Affine Quantum Groups

Lecture 4-5

Quantum affine sl_2

Three presentations of $U_q(\hat{\mathfrak{g}})$

- Drinfeld-Jimbo presentation as for any Kac-Moody.

$$E_0, \dots, E_r, K_0^{\pm 1}, \dots, K_r^{\pm 1}, F_0, \dots, F_r$$

- New Drinfeld presentation — loop presentation

$$X_1^+[n], \dots, X_r^+[n], \quad \dots \quad, X_1^-[n] \dots, \quad X_r^-[n]$$

$$X_i^+(z) = \sum X_i^+[n] z^{-n} \quad \dots \quad X_i^-(z) = \sum X_i^-[n] z^{-n}$$

- RLL presentation $\qquad L^+(z), L^-(z)$ – matrices with coefficients in algebra

$$R(z/w) L_1^+(z) L_2^+(w) = L_2^+(w) L_1^+(z) R(z/w)$$

$R(z)$ – finite dimensional R-matrix

Faddev-Reshetikhin – Takhtajan

Semenov-Tian-Shansky

Drinfeld-Jimbo presentation

- $\boxed{U_q(\hat{sl}_2)}$ Generators $E_0, E_1, K_0, K_1, F_0, F_1$ Cartan matrix $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

Relations $[E_i, F_j] = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q - q^{-1}}$

$K_i E_j = q^{a_{ij}} E_j K_i$, $K_i F_j = q^{-a_{ij}} F_j K_i$, $K_i K_i^{-1} = K_i^{-1} K_i = 1$, $K_i K_j = K_j K_i$

$E_i^3 E_j - (q^{-2}+1+q^2) E_i^2 E_j E_i + (q^{-2}+1+q^2) E_i E_j E_i^2 - E_j E_i^3 = 0$

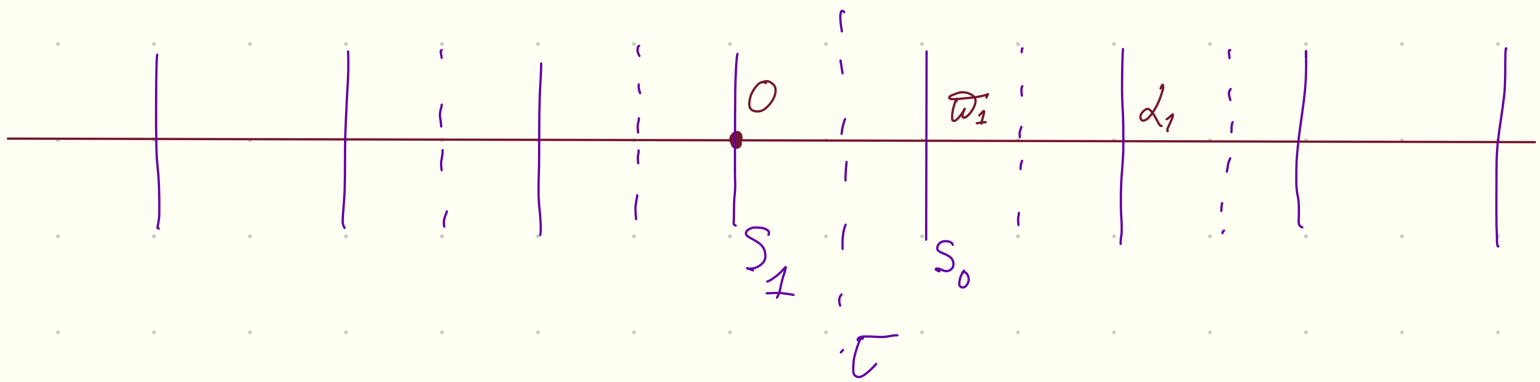
$F_i^3 F_j - (q^{-2}+1+q^2) F_i^2 F_j F_i + (q^{-2}+1+q^2) F_i F_j F_i^2 - F_j F_i^3 = 0$ $\quad i \neq j$

- $\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i$, $\Delta K_i = K_i \otimes K_i$, $\Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i$
- $K = K_1 K_0$ — is centrae
- One can add d or q^{2d}, $d \in \mathbb{C}$ relations

$[d, E_1] = [d, F_1] = [d, K_1] = 0$, $[d, E_0] = E_0$ $[d, F_0] = -F_0$, $[d, K_0] = 0$

Affine braid group

- Weyl group $W^{ae} = \langle s_0, s_1, \tau \mid \tau s_1 \tau^{-1} = s_0,\ \tau s_0 \tau^{-1} = s_1,\ \tau^2 = s_0^2 = s_1^2 = e \rangle$



Translations $Q = \{ (s_0 s_1)^n \mid n \in \mathbb{Z} \} \subset W^{a}$

$P = \{ (\tau s_1)^n \mid n \in \mathbb{Z} \} \subset W^{ae}$

no braid relation!

- Braid group $Br^{ae} = \langle T_0, T_1, \tau \mid \tau T_0 \tau^{-1} = T_1,\ \tau T_1 \tau^{-1} = T_0,\ \tau^2 = e \rangle$

Translations $\langle Y^{\lambda} \rangle = \langle (\tau T_1)^n \mid n \in \mathbb{Z} \rangle \subset Br^{ae}$

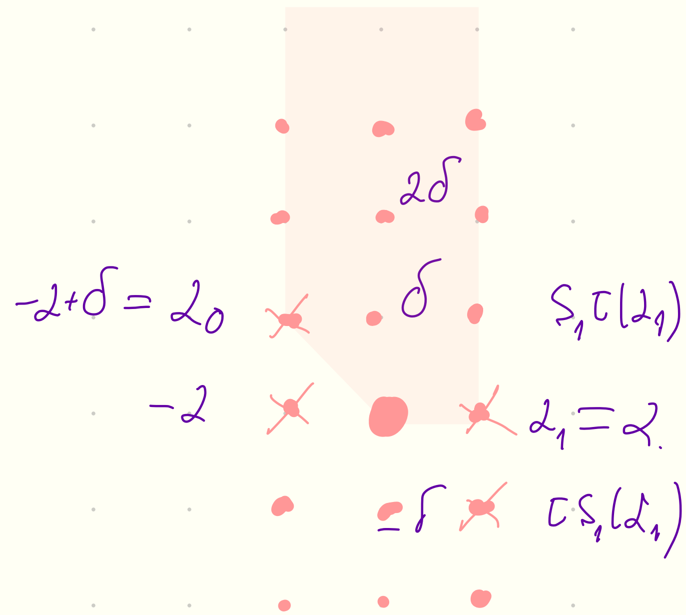
another choice $\langle (\tau T_0)^n \mid n \in \mathbb{Z} \rangle$

Root generators

- Br^{ae} acts on $U_q(\hat{sl}_2)$

  T_0, T_1 – Lusztig automorphisms

  τ: $E_0 \leftrightarrow E_1$, $F_0 \leftrightarrow F_1$, $K_0 \leftrightarrow K_1$



- Def $E_{\alpha+n\delta} = (\tau T_1)^{-n} E_1$ $\quad n \geq 0$

  $E_{-\alpha+(n+1)\delta} = (\tau T_1)^{n} E_0$ $\quad n \geq 0$

- Question: how to define E_δ?

  For $q=1$ E_δ is commutator of E_1, E_0

  Natural – some q-commutator

  Recall $\mathrm{ad}_{q,X} Y = XY - q^{(\lambda,\mu)} YX$ $\qquad \mathrm{ad}_{q,E_i}^{1-a_{ij}} E_j = 0$ – Serre relation two natural options

$$E_\delta \begin{cases} ad_{q,E_1} E_0 = E_0 E_1 - q^{-2} E_1 E_0 \\ ad_{q,E} E_1 = E_1 E_0 - q^{-2} E_0 E_1. \end{cases}$$

- Proposition $\quad \sigma T_1 (E_0 E_1 - q^{-2} E_1 E_0) = E_0 E_1 - q^{-2} E_1 E_0$

Pf $\quad \sigma T_1 (E_1) = \sigma(-F_1 K_1) = -F_0 K_0$

$$\sigma T_1 (E_0) = \sigma\left(E_1^{(2)} E_0 - q^{-1} E_1 E_0 E_1 + q^{-2} E_0 E_1^{(2)} \right)$$
$$= E_0^{(2)} E_1 - q^{-1} E_0 E_1 E_0 + q^{-2} E_1 E_0^{(2)}$$

$$\sigma T_1 (E_0 E_1 - q^{-2} E_1 E_0) = (q^{-2} F_0 K_0 \, \sigma T_1 (E_0) - \sigma T_1 (E_0) F_0 K_0)$$

$$= ([F_0, \sigma T_1 (E_0)] K_0 = \frac{-1}{[2]_q} \Big(\frac{K_0 - K_0^{-1}}{q-q^{-1}} E_0 E_1 + E_0 \frac{K_0 - K_0^{-1}}{q-q^{-1}} E_1 -$$

$$-q^{-1}[2]_q \left(\frac{K_0 - K_0^{-1}}{q-q^{-1}} E_1 E_0 + E_0 E_1 \frac{K_0 - K_0^{-1}}{q-q^{-1}} \right) + q^{-2} E_1 \frac{K_0 - K_0^{-1}}{q-q^{-1}} E_0 + q^{-2} E_1 E_0 \frac{K_0 - K_0^{-1}}{q-q^{-1}} \Big) K_0$$

$$= \frac{-1}{[2]_q (q-q^{-1})} \Big(E_0 E_1 K_0^{-1} (-1 - q^2 + q^{-1}[2]_q) + E_0 E_1 K_0 (1 + q^{-2} - q^{-1}[2]_q) $$

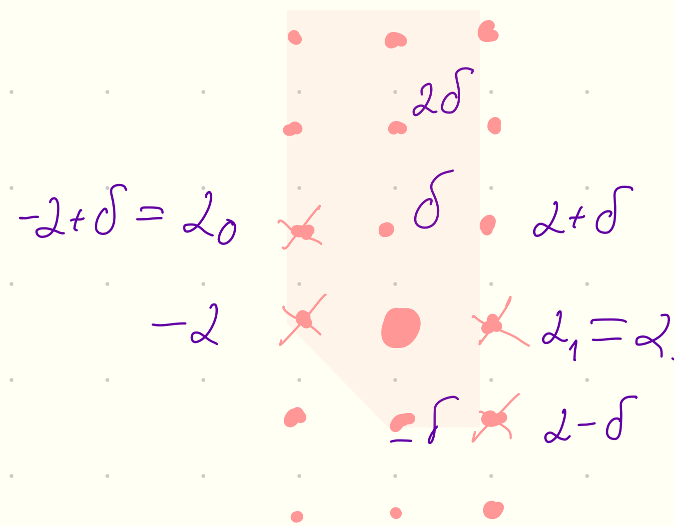
$$+E_1E_0K_0^{-1}(\ldots) + E_0E_1K_0(\cdots)\big)K_0 = \frac{1}{q^{-2}-q^{2}}\Big(E_0E_1K_0^{-1}(q^{-2}-q^{2})$$
$$+E_1E_0K_0^{-1}(1-q^{-4})\Big)K_0 = E_0E_1 - q^{-2}E_1E_0 \qquad \square$$

- **Def** Imaginary root generators

$$E_{n\delta} = E_{-\alpha+\delta}E_{\alpha+(n-1)\delta} - q^{-2}E_{\alpha+(n-1)\delta}E_{-\alpha+\delta}$$

- Compute $[E_\delta, E_\alpha] = [E_0E_1 - q^{-2}E_1E_0, E_1]$

$$= E_0E_1^2 - (1+q^{-2})E_1E_0E_1 + q^{-2}E_1^2E_0$$
$$= E_0E_1^2 - q^{-1}[2]_q E_1E_0E_1 + q^{-2}E_1^2E_0$$
$$= [2]_q\left(E_0E_1^{(2)} - q^{-1}E_1E_0E_1 + q^{-2}E_1^{(2)}E_0\right)$$
$$= [2]_q\, T_1^{-1}(E_0) = [2]_q\, T_1^{-1}\tau(E_1) = [2]_q\, E_{\alpha+\delta}$$



Lemma $[E_\delta, E_{\alpha+n\delta}] = [2]_q E_{\alpha+(n+1)\delta}$

Pf $E_{\alpha+n\delta} = (\tau T_1)^{-n} E_1$. Induction. Base $n=0$, above

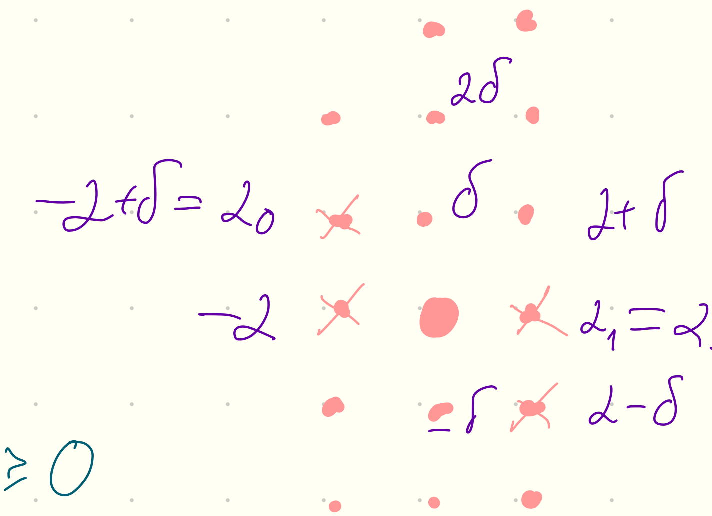
Step: use $(\tau T_1)^{-1}$ $\square$

- **Lemma** $[E_\delta, E_{-\alpha+n\delta}] = -[2]_q E_{-\alpha+(n+1)\delta}$.

Pf Similarly ... $\square$

- Let $U_q(\hat{n}_+)$ - subalgebra generated by E_0, E_1

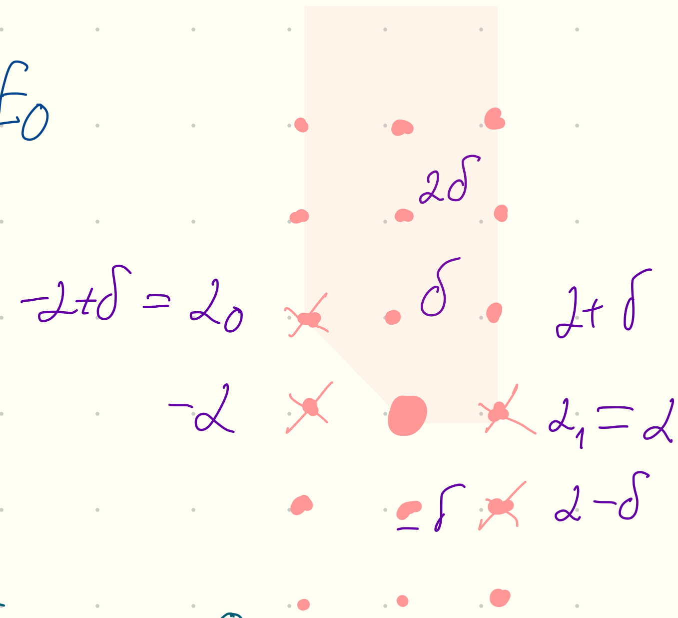
Corollary $E_{\alpha+n\delta}, E_{-\alpha+(n+1)\delta} \in U_q(\hat{n}_+)$ $n \geq 0$



Relations Between $E_{\alpha+n\delta}$

- $E_\alpha = E_1$, $\quad E_{\alpha+\delta} = E_0 E_1^{(2)} - q^{-1} E_1 E_0 E_1 + q^{-2} E_1^{(2)} E_0$

$E_{\alpha+\delta} E_\alpha - q^2 E_\alpha E_{\alpha+\delta}$ = Serre $= 0$



- Problem (urgent) For $n \geq 0$

$$E_{\alpha+(n+1)\delta} E_\alpha - q^2 E_{\alpha+n\delta} E_{\alpha+\delta} + E_{\alpha+\delta} E_{\alpha+n\delta} - q^2 E_\alpha E_{\alpha+(n+1)\delta} = 0$$

Hint Induction. Base $n=0$.

Step Use $[E_\delta, -]$ and $(\mathcal{C}T_1)^{-1}$

- Corollary $E_{\alpha+(n+1)\delta} E_{\alpha+m\delta} - q^2 E_{\alpha+n\delta} E_{\alpha+(m+1)\delta} + E_{\alpha+(m+1)\delta} E_{\alpha+n\delta} - q^2 E_{\alpha+m\delta} E_{\alpha+(n+1)\delta} = 0$

Pf For $n \geq m$ follows from problem applying $(\mathcal{C}T_1)^{\#}$

Relation is symmetris w.r.to $n \Leftrightarrow m \Rightarrow$ $\square$

- Half-current

<u>Def</u> $e^+(z) = \sum_{n\geq 0} E_{\alpha+n\delta}\, z^{-n}$

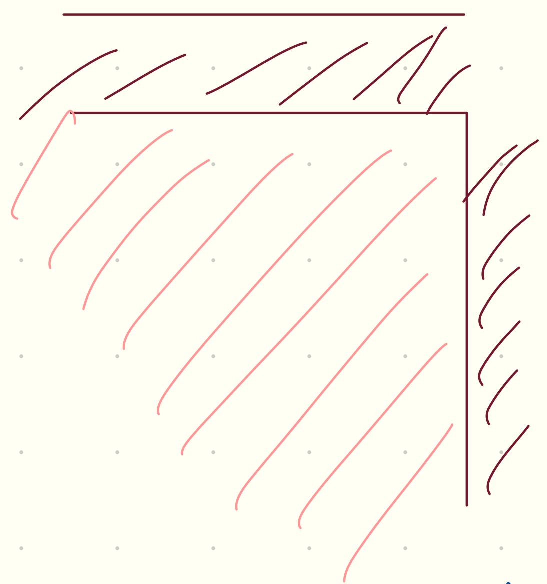
Relation

$$e^+(z)e^+(w)(z-q^2w) + e^+(w)e^+(z)(w-q^2z)$$

$$= \sum_{n,m\geq -1} \#_{n,m}\, z^{-n} w^{-m}$$

= $\#_{n,m} = 0$ for $n,m \geq 0$.

= Boundary term



- <u>Problem</u>*

$$e^+(z)e^+(w)(z-q^2w) + e^+(w)e^+(z)(w-q^2z) = (1-q^2)\left(z\, e^+(w)^2 + w\, e^+(z)^2\right)$$

Relations Between $E_{-2+n\delta}$

- **Prop** $E_{-2+2\delta}E_{-2+\delta} - q^{-2}E_{-2+\delta}E_{-2+2\delta} = 0$

$$E_{-2+(n+1)\delta}E_{-2+m\delta} - q^{-2}E_{-2+n\delta}E_{-2+(m+1)\delta} + E_{-2+(m+1)\delta}E_{-2+n\delta} - q^{-2}E_{-2+m\delta}E_{-2+(n+1)\delta} = 0$$

- Half currents $e^{-}(z) = \sum_{n\geq 0} E_{-2+n\delta} z^{n}$

$$e^{-}(z)e^{-}(w)(z-q^{-2}w) + e^{-}(w)e^{-}(z)(w-q^{-2}z) = \text{Boundary term}$$

$$= (1-q^{-2})(z\, e^{-}(w)^{2} - w\, e^{-}(z)^{2})$$

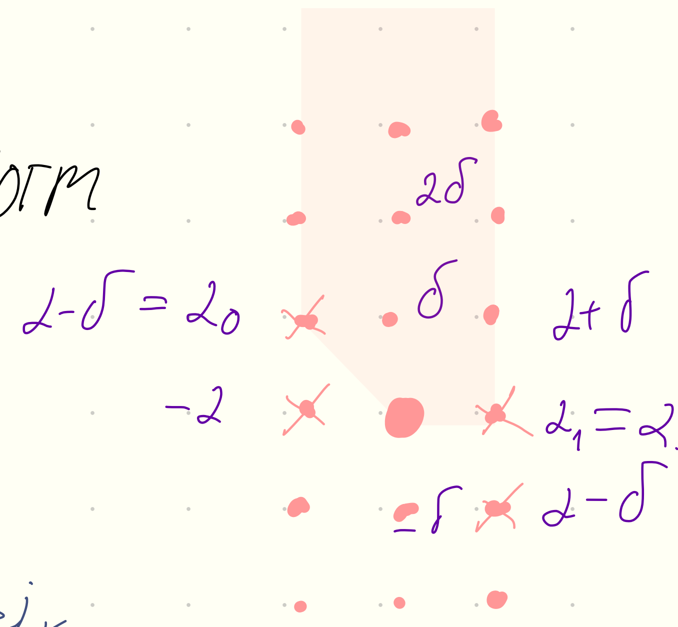
PBW property

- **Lemma** Any product of the form

$E_{\alpha+i_1\delta}\, E_{\alpha+i_2\delta} \cdots E_{\alpha+i_k\delta}$ is

linear combination of

$E_{\alpha+j_1\delta}\, E_{\alpha+j_2\delta} \cdots E_{\alpha+j_k\delta}$ $\quad j_1 \geq j_2 \geq \dots \geq j_k$



Pf Using $[E_{\alpha+(n+1)\delta}, E_{\alpha+m\delta}]_{q^2} = [E_{\alpha+n\delta}, E_{\alpha+(m+1)\delta}]_{q^2}$ we can permute or move closer any $E_{\alpha+n\delta}$, $E_{\alpha+m\delta}$ $\square$

- Similarly for $E_{-\alpha+n\delta}$.

Key Theorem

- **Lemma** The following are equivalent

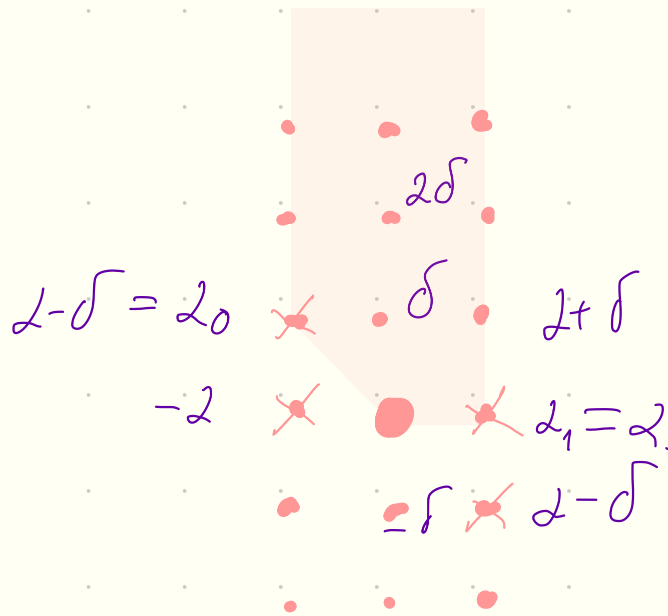
ⓐ $\tau T_1(E_{p\delta}) = E_{p\delta}$ $\quad \forall\ 1 \le p \le n$

ⓑ $[E_{-\alpha+(p-r)\delta}, E_{\alpha+r\delta}]_{q^{-2}} = E_{p\delta}$ $\quad 0 \le r < p \le n.$

ⓒ $[E_\delta, E_{p\delta}] = 0$ $\quad p \le n$

- **Th** ⓐ, ⓑ, ⓒ above holds $\forall n$.

Pf Induction by n $\square$



$E_{n\delta}$ and $E_{\alpha+m\delta}$ relations

- Recall $$E_\delta E_{\alpha+n\delta} = [2]_q E_{\alpha+(n+1)\delta} + E_{\alpha+n\delta}E_\delta$$

- More general computatio
  (use $E_{m\delta} = [E_{-\alpha+p\delta}, E_{\alpha+(m-p)\delta}]_{q^{-2}}$)

$$E_{(n+1)\delta}E_1 - E_1 E_{(n+1)\delta} = E_{-\alpha+n\delta}E_{\alpha+\delta}E_\alpha - q^{-2}E_{\alpha+\delta}E_{-\alpha+n\delta}E_\alpha - E_\alpha E_{-\alpha+n\delta}E_{\alpha+\delta} + q^{-2}E_\alpha E_{\alpha+\delta}E_{-\alpha+n\delta}$$

$$= q^2 E_{-\alpha+n\delta}E_\alpha E_{\alpha+\delta} - q^{-2}E_{\alpha+\delta}E_{n\delta} - q^{-4}E_{\alpha+\delta}E_\alpha E_{-\alpha+n\delta} + q^2 E_{n\delta}E_{\alpha+\delta} - q^2 E_{-\alpha+n\delta}E_\alpha E_{\alpha+\delta}$$
$$+ q^{-4}E_{\alpha+\delta}E_\alpha E_{-\alpha+n\delta} = q^2 E_{n\delta}E_{\alpha+\delta} - q^{-2}E_{\alpha+\delta}E_{n\delta}$$

Finally $$E_{(n+1)\delta}E_\alpha - q^2 E_{n\delta}E_{\alpha+\delta} = E_\alpha E_{(n+1)\delta} - q^{-2}E_{\alpha+\delta}E_{n\delta}$$

- Using (T_1) we can $E_\alpha \longrightarrow E_{\alpha+m\delta}$

$$E_{(n+1)\delta}E_{\alpha+m\delta} - q^2 E_{n\delta}E_{\alpha+(m+1)\delta} = E_{\alpha+m\delta}E_{(n+1)\delta} - q^{-2}E_{\alpha+(m+1)\delta}E_{n\delta}$$

- PBW we can permute or move closer $E_{n\delta}$ and $E_{\alpha+m\delta}$
- Loop form $\quad e_\delta(z) = 1 + (q-q^{-1}) \sum_{p>0} E_{p\delta} z^{-p}$

$$(z-q^2 w)\, e_\delta(z)\, e^+(w) = (z-q^{-2}w)\, e^+(w)\, e_\delta(z)$$

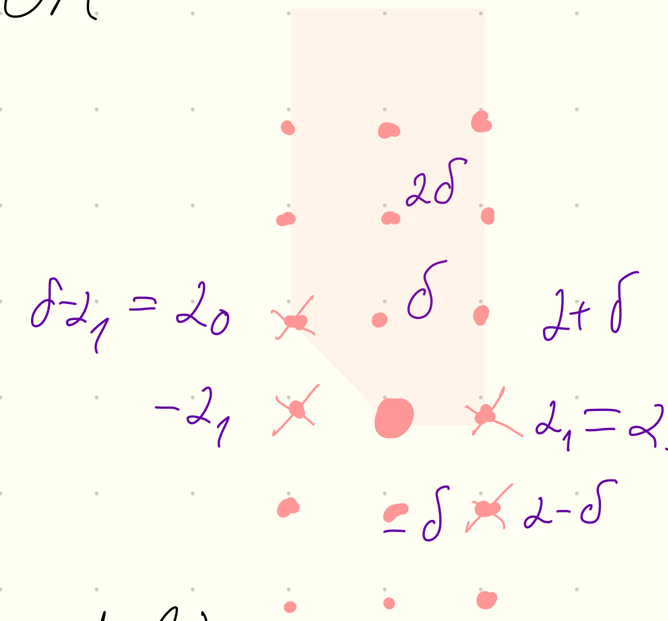
- Similarly

$$E_{(n+1)\delta} E_{-\alpha+m\delta} - q^{-2} E_{n\delta} E_{-\alpha+(m+1)\delta} = E_{-\alpha+m\delta} E_{(n+1)\delta} - q^2 E_{-\alpha+(m+1)\delta} E_{n\delta}$$

$$(z-q^{-2}w)\, e_\delta(z)\, e^+(w) = (z-q^2 w)\, e^+(w)\, e_\delta(z)$$

$E_{n\delta}$ and $E_{m\delta}$ relation

- **Prop** $[E_{n\delta}, E_{m\delta}] = 0 \quad n, m > 0$



Fact $X \in U(\hat{n}^+)$, $[X, F_0] = [X, F_1] = 0$
hence $X = 0$

Pf For $q=1$ no such element
(otherwise in lowest weight Verma module
M_λ vector $X v_\lambda$ - singular)

Let $X = [E_{n\delta}, E_{m\delta}]$, $X \in U(\hat{n}^+)$

$cT_1(X) = X \qquad cT_1(F_0) = -E_1 K_1 \Rightarrow cT_1([X, F_0]) = -[X, E_1] K_1$

For F_1 similarly

Hence sufficient to check $[X, E_0] = [X, E_1] = 0$

Induction on $m+n$. Step

$$[[E_{m\delta}, E_{n\delta}], E_\alpha] = [[E_{m\delta}, E_\alpha], E_{n\delta}] + [E_{m\delta}, [E_{n\delta}, E_\alpha]] =$$

$$[q^2 E_{(m-1)\delta} E_{\alpha+\delta} - q^{-2} E_{\alpha+\delta} E_{(m-1)\delta}, E_{n\delta}] + [E_{m\delta}, q^2 E_{(n-1)\delta} E_{\alpha+\delta} - q^{-2} E_{\alpha+\delta} E_{(n-1)\delta}]$$

$$= E_{(m-1)\delta} E_{\alpha+2\delta} E_{(n-1)\delta} - q^4 E_{(m-1)\delta} E_{(n-1)\delta} E_{\alpha+2\delta} - q^4 E_{\alpha+2\delta} E_{(n-1)\delta} E_{(m-1)\delta} + E_{(n-1)\delta} E_{\alpha+2\delta} E_{(m-1)\delta}$$

$$+ q^4 E_{(n-1)\delta} E_{(m-1)\delta} E_{\alpha+2\delta} - E_{(n-1)\delta} E_{\alpha+2\delta} E_{(m-1)\delta} - E_{(m-1)\delta} E_{\alpha+2\delta} E_{(n-1)\delta} + q^{-4} E_{\alpha+2\delta} E_{(m-1)\delta} E_{(n-1)\delta}$$

$$= q^4 [E_{(n-1)\delta}, E_{(m-1)\delta}] E_{\alpha+2\delta} - q^{-4} E_{\alpha+2\delta} [E_{(m-1)\delta}, E_{(n-1)\delta}] = 0$$

○ – use $E_{n\delta}$ and $E_{\alpha+m\delta}$ relations

○ – use $E_{n\delta}$ and $E_{\alpha+m\delta}$ relations

$\square$

- Th (PBW) Elements

$$\{ E_{-\alpha+\delta}^{a_1} E_{-\alpha+2\delta}^{a_2} \cdot \ldots \cdot E_{\delta}^{b_1} E_{2\delta}^{b_2} \cdot \ldots \cdot \ldots E_{\alpha+2\delta}^{c_2} E_{\alpha+\delta}^{c_1} E_{\alpha}^{c_0} \}$$

form a Basis in $U_q(\hat{n}_+)$

- Remark In f.d. case $w_0 = s_{i_1} \cdot \ldots \cdot s_{i_N}$

$\beta_1 = s_{i_N} \cdots s_{i_2}(\alpha_{i_1}), \ \ldots, \ \beta_{N-1} = s_{i_N}(\alpha_{i_{N-1}}), \ \beta_N = \alpha_{i_N}$

Elements $E_{\beta_1}^{a_1} \cdots E_{\beta_N}^{a_N}$ form a basis in $U(n_+)$

For $\hat{sl}_2$ $w_0 \sim s_0 s_1 s_0 s_1 \ldots \ldots s_0 s_1 s_0 s_1$

$$-\alpha+\delta < -\alpha+2\delta < \ldots < \underset{\vdots}{\overset{\delta}{2\delta}} < \ldots < \alpha+2\delta < \alpha+\delta < \alpha$$

Automorphisms, $U_q(\hat{n}_-)$

- $T_i(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} E_i^{(-a_{ij}-r)} E_j E_i^{(r)}$ $\quad$ $T_i(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{r} F_i^{(r)} F_j F_i^{(-a_{ij}-r)}$
- $T_i^{-1}(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q_i^{-r} E_i^{(r)} E_j E_i^{(-a_{ij}-r)}$ $\quad$ $T_i^{-1}(F_j) = \sum (-1)^{r-a_{ij}} q_i^{r} F_i^{(-a_{ij}-r)} F_j F_i^{(r)}$
- $\varphi(E_i) = F_i,\ \varphi(F_i) = E_i,\ \varphi(K_i) = K_i,\ \varphi(q) = q^{-1}$ automorphism
- $\varphi T_i = T_i^{-1}\varphi$ $\quad$ $\sigma\varphi\sigma T_1 = \varphi T_1 = T_1^{-1}\varphi = T_1^{-1}\sigma\sigma\varphi = (\sigma T_1)^{-1}\sigma\varphi$

  $\varphi\sigma = \sigma\varphi$ $\Rightarrow$ i.e. $\sigma\varphi$ preserves translations

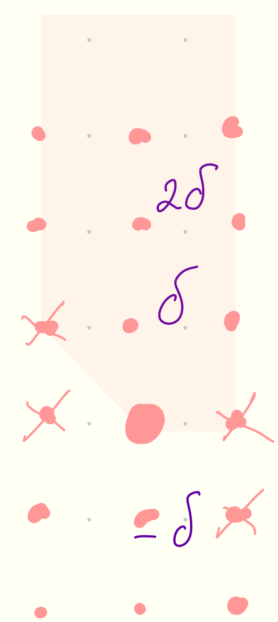
- Define root vectors in $U_q(\hat{n}_-)$ by

$$\sigma\varphi(E_{\alpha+n\delta}) = \sigma\varphi((\sigma T_1)^{-n}E_1) = (\sigma T_1)^n F_0 = F_{\alpha-(n+1)\delta}$$

$$\sigma\varphi(E_{-\alpha+(n+1)\delta}) = \sigma\varphi((\sigma T_1)^n E_0) = (\sigma T_1)^{-n} F_1 = F_{-\alpha-n\delta}$$

$$\sigma\varphi(E_{n\delta}) = F_{-n\delta}$$

$\Rightarrow$ PBW property



Full currents

- $X^+[n] = (\sigma T_1)^{-n} E_1, \qquad X^-[n] = (\sigma T_1)^{n} F_1 \qquad n \in \mathbb{Z}$

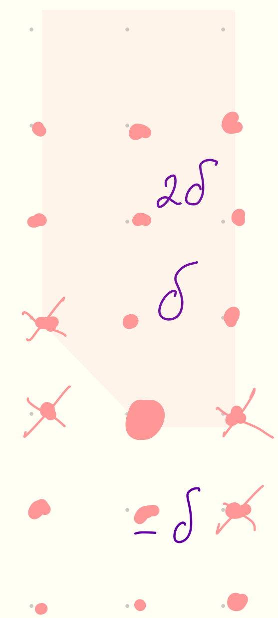
- Relation to root generators

$n \geq 0 \qquad X^+[n] = E_{\alpha+n\delta}$ $\qquad K = K_0 K_1$

$X^+[-1] = \sigma T_1(E_1) = \sigma(-F_1 K_1) = -F_0 K_0 = -(F_{\alpha-\delta} K) K_1^{-1}$

$X^+[-2] = \sigma T_1(-F_0 K_0) = -(F_{\alpha-2\delta} K^2) K_1^{-1}$

$X^+[-n] = -(F_{\alpha-n\delta} K^n) K_1^{-1}$ $\qquad \sum_{n>0} X^+[-n] z^n = -f_+(Kz) K_1^{-1}$



Remark (a) $X^+[-n] \notin U_q(\hat{n}_-)$ but $X^+[-n] \notin U_q(\hat{b}_-)$

(b) $\sum_{n>0} X^+[-n] z^n = -f^+(zK) K_1^{-1}$, where $f^-(z) = \sum_{n>0} F_{\alpha-n\delta} z^n$

shift of variable

- $n \geq 0 \qquad X^-[-n] = F_{-\alpha-n\delta}$

$X^-[1] = (\sigma T_1) F_1 = \sigma(-K_1^{-1} E_1) = -K_0^{-1} E_0 = -K_1 K^{-1} E_{-\alpha+\delta}$

$X^-[n] = -K_1 K^{-n} E_{-\alpha+n\delta}$ $\qquad \sum_{n>0} X^-[n] z^{-n} = -K_1 e^-(zK)$

$X^+ X^+$ relation

- We know $E_{\alpha+(n+1)\delta} E_{\alpha+m\delta} - q^2 E_{\alpha+n\delta} E_{\alpha+(m+1)\delta} + E_{\alpha+(m+1)\delta} E_{\alpha+n\delta} - q^2 E_{\alpha+m\delta} E_{\alpha+(n+1)\delta} = 0$
- Hence $X^+[n+1] X^+[m] - q^2 X^+[n] X^+[m+1] - X^+[m+1] X^+[n] - q^2 X^+[m] X^+[n+1] = 0$
- $X^+(z) = \sum X^+[n] z^{-n}$

$$X^+(z) X^+(w)(z - q^2 w) + X^+(w) X^+(z)(w - q^2 z) = 0$$

- Similarly $X^-(z) = \sum X^-[n] z^{-n}$

$$X^-(z) X^-(w)(z - q^{-2} w) + X^-(w) X^-(z)(w - q^{-2} z) = 0$$

X^+ X^- relation

- For $n \geq 0,\ m > 0$ $\quad [X^+[n], X^-[m]] = -E_{\alpha+n\delta} K_1 K^{-m} E_{-\alpha+m\delta} + K_1 K^{-m} E_{-\alpha+m\delta} E_{\alpha+n\delta}$

$$= K^{-m} K_1 \left(E_{-\alpha+m\delta} E_{\alpha+n\delta} - q^{-2} E_{\alpha+n\delta} E_{-\alpha+m\delta} \right) = K^{-m} K_1 E_{(m+n)\delta}$$

using σT_1 true for $\forall n, m,\ n+m > 0$

- $[X^+[n], X^-[-n]] = (\sigma T_1)^{-n} [E_1, F_1] = (\sigma T_1)^{-n} \left(\frac{K_1 - K_1^{-1}}{q - q^{-1}} \right) = \frac{K_1 K^n - K_1^{-1} K^{-n}}{q - q^{-1}}$

- Denote $K_1^{-1} \psi^+(z) = 1 + (q - q^{-1}) \sum_{n>0} E_{n\delta} z^{-n} = \exp\left(\sum_{n>0} (q - q^{-1}) h_n z^{-n} \right)$

$$K_1 \psi^-(z) = 1 + (q^{-1} - q) \sum_{n>0} F_{-n\delta} z^n = \exp\left(\sum_{n>0} (q^{-1} - q) h_{-n} z^n \right)$$

- Then $[X^+(z), X^-(w)] = K_1 \sum_{n>0} E_{n\delta} \sum_m K^{-m} w^{-m} z^{n-m} +$

$$+ \frac{K_1}{q - q^{-1}} \sum_m z^{-m} w^m K^m + \ldots = \frac{1}{q - q^{-1}} \left(\psi^+(z)\, \delta\!\left(\frac{Kw}{z} \right) - \psi^-(w)\, \delta\!\left(\frac{w}{Kz} \right) \right)$$

- Problem Finish the proof. Hint Use $\sigma \varphi$

$h_r \quad X^{\pm}$ relation

- We had $(z-q^2w)\,e_\delta(z)\,e^+(w) = (z-q^{-2}w)\,e^+(w)\,e_\delta(z)$

$\psi_+(z) = K_1\, e_\delta(z) \qquad X^+(w) \sim e^+(w)$

$$(z-q^2w)\,\psi^+(z)\,X^+(w) + (w-q^2z)\,X^+(w)\,\psi^+(z) = 0$$

- Similarly $(z-q^{-2}w)\,e_\delta(z)\,e^-(w) = (z-q^2w)\,e^-(w)\,e_\delta(z)$

$\psi_+(z) = K_1\, e_\delta(z) \qquad X^-(w) \sim -K_1\, e^-(wK)$

$$(z-q^{-2}w)\,\psi^+(zK)\,X^-(w) + (w-q^{-2}z)\,X^-(w)\,\psi^+(zK) = 0$$

- <u>Problem</u> (urgent) $[h_r, X^+(w)] = \frac{[2r]}{r} w^r\, X^+(w)$

$$[h_r, X^-(w)] = -K^r \frac{[2r]}{r} w^r\, X^-(w)$$

- Apply $\tau\Phi$ Compute $\tau\Phi(X^+[n]) = \tau\Phi((\tau T_1)^{-n}E_1) =$
$= (\tau T_1)^n F_0 = (\tau T_1)^n(-X^+[-1]K^{-1}K_1) = -X^+[-n-1]K^{-n-1}K_1$

Hence $\tau\Phi(X^+(z)) = \sum -X^+[-n-1]z^{-n}K^{-n-1}K_1 = -zX^+(z^{-1}K^{-1})K_1$

Similarly $\tau\Phi(X^-(z)) = -K_1^{-1}z^{-1}X^-(z^{-1}K^{-1})$

Problem Show that $\tau\Phi(h_r) \doteq h_{-r}$

$\tau\Phi(\psi^+(z)) = K\psi^-(z^{-1})$ $\tau\Phi(\psi^-(z)) = K^{-1}\psi^+(z^{-1})$

- Applying $\tau\Phi$ to $[h_r, X^\pm]$ we get

$[h_{-r}, X^+(w)] = \frac{[2r]}{r}K^{-r}w^{-r}X^+(w)$
$[h_{-r}, X^-(w)] = -\frac{[2r]}{r}w^{-r}X^-(w)$

- Last computation

$$[h_{-r}, E_{p\delta}] = [h_{-r}, K_1^{-1}[X^+[p], X^-[0]]] = \cdots$$

$$\cdots = -\frac{[2r]}{r}(K^r - K^{-r}) \begin{cases} E_{(p-r)\delta} & p-r>0 \\ 1/(q-q^{-1}) & p=r \\ 0 & p-r<0 \end{cases}$$

- Hence $[h_{-r}, \psi^+(z)] = -\frac{[2r]}{r}(K^r - K^{-r})\, z^{-r}\psi^+(z)$

(Recall $K_1^{-1}\psi^+(z) = 1 + (q-q^{-1})\sum_{n>0} E_{n\delta} z^{-n} = \exp\left(\sum_{n>0}(q-q^{-1}) h_n z^{-n}\right)$)

- Hence $[h_r, h_s] = \frac{[2r]}{r}\frac{K^r - K^{-r}}{q-q^{-1}}\delta_{r+s}$

New realization

- <u>Th</u> $U_q(\hat{sl}_2)$ has presentation with generators $X^+[n]$, $X^-[n]$, $n \in \mathbb{Z}$, h_r, h_{-r} $r \in \mathbb{Z}_{>0}$, $K_0^{\pm 1} K_1^{\pm 1}$ and relations
   - $$[X^+(z), X^-(w)] = \frac{1}{q-q^{-1}} \left(\psi^+(z)\, \delta\!\left(\frac{Kw}{z}\right) - \psi^-(w)\, \delta\!\left(\frac{w}{Kz}\right) \right)$$
   - $$[h_r, h_s] = \frac{[2r]}{r} \frac{K^r - K^{-r}}{q-q^{-1}} \delta_{r+s}$$
   - $$[h_r, X^+(w)] = \frac{[2r]}{r} w^r X^+(w) \qquad [h_{-r}, X^+(w)] = \frac{[2r]}{r} K^{-r} w^{-r} X^+(w)$$
     $$[h_r, X^-(w)] = -K^r \frac{[2r]}{r} w^r X^-(w) \qquad [h_{-r}, X^-(w)] = -\frac{[2r]}{r} w^{-r} X^-(w)$$
   - $$X^+(z) X^+(w)(z - q^2 w) + X^+(w) X^+(z)(w - q^2 z) = 0$$
     $$X^-(z) X^-(w)(z - q^{-2} w) + X^-(w) X^-(z)(w - q^{-2} z) = 0$$

Here $\psi^{\pm}(z) = K_1^{\pm 1} \exp\left(\pm \sum_{\pm n > 0} (q - q^{-1}) h_n z^{-n}\right)$

Evaluation representations

- **Problem** $\exists$ evaluation homomorphism

$$ev_u : U_q(\widehat{sl_2}) \to U_q(sl_2) \qquad \begin{array}{ll} E_1 \mapsto E & E_0 \mapsto uF \\ F_1 \mapsto F & F_0 \mapsto u^{-1}E \end{array}$$

- **Remark** Not Hopf algebra homomorphism
- **Remark** One can consider u as formal variable

Hence

$$ev_u : U_q(\widehat{sl_2}) \to U(sl_2)[u^{\pm 1}]$$

- **Def** For V rep. of $U_q(sl_2)$ let $V(u)$ be $ev_u : U_q(\widehat{sl_2}) \to U_q(sl_2) \to End(V)$

- **Example** $\mathbb{C}^2(u)$

$$E_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F_1 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad K_1 \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

$$E_0 \mapsto \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \quad F_0 \mapsto \begin{pmatrix} 0 & u^{-1} \\ 0 & 0 \end{pmatrix} \quad K_0 \mapsto \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$$

- **Problem** Find formulas for $X^+[n]$, $X^-[n]$, h_r, h_{-r} in $\mathbb{C}^2(u)$

- **Problem** Check formula for R in $\mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)$.

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q(u_1-u_2)}{u_1-q^2u_2} & \frac{u_2(1-q^2)}{u_1-q^2u_2} & 0 \\ 0 & \frac{u_1(1-q^2)}{u_1-q^2u_2} & \frac{q(u_1-u_2)}{u_1-q^2u_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(Basis $v_+\otimes v_+$, $v_+\otimes v_-$, $v_-\otimes v_+$, $v_-\otimes v_-$)

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