial_t$$

- $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \qquad \tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \Phi^a} \tilde{\mathfrak{g}}_\alpha$

$\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d, \qquad \Phi^a = \{\alpha + n\delta \mid (\alpha, n) \in (\Phi \cup \{0\}) \times \mathbb{Z} \setminus \{0, 0\}\}$

$\tilde{\mathfrak{g}}_{\alpha + n\delta} = \mathfrak{g}_\alpha \otimes t^n \qquad \tilde{\mathfrak{g}}_{n\delta} = \mathfrak{h} \otimes t^n.$

- $\tilde{\mathfrak{h}}_{\mathbb{Z}}$ – weight space for $\tilde{\mathfrak{h}}$

$\alpha \in \mathfrak{h}^* \subset \tilde{\mathfrak{h}}^*$ $\quad \delta|_{\mathfrak{h}} = \delta(K) = 0, \quad \delta(d) = 1$

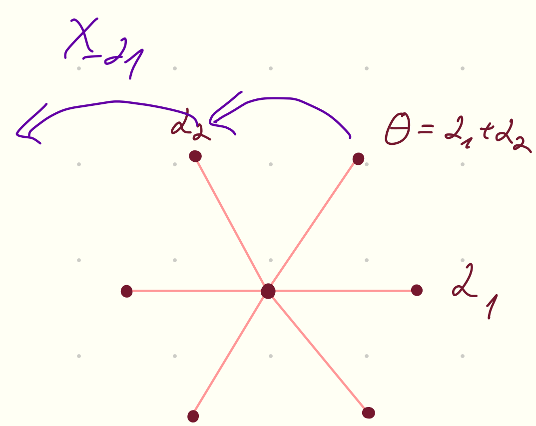
$\varpi_0|_{\mathfrak{h}} = \varpi_0|_d = 0 \quad \varpi_0(K) = 1$

	$\mathfrak{h}$	K	d
$\mathfrak{h}^*$	$*$	0	0
δ	0	0	1
ϖ_0	0	1	0

$\tilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\delta + \mathbb{C}\varpi_0$

$\varpi_1, \dots, \varpi_r \in \mathfrak{h}^*$ – fundamental weights for Φ

- $\theta \in \Phi$ – highest weight of adjoint rep.



X_θ satisfies $\mathrm{ad}_{X_{-\alpha_i}}^{(\theta, \alpha_i^\vee)+1} X_\theta = 0$

(General property of f.d. reps $\quad v_\lambda \in V_\lambda$ h.w. vec $\quad X_{-\theta}^{(\alpha_i^\vee, \lambda)+1} v = 0$)

$i = 1, \dots, r$ $\quad$ $E_i = X_{\alpha_i}[0]$, $F_i = X_{-\alpha_i}[0]$

$\{\alpha_1 \dots \alpha_r\} = \Pi$ - simple roots

$E_0 = X_{-\theta}[1]$, $\quad$ $F_0 = X_{\theta}[-1]$

$H_i = H_{\alpha_i}[0]$, $\quad$ $H_0 = K - H_\theta$

<u>Problem</u> (urgent) Show that $E_0, \dots, E_r, H_0, \dots, H_r, F_0, \dots, F_r$ are Chevalley generators of Kac-Moody Lie algebra

$[H_i, E_j] = a_{ij} E_j$, $\quad$ $[H_i, F_j] = -a_{ij} F_j$ $\quad$ $[E_i, F_j] = \delta_{ij} H_i$

$ad_{E_i}^{a_{ij}+1} E_j = 0$ $\quad$ $ad_{F_i}^{-a_{ij}+1} F_j = 0$ — Serre relations

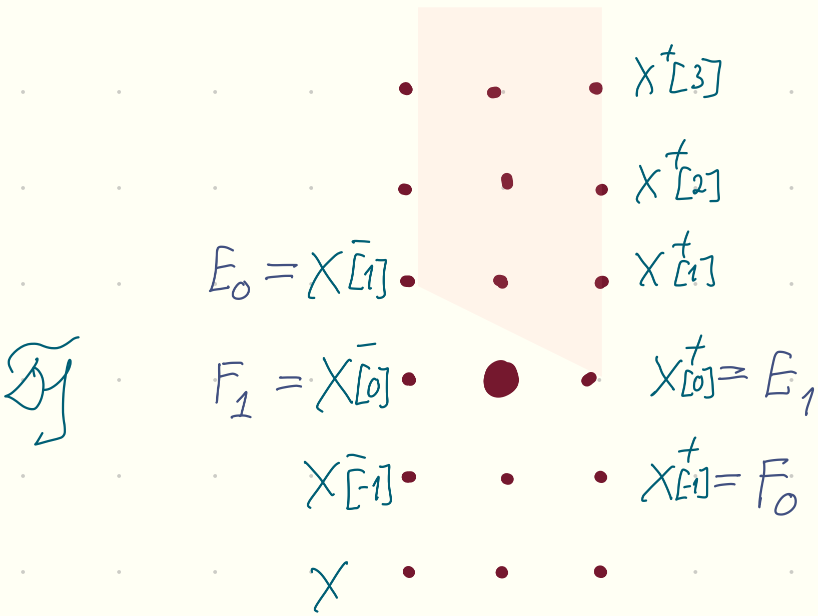
<u>Obvious hint</u>: Sufficient to check relations with E_0, F_0

- $\alpha_0 = \delta - \theta$ $\qquad$ $a_{0j} = \dfrac{2(\delta-\theta, \alpha_0)}{(\delta-\theta, \delta-\theta)} = \dfrac{-2(\theta, \alpha_j)}{(\theta, \theta)} = -(\theta, \alpha_j)$

normalization $\qquad$ $(\theta, \theta) = 2$

- Examples

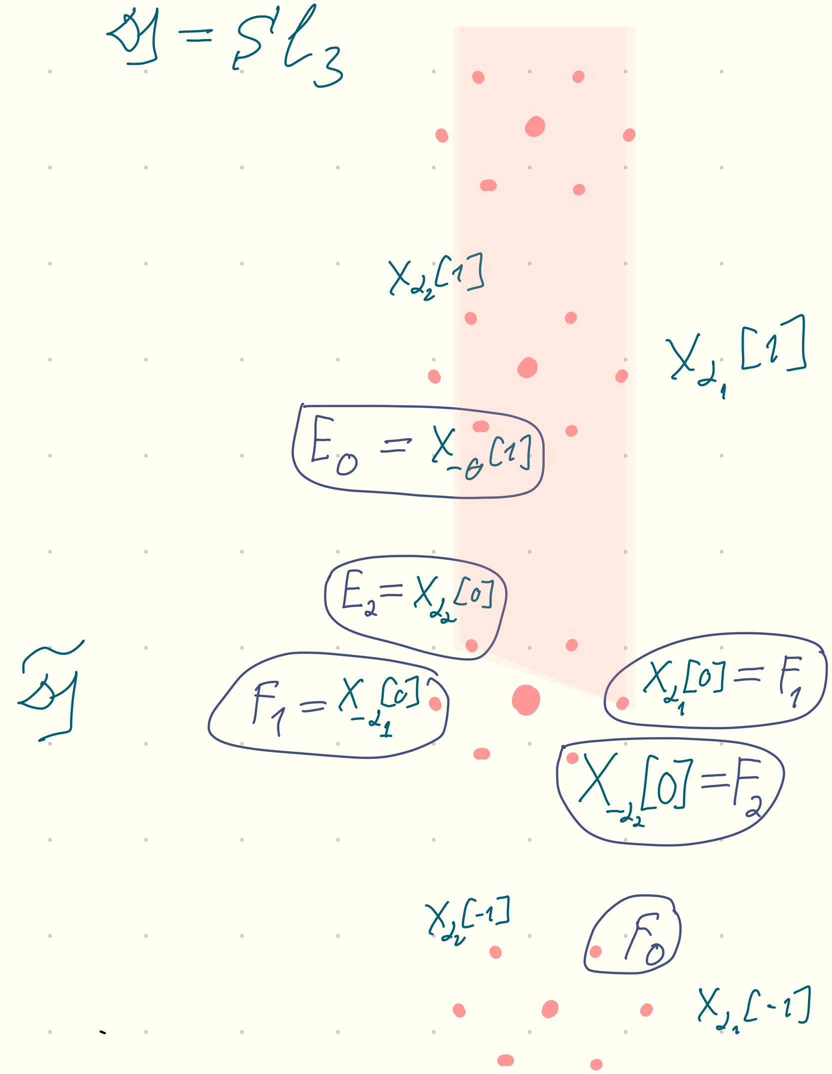
$$\mathfrak{g} = \mathfrak{sl}_2 = \langle X^-, X^0, X^+ \rangle$$



Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

$$\mathfrak{g} = \mathfrak{sl}_3$$



Cartan matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

- In general $\tilde{\mathfrak{g}} = \tilde{n}_- \oplus \hat{\mathfrak{h}} \oplus \tilde{n}_+$

$\tilde{n}_+$ — subalgebra generated by $E_0, \dots, E_r$

$\tilde{n}_+ = \bigoplus_{\alpha \in \Phi_+^a} \tilde{\mathfrak{g}}_\alpha$ $\quad \Phi_+^a = \left\{ \alpha + n\delta \;\middle|\; \begin{matrix} n > 0 \\ \text{or} \\ n = 0 \quad \alpha \in \Phi_+ \end{matrix} \right\}$

$\tilde{n}_-$ — subalgebra generated by $F_0, \dots, F_r$

Affine Weyl group

- W^a genereted by $s_0, \dots, s_r$, $s_i^2 = e$

  braid relations $\underbrace{s_i s_j s_i \dots}_{m_{ij}} = \underbrace{s_j s_i s_j \dots}_{m_{ij}}$

- For Weyl group (but <u>not</u> for braid group) braid relation equiv. to $(s_i s_j)^{m_{ij}} = e$

- $\tilde{\mathfrak{h}}^* = \langle \alpha_1, \dots, \alpha_r, \delta, \varpi_0 \rangle$

$r+2$ dimensional rep of W^a generated by reflections w.r.t. simple roots

$\alpha_1, \dots, \alpha_r, \alpha_0$

$\alpha_0 = \delta - \theta$

$$s_\alpha(v) = v - 2\frac{(\alpha, v)}{(\alpha, \alpha)}\alpha$$

$$\begin{array}{c|cccc}
 & \alpha_1 \dots & \alpha_r & \delta & \varpi_0 \\
\hline
\alpha_1 & & & 0 & 0 \\
\vdots & & A & \vdots & \\
\alpha_r & & & 0 & 0 \\
\delta & 0 & 0 & 0 & 1 \\
\varpi_0 & 0 & 0 & 1 & 0
\end{array}$$

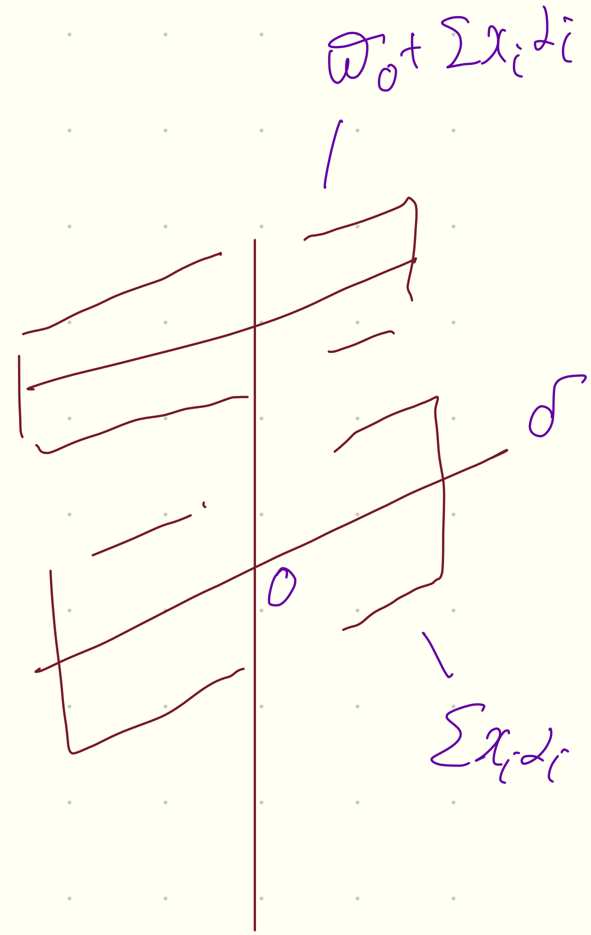
- Reflection w.r.t. $\alpha + n\delta \in \Phi^a$ belong to W^a
- δ is orthogonal to $\alpha_1 \dots \alpha_r, \alpha_0 \Rightarrow \delta$ is invariant
- smaller reps:

ⓐ $r+1$ dim. rep of W^a

$$\langle \alpha_1, \dots, \alpha_r, \alpha_0 \rangle = \langle \alpha_1, \dots, \alpha_r, \delta \rangle$$

ⓑ $\langle \varpi_0 + \sum_{i=0}^{r} x_i \alpha_i \rangle / \langle \delta \rangle$

r-dim <u>affine</u> rep of W^a



- <u>Example</u> $\mathfrak{g} = \mathfrak{sl}_2$

@ Vector space $\langle \alpha_0, \alpha_1 \rangle = \langle \alpha, \delta \rangle$

Scalar product
$$\begin{array}{c|cc} & \alpha & \delta \\ \hline \alpha & 2 & 0 \\ \delta & 0 & 0 \end{array}$$

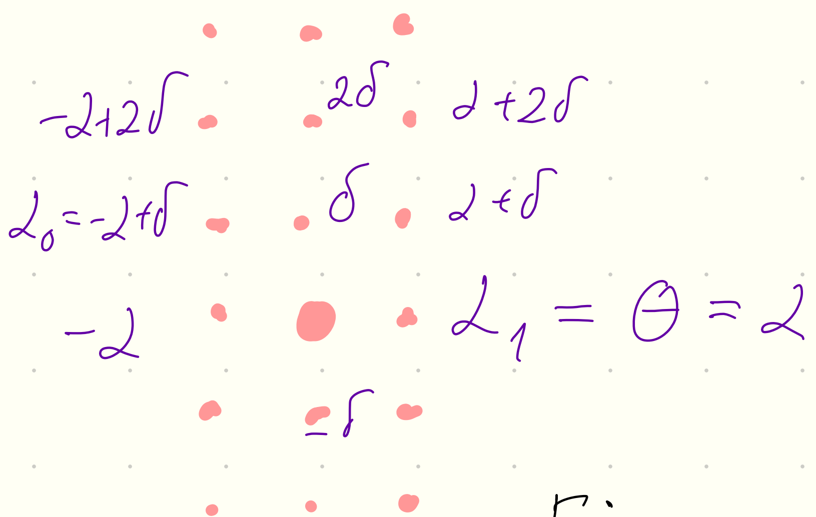


Fig.

$$S_1 \quad \begin{array}{l} \alpha_1 \\ \alpha_0 \end{array} \longmapsto \begin{array}{l} -\alpha_1 = -\alpha \\ \alpha_0 + 2\alpha_1 = \alpha + \delta \end{array} \qquad \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$S_0 \quad \begin{array}{l} \alpha_1 \\ \alpha_0 \end{array} \longmapsto \begin{array}{l} \alpha_1 + 2\alpha_0 \\ -\alpha_0 \end{array} \qquad \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

$$S_0(\alpha_1) = \alpha_1 - (\alpha_0, \alpha_1)\alpha_0$$
$$= \alpha_1 + 2\alpha_0 = 2\delta - \alpha = 2\alpha_0 + \alpha_1$$

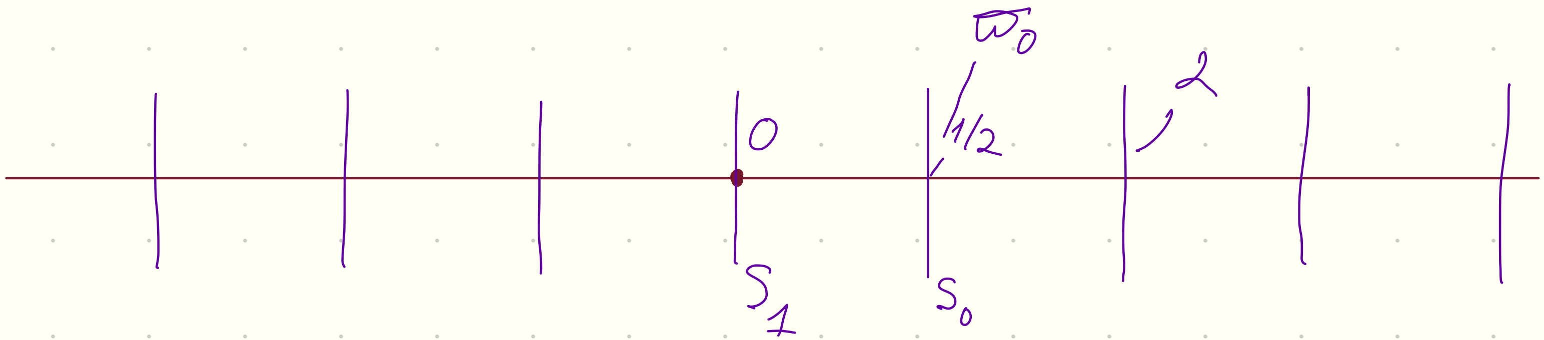
ⓑ Space $\{\varpi_0 + x\alpha + y\delta\} / \langle \delta \rangle$

$\varpi_0 + x\alpha + \mathbb{C}\delta$ — representative

$s_1(\varpi_0 + x\alpha + \mathbb{C}\delta) = \varpi_0 - x\alpha + \mathbb{C}\delta$ $\qquad x \mapsto -x$

$$s_0(\varpi_0 + x\alpha + \mathbb{C}\delta) = \varpi_0 + x\alpha - (1-2x)(\delta - \alpha) + \mathbb{C}\delta$$
$$= \varpi_0 + (1-x)\alpha + \mathbb{C}\delta$$

$x \mapsto 1-x$



$s_0 s_1$ — infinite order, i.e. $m_{0,1} = \infty$

- <u>Theorem</u> W^a acts on $\langle \alpha_1, \dots, \alpha_r \rangle$ generated by reflections $s_{\alpha,n}$ w.r.t. $H_{\alpha,n} = \{v \mid (\alpha, v) = n\}$ affine hyperplane, $\alpha \in \Phi^+, n \in \mathbb{Z}$

(For not simply laced case $H_{\alpha,n} = \{v \mid (\alpha^\vee, v) = n\}$)

- Simple reflections $\quad s_0 - H_0 = \{v \mid (v,\theta) = 1\}$

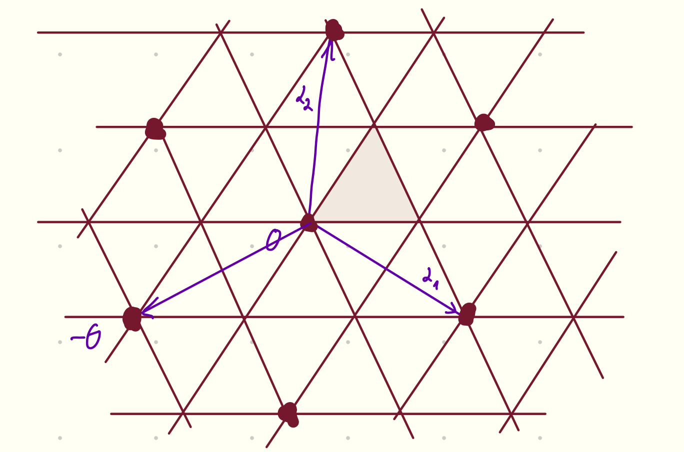
  $s_i \quad H_i = \{v \mid (v, \alpha_i) = 0\}$

  $1 \le i \le r$

- <u>Def</u> <u>Alcove</u> – connected component in $P \otimes_{\mathbb{Z}} \mathbb{R}$. complement to $H_{\alpha, n}$

- $H_0, \dots, H_r$ bounds <u>fundamental alcove</u> $\mathcal{A}$

- <u>Th</u> W^a acts action on set of alcoves is <u>free</u> and <u>transitive</u>.



A_2

- Theorem-Definition.

The following definitions of $\ell(w)$, $w \in W^a$ are equivalent

ⓐ $\ell(w)=k$ — length of shortest expression $w=s_{i_1}\dots s_{i_k}$ (reduced expression)

ⓑ $w(A)=A'$ $\ell(w)$ — number of hyperplanes separating A and A'

ⓒ $\ell(w)=\#\{\beta\in\Phi^a_+ \mid w(\beta)\in\Phi^a_-\}$ $t_\alpha(x)=x+\alpha$

- Th $W^a\simeq W\ltimes Q$

Pf $W\subset W^a$ — subgroup preserving 0

Any element $w\in W^a$ has linear part from W. Hence $W^a=W\ltimes T$

$Q\subset T$, for any $\alpha\in\Phi$: $t_\alpha=s_{\alpha,1}s_{\alpha,0}$ $t_\alpha(x)=x+\alpha$

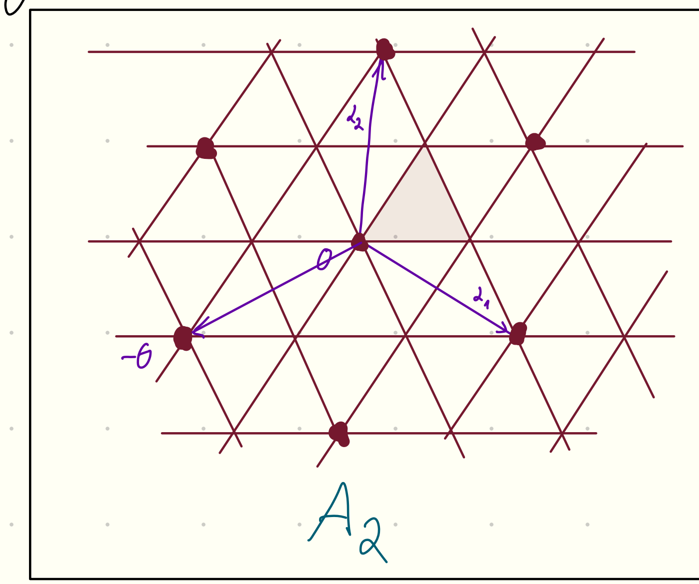
$T\subset Q$ since any $s_{\alpha,n}$ shifts by element of Q $\square$

- Remark In $r+1$-dim. rep. of W^a gen $\langle\alpha_1,\dots,\alpha_r,\delta\rangle$ we have $t_\lambda(\beta)=\beta+(\lambda,\alpha)\delta$ Indeed, easy to see for $s_{\alpha,1}$, $s_{\alpha,0}$.

- Affine extended group
  For any $\lambda \in P$, t_λ preserves set of hyperplanes $H_{\alpha,n}$.

<u>Def</u> $W^{ae} = W \ltimes P$

- Ω – subgroup of W^{ae} preserving A



<u>Prop</u> a) $W^{ae} \simeq \Omega \ltimes W^a$
b) $\Omega \simeq W^{ae}/W^a \simeq P/Q$

<u>Example</u> $\Phi = A_2 \Longrightarrow \Omega = \mathbb{Z}/3\mathbb{Z}$

- $\forall w \in \Omega$, w preserves A $\Downarrow$

w permute facets of A
w permute $\{0, 1, \dots, r\} \Longrightarrow w S_i w^{-1} = S_{w(i)}$

- For any $w \in W^{ae}$, $\ell(w)$ – number of hyperplanes separating A, $w(A)$. <u>Example</u> $w \in \Omega$, $\ell(w) = 0$.

Affine Braid group

- **Def** Br^a generated by $T_0, T_1, \dots, T_r$ subject of Braid relations

- **Th** (Matsumoto) $w \in W^a$, $w = s_{i_1} \dots s_{i_k}$ reduced decomposition. Then $T_w = T_{i_1} \dots T_{i_k}$ does not depend of choise of decomp.

- **Corol** If $\ell(ww') = \ell(w) + \ell(w')$ then $T_{ww'} = T_w T_{w'}$

- Br^{ae} generated by $\Omega, T_0, \dots, T_r$ subject of Braid relations, Ω relations and
$\forall\ \omega \in \Omega \quad \omega T_i \omega^{-1} = T_{\omega(i)}$

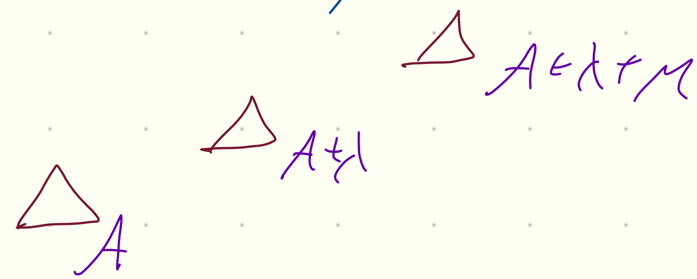
The same corollary holds for Br^{ae}

- <u>Def</u> For $\lambda \in P^-$ let $Y^\lambda = T_{t_\lambda} \in Br^{ae}$

  For $\lambda = P$, if $\lambda = \lambda_1 - \lambda_2$, $\lambda_1, \lambda_2 \in P^+$ let $Y^\lambda = T_{t_{\lambda_1}} T_{t_{\lambda_2}}^{-1}$

- <u>Prop</u> $\lambda, \mu \in P^+$ then $T_{t_\lambda} T_{t_\mu} = T_{t_{\lambda+\mu}} = T_{t_\mu} T_{t_\lambda}$

<u>Pf</u> $\ell(t_{\lambda+\mu}) = \ell(t_\lambda) + \ell(t_\mu)$



$\square$

- <u>Corol</u> a) Definition of Y^λ is correct

  b) $\lambda \mapsto Y^\lambda$ homomorphism $P \to Br^{ae}$.

<u>Pf</u> If $\lambda = \lambda_1 - \lambda_2 = \lambda_1' - \lambda_2'$ $\Rightarrow$

$$T_{t_{\lambda_1'}} T_{t_{\lambda_2}} = T_{t_{\lambda_2+\lambda_1'}} = T_{t_{\lambda_1+\lambda_2'}} = T_{t_{\lambda_1}} T_{t_{\lambda_2'}} \Rightarrow$$

$$T_{t_{\lambda_1'}} T_{t_{\lambda_2'}}^{-1} = T_{t_{\lambda_1}} T_{t_{\lambda_2}}^{-1}$$

$\square$

- **Problem** a) $s_i\lambda=\lambda \Longrightarrow T_i Y^\lambda = Y^\lambda T_i$ $\quad i=1,\dots r$

b) $(\alpha_i^\vee,\lambda)=1 \Longrightarrow T_i Y^\lambda T_i = T^{s_i\lambda}$

- **Problem** For $\Phi=A_n$ elements $Y_i=T_{i-1}^{-1}\cdots T_1^{-1}\tau T_{n-1}\cdots T_i$ generate $P\subset Br^{ae}$. Here $\tau\in\Omega$, $\tau T_i\tau^{-1}=T_{i+1}$, $\tau^{n+1}=e$.

a) For $n=1,2$ compute Y^{ϖ_i}, and show above.

b)* Show for any n.

- **Problem*** $P\subset W^{ae}$ – subgroup of elements with finitely many conjugates