Affine Quantum Groups

Lecture 7

RLL realization for $U_q(\widehat{sl}_2)$

Where we are

- Drinfeld-Jimbo $U_q(\hat{sl}_2)$ G $E_0, E_1, K_0^{\pm 1}, K_1^{\pm 1}, F_0, F_1$

- Generators $E_0, E_1, K_0^{\pm 1}, K_1^{\pm 1}, F_0, F_1$

- Coproduct

$$\Delta E_0 = E_0 \otimes K_0 + 1 \otimes E_0, \qquad \Delta E_1 = E_1 \otimes K_1 + 1 \otimes E_1$$

$$\Delta E_\beta = E_\beta \otimes K_\beta + 1 \otimes E_\beta + \text{"low terms"}$$

(not explicit formula)

$\Downarrow$

Triangularity

$\Downarrow$

Universal R matrix

- New Drinfeld $\qquad X^+[n],\ h_r,\ K_0, K_1,\ X^-[n],\ n \in \mathbb{Z}$

- New coproduct

$$\Delta X^+[n] = X^+[n] \otimes K_{\alpha+n\delta} + 1 \otimes X^+[n] + \sum X^+[n-m] \otimes E_{m\delta} K_{\alpha+(n-m)\delta}$$

$E_{\alpha+n\delta}$

- Today $\quad$ RLL
  - algebra $\quad$ new Drinfeld
  - coalgebra $\quad$ Drinfeld-Jimbo

RTT realization

- $U(R)$ ~ Hopf algebra with generators

$$e^+_{ij}[n], \quad , \quad e^-_{ji}[-n], \qquad 1 \le i \le j \le 2$$

$$e^+_{ji}[n+1] \quad e^-_{ij}[-n-1] \qquad 1 \le i < j \le 2$$

$$e^{\pm}_{ij} = \sum e^{\pm}_{ij}[n]\, z^{-n}$$

$$L^+(z) = \begin{pmatrix} e^+_{11}(z) & e^+_{12}(z) \\ e^+_{21}(z) & e^+_{22}(z) \end{pmatrix} \qquad L^-(z) = \begin{pmatrix} e^-_{11}(z) & e^-_{12}(z) \\ e^-_{21}(z) & e^-_{22}(z) \end{pmatrix}$$

- with relations $e^+_{ii}[0]e^-_{ii}[0] = 1$

$$R(z/w)\, L^{\pm}_1(z)\, L^{\pm}_2(w) = L^{\pm}_2(w)\, L^{\pm}_1(z)\, R(z/w)$$ — RLL relation

$$R(K^{-1}\tfrac{z}{w})\, L^-_1(z)\, L^+_2(w) = L^+_2(w)\, L^-_1(z)\, R(K\tfrac{z}{w})$$

(notations L^+, L^- permuted vs [DF])

- Coproduct
$$\Delta L^-(z) = (1 \otimes L^-(K_{(1)}^{-1} z))(L^-(z) \otimes 1)$$
$$\Delta L^+(z) = (1 \otimes L^+(z))(L^+(K_{(2)}^{-1} z) \otimes 1)$$

More explicitly
$$\Delta \ell_{ij}^-(a) = \ell_{kj}^-(a) \otimes \ell_{ik}^-(a K_{(1)}^{-1})$$
$$\Delta \ell_{ij}^+(a) = \ell_{kj}^+(a K_{(2)}^{-1}) \otimes \ell_{ik}(a)$$

$$R(z/w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q(z-w)}{z-q^2 w} & \frac{w(1-q^2)}{z-q^2 w} & 0 \\ 0 & \frac{z(1-q^2)}{z-q^2 w} & \frac{q(z-w)}{z-q^2 w} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R^{-1}(z/w) = R^t(w/z)$$

$U_q(\widehat{\mathfrak{gl}}_2)$

- Algebra generated by

$$K_1^{\pm}(z)=\sum_{\pm r\geq 0} K_1[r]z^{-r},\quad K_2^{\pm}(z)=\sum_{\pm r\geq 0} K_2[r]z^{-r},\quad X^{\pm}(z)=\sum_{n\in\mathbb{Z}} X^{\pm}[n]z^{-n}$$

- Relations

  - $K_i^+[0]K_i^-[0]=K_i^-[0]K_i^+[0]=1 \qquad i,j=1,2$
  - $K_i^{\pm}(z)K_j^{\pm}(w)=K_j^{\pm}(w)K_i^{\pm}(z)$
  - $K_i^+(z)K_i^-(w)=K_i^-(w)K_i^+(z)$
  - $\dfrac{zK^{\pm 1}-w}{zK^{\pm 1}q^{-1}-wq}K_1^{\mp}(z)K_2^{\pm}(w)=K_2^{\pm}(w)K_1^{\mp}(z)\dfrac{zK^{\mp 1}-w}{zK^{\mp 1}q^{-1}-wq}$
  - $X^+(z)X^+(w)(z-q^2w)+X^+(w)X^+(z)(w-q^2z)=0$
  - $X^-(z)X^-(w)(z-q^{-2}w)+X^-(w)X^-(z)(w-q^{-2}z)=0$

- $[X^+(z), X^-(w)] = \frac{1}{q-q^{-1}}\left(\mathcal{K}_2^+(z)\mathcal{K}_1^+(z)^{-1}\delta\left(\frac{Kw}{z}\right) - \mathcal{K}_2^-(w)\mathcal{K}_1^-(w)^{-1}\delta\left(\frac{w}{Kz}\right)\right)$

- $\mathcal{K}_1^-(z)X^+(w) = \dfrac{zq^{-1}-wKq}{z-wK}X^+(w)\mathcal{K}_1^-(z)$

$$\mathcal{K}_2^-(z)X^+(w) = \frac{zq-wKq^{-1}}{z-wK}X^+(w)\mathcal{K}_2^-(z)$$

$$\mathcal{K}_1^+(z)X^+(w) = \frac{zq^{-1}-wq}{z-w}X^+(w)\mathcal{K}_1^+(z)$$

$$\mathcal{K}_2^+(z)X^+(w) = \frac{zq-wq^{-1}}{z-w}X^+(w)\mathcal{K}_2^+(z)$$

$$\mathcal{K}_1^-(z)X^-(w) = \frac{z-w}{zq^{-1}-wq}X^-(w)\mathcal{K}_1^-(z)$$

$$\mathcal{K}_2^-(z)X^-(w) = \frac{z-w}{zq-wq^{-1}}X^-(w)\mathcal{K}_2^-(z)$$

$$\mathcal{K}_1^+(z)X^-(w) = \frac{z-wK}{zq^{-1}-wKq}X^-(w)\mathcal{K}_1^+(z)$$

$$\mathcal{K}_2^+(z)X^-(w) = \frac{z-wK}{zq-wKq^{-1}}X^+(w)\mathcal{K}_2^+(z)$$

- Remark Let $\psi^{\pm}(z)=\mathcal{K}_2^{\pm}(z)\mathcal{K}_1^{\pm}(z)$. Subalgebra generated by $X^{\pm}(z)$, $\mathcal{K}^{\pm}(z)$ is isomorphic to $U_q(\widehat{sl}_2)$ in new Drinfeld realization

Currents $\mathcal{K}_1^{+}(qz)\mathcal{K}_2^{+}(q^{-1}z)$ and $\mathcal{K}_1^{-}(qz)\mathcal{K}_2^{-}(q^{-1}z)$ commute with $U_q(\widehat{sl}_2)$

- $\boxed{\text{Theorem } \mathcal{U}(R) \text{ and } U_q(\mathfrak{gl}_2) \text{ are isomorphic}}$

Gauss decomposition

$$L^+(z) = \begin{pmatrix} \ell^+_{11}(z) & \ell^+_{12}(z) \\ \ell^+_{21}(z) & \ell^+_{22}(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (q^{-1}-q)\,\mathcal{E}^+(z) & 1 \end{pmatrix} \begin{pmatrix} \mathcal{K}^+_1(z) & 0 \\ 0 & \mathcal{K}^+_2(z) \end{pmatrix} \begin{pmatrix} 1 & (q^{-1}-q)\,\mathcal{F}^+(z) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{K}^+_1(z) & (q^{-1}-q)\,\mathcal{K}^+_1(z)\,\mathcal{F}^+(z) \\ (q^{-1}-q)\,\mathcal{E}^+(z)\mathcal{K}^+_1(z) & \mathcal{K}^+_2(z) + (q^{-1}-q)^2\,\mathcal{E}^+(z)\mathcal{K}^+_1(z)\,\mathcal{F}^+(z) \end{pmatrix}$$

$$L^-(z) = \begin{pmatrix} \ell^-_{11}(z) & \ell^-_{12}(z) \\ \ell^-_{21}(z) & \ell^-_{22}(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (q-q^{-1})\,\mathcal{E}^-(z) & 1 \end{pmatrix} \begin{pmatrix} \mathcal{K}^-_1(z) & 0 \\ 0 & \mathcal{K}^-_2(z) \end{pmatrix} \begin{pmatrix} 1 & (q-q^{-1})\,\mathcal{F}^-(z) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{K}^-_1(z) & (q-q^{-1})\,\mathcal{K}^-_1(z)\,\mathcal{F}^-(z) \\ (q-q^{-1})\,\mathcal{E}^-(z)\mathcal{K}^-_1(z) & \mathcal{K}^-_2(z) + (q^{-1}-q)^2\,\mathcal{E}^-(z)\,\mathcal{K}^-_1(z)\,\mathcal{F}^-(z) \end{pmatrix}$$

Relations

- $R(z/w)\, L_1^{\pm}(z)\, L_2^{\pm}(w) = L_2^{\pm}(w)\, L_1^{\pm}(z)\, R(z/w)$

- Denote basis in $\mathbb{C}^2$ by ξ_1, ξ_2.
  Ordering in basis in $\mathbb{C}^2 \otimes \mathbb{C}^2$: $\xi_1 \otimes \xi_1,\ \xi_1 \otimes \xi_2,\ \xi_2 \otimes \xi_1,\ \xi_2 \otimes \xi_2$.

- Compute matrix element $\xi_1 \otimes \xi_1 \longmapsto \xi_1 \otimes \xi_1$

$$\xi_1 \otimes \xi_1 \xmapsto{R} \xi_1 \otimes \xi_1 \xmapsto{L_1} K_1^+(z)\, \xi_1 \otimes \xi_1 + \#\, \xi_2 \otimes \xi_1 \xmapsto{L_2} K_1^+(w) K_1^+(z)\, \xi_1 \otimes \xi_1 + \ldots$$

$$\xi_1 \otimes \xi_1 \xmapsto{L_2} K_1^+(w)\, \xi_1 \otimes \xi_1 + \#\, \xi_1 \otimes \xi_2 \xmapsto{L_1} K_1^+(z) K_1^+(w)\, \xi_1 \otimes \xi_1 + \ldots \xmapsto{R} K_1^+(z) K_1^+(w)\, \xi_1 \otimes \xi_1 + \ldots$$

Hence $K_1^{\pm}(z)\, K_1^{\pm}(w) = K_1^{\pm}(w)\, K_1^{\pm}(z)$

Similarly $\mathcal{K}_1^+(z)\mathcal{K}_1^-(w) = \mathcal{K}_1^-(w)\mathcal{K}_1^+(z)$

- $$L^\pm(z)^{-1} = \begin{pmatrix} 1 & \pm(q-q^{-1})\mathcal{F}^\pm(z) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathcal{K}_1^\pm(z)^{-1} & 0 \\ 0 & \mathcal{K}_2^\pm(z)^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \pm(q-q^{-1})\mathcal{E}^\pm(z) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{K}^\pm(z)^{-1} + (q-q^{-1})^2\mathcal{F}^\pm(z)\mathcal{K}_2^\pm(z)\mathcal{E}^\pm(z) & \pm(q-q^{-1})\mathcal{F}^\pm(z)\mathcal{K}_2^\pm(z) \\ \pm(q-q^{-1})\mathcal{K}_2^\pm(z)\mathcal{E}^\pm(z) & \mathcal{K}_2^\pm(z)^{-1} \end{pmatrix}$$

- $$L_1^\pm(z)^{-1}L_2^\pm(w)^{-1}R(z/w) = R(z/w)L_2^\pm(w)^{-1}L_1^\pm(z)^{-1}$$
$$L_1^-(z)^{-1}L_2^+(w)^{-1}R\left(\kappa^{-1}\frac{z}{w}\right) = R\left(\kappa\frac{z}{w}\right)L_2^-(w)^{-1}L_1^+(z)^{-1}$$

Follows from $RLL = LLR$

- Compute $\xi_2\otimes\xi_2 \mapsto \xi_2\otimes\xi_2$

$$\mathcal{K}_2^\pm(z)\mathcal{K}_2^\pm(w) = \mathcal{K}_2^\pm(w)\mathcal{K}_2^\pm(z)$$

$$\mathcal{K}_2^+(z)\mathcal{K}_2^-(w) = \mathcal{K}_2^-(w)\mathcal{K}_2^+(z)$$

- $L_2^\pm(w)^{-1} R(z/w) L_1^\pm(z) = L_1^\pm(z) R(z/w) L_2^\pm(w)^{-1}$

  $L_2^+(w)^{-1} R(k^{-1} z/w) L_1^-(z) = L_1^-(z) R(k\, z/w) L_2^+(w)^{-1}$

  $L_2^+(w) R(k\, z/w)^{-1} L_1^-(z)^{-1} = L_1^-(z)^{-1} R(k^{-1} z/w)^{-1} L_2^+(w)$

  Follows from $RLL = LLR$

- Compute $\xi_1 \otimes \xi_2 \mapsto \xi_1 \otimes \xi_2$ and $\xi_2 \otimes \xi_1 \mapsto \xi_2 \otimes \xi_1$

$$K_1^\pm(z) K_2^\pm(w)^{-1} = K_2^\pm(w)^{-1} K_1^\pm(z)$$

$$\frac{z - wk}{zq^{-1} - wkq} K_2^+(w)^{-1} K_1^-(z) = K_1^-(z) K_2^+(w)^{-1} \frac{zk - w}{zkq^{-1} - wq}$$

$$\frac{zk - w}{zkq - wq^{-1}} K_1^+(w) K_2^-(z)^{-1} = K_2^-(z)^{-1} K_1^+(w) \frac{z - wk}{zq - wkq^{-1}}$$

All relations between $K_1^\pm(z), K_2^\pm(z)$ found.

$K - \mathcal{E}$ relations

- $R(z/w)\, L_1^{\pm}(z)\, L_2^{\pm}(w) = L_2^{\pm}(w)\, L_1^{\pm}(z)\, R(z/w)$

Compute $\xi_1 \otimes \xi_1 \longmapsto \xi_1 \otimes \xi_2$

$$
\begin{tikzcd}[column sep=small]
 & \xi_1 \otimes \xi_1 \arrow[dl, "L_2"'] \arrow[dr, "R"] & \\
e^+_{11}\xi_1\otimes\xi_1 + e^+_{21}\xi_1\otimes\xi_2 \arrow[d, "L_1"'] & & R^{11}_{11}\xi_1\otimes\xi_1 \arrow[d, "L_1^+"] \\
e^+_{21}e^+_{11}\xi_2\otimes\xi_1 + e^+_{11}e^+_{21}\xi_1\otimes\xi_2 \arrow[d, "R"'] & & R^{11}_{11}e^+_{11}\xi_1\otimes\xi_1 + \# \arrow[d, "L_2^+"] \\
\left(R^{12}_{21}e^+_{21}e^+_{11} + R^{12}_{12}e^+_{11}e^+_{21}\right)\xi_1\otimes\xi_2 + \# & & R^{11}_{11}e^+_{21}e^+_{11}\xi_1\otimes\xi_2 + \#
\end{tikzcd}
$$

- Hence $\mathcal{E}^+(w)\,\mathcal{K}_1^+(w)\,\mathcal{K}_1^+(z) = \dfrac{w(q^{-1}-q)}{q^{-1}z-qw}\,\mathcal{E}^+(z)\,\mathcal{K}_1^+(z)\,\mathcal{K}_1^+(w)$

$$+ \frac{z-w}{q^{-1}z-qw}\,\mathcal{K}_1^+(z)\,\mathcal{E}^+(w)\,\mathcal{K}_1^+(w)$$

Using commutativity $\mathcal{K}_1^+(z)$, $\mathcal{K}_1^+(w)$

$$\mathcal{K}_1^+(z)\,\mathcal{E}^+(w) = \frac{q^{-1}z - qw}{z-w}\,\mathcal{E}^+(w)\,\mathcal{K}_1^+(z) + \underbrace{\frac{w\,(q-q^{-1})}{z-w}\,\mathcal{E}^+(z)\,\mathcal{K}_1^+(z)}_{\text{local term}}$$

- Similarly

$$\mathcal{K}_1^-(z)\,\mathcal{E}^-(w) = \frac{q^{-1}z - qw}{z-w}\,\mathcal{E}^-(w)\,\mathcal{K}_1^-(z) + \underbrace{\frac{w\,(q-q^{-1})}{z-w}\,\mathcal{E}^-(z)\,\mathcal{K}_1^-(z)}_{\text{local term}}$$

- <u>Problem</u> ⓐ Find $\mathcal{K}_1^+$, $\mathcal{F}^+$ relation

  ⓑ Find $\mathcal{F}^+$, $\mathcal{F}^+$ relation

- $R(K^{-1}\frac{z}{w}) L_1^-(z) L_2^+(w) = L_2^+(w) L_1^-(z) R(K\frac{z}{w})$

Compute $\xi_1 \otimes \xi_1 \longmapsto \xi_1 \otimes \xi_2$

$$R_{21}^{12}(K^{-1}z/w)\, e_{21}^-(z) e_{11}^+(w) + R_{12}^{12}(K^{-1}z/w)\, e_{11}^-(z) e_{21}^+(w) = R_{11}^{11}(K\tfrac{z}{w}) e_{21}^+(w) e_{11}^-(z)$$

$$-\frac{wK(q^{-1}-q)}{zq^{-1}-wKq}\mathcal{E}^-(z)\mathcal{K}_1^-(z)\mathcal{K}_1^+(w) + \frac{z-wK}{zq^{-1}-wKq}\mathcal{K}_1^-(z)\mathcal{E}^+(w)\mathcal{K}_1^+(w) = \mathcal{E}^+(w)\mathcal{K}_1^+(w)\mathcal{K}_1^-(z)$$

Hence

$$\mathcal{K}_1^-(z)\mathcal{E}^+(w) = \frac{zq^{-1}-wKq}{z-wK}\mathcal{E}^+(w)\mathcal{K}_1^-(z) - \frac{wK(q-q^{-1})}{z-wK}\mathcal{E}^-(z)\mathcal{K}_1^-(z)$$

- Recall

$$\mathcal{K}_1^-(z)\mathcal{E}^-(w) = \frac{q^{-1}z-qw}{z-w}\mathcal{E}^-(w)\mathcal{K}_1^-(z) + \frac{w(q-q^{-1})}{z-w}\mathcal{E}^-(z)\mathcal{K}_1^-(z)$$

$\underline{X^+(w) = \mathcal{E}^+(w) + \mathcal{E}^-(Kw)}$ Then

$$\mathcal{K}_1^-(z) X^+(w) = \frac{zq^{-1}-wKq}{z-wK} X^+(w)\mathcal{K}_1^-(z)$$

$\mathcal{E}$-$\mathcal{E}$ relation

- $R(z/w)\, L_1^{\pm}(z)\, L_2^{\pm}(w) = L_2^{\pm}(w)\, L_1^{\pm}(z)\, R(z/w)$

$\xi_1 \otimes \xi_1 \longmapsto \xi_2 \otimes \xi_2$ $\qquad e_{21}^{\pm}(z)\, e_{21}^{\pm}(w) = e_{21}^{\pm}(z)\, e_{21}^{\pm}(w)$

$$\mathcal{E}^{\pm}(z)\mathcal{K}_1^{\pm}(z)\mathcal{E}^{\pm}(w)\mathcal{K}_1^{\pm}(w) = \mathcal{E}^{\pm}(w)\mathcal{K}_1^{\pm}(w)\mathcal{E}^{\pm}(z)\mathcal{K}_1^{\pm}(z)$$

$$\left(\frac{q^{-1}z - qw}{z-w}\mathcal{E}^{\pm}(z)\mathcal{E}^{\pm}(w) + \frac{w(q-q^{-1})}{z-w}\mathcal{E}^{\pm}(z)^2\right)\mathcal{K}_1^{\pm}(z)\mathcal{K}_1^{\pm}(w)$$

$$\left(\frac{q^{-1}w - qz}{w-z}\mathcal{E}^{\pm}(w)\mathcal{E}^{\pm}(z) + \frac{z(q-q^{-1})}{w-z}\mathcal{E}^{\pm}(w)^2\right)\mathcal{K}_1^{\pm}(w)\mathcal{K}_1^{\pm}(z)$$

Hence

$$\underline{(z-q^2w)\mathcal{E}^{\pm}(z)\mathcal{E}^{\pm}(w) + (w-q^2z)\mathcal{E}^{\pm}(w)\mathcal{E}^{\pm}(z) = (1-q^2)\left(w\,\mathcal{E}^{\pm}(z)^2 + z\,\mathcal{E}^{\pm}(w)^2\right)}$$

agrees with half current relations

Other relations are similar

From universal R matrix

- Above we constructed map $U(\widehat{sl}_2) \to U(R)$

  Inverse map can be constructed using universal R matrix. We work for $U_q(\widehat{sl}_2)$

- Recall $R = q^{\hbar(d\otimes k + k\otimes d)} \check{R}^{-} \check{R}^{0} R^{+}$ where

$$R^{-} = \prod_{r>0} \left(\sum_{n\geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} (K_2 E_{-2+r\delta})^n \otimes (F_{2-r\delta} K_{-2})^n \right)$$

$$\check{R}^{0} = q^{\frac{1}{2} H_1 \otimes H_1} \left(\prod_{r>0} \sum_{n\geq 0} \frac{1}{n!} \left(\frac{r(q-q^{-1})}{[2r]} \right)^n h_r^n \otimes h_{-r}^n \right), \quad R^{+} = \prod_{r\geq 0} \left(\sum_{n\geq 0} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E_{2+r\delta}^n \otimes F_{-2-r\delta}^n \right)$$

- Let $q^{d\otimes k} L^{-} = (\rho_v \otimes id) R$ $\qquad L^{+} q^{-k\otimes d} = (id \otimes \rho) R^{-1}$

- Yang-Baxter $R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$. apply $\rho_{u_1} \otimes \rho_{u_2} \otimes id$

$$LHS = f(a_1/a_2)\, R(a_1/a_2)\, q^{d_1\otimes k} L_1^-(a_1)\, q^{d_2\otimes k} L_2^-(a_2)$$
$$= q^{(d_1+d_2)\otimes k} f(a_1/a_2) R(a_1/a_2) L_1^-(a_1) L_2^-(a_2)$$

$$RHS = q^{d_2\otimes k} L_2^-(a_2)\, q^{d_1\otimes k} L_1^-(a_1)\, f(a_1/a_2)\, R(a_1/a_2)$$
$$= q^{(d_1+d_2)\otimes k} f(a_1/a_2) L_2^-(a_2) L_1^-(a_1) R(a_1/a_2)$$

Hence $R(a_1/a_2) L_1^-(a_1) L_2^-(a_2) = L_2^-(a_2) L_1^-(a_1) R(a_1/a_2)$

Here we used $q^d E_{\pm 2+n\delta}\, q^{-d} = q^n E_0$ hence $q^{d\otimes k} a\, q^{-d\otimes k} = Ka$

- Problem Show $R(a_1/a_2) L_1^+(a_1) L_2^+(a_2) = L_2^+(a_2) L_1^+(a_1) R(a_1/a_2)$

- $R_{13} R_{12} R_{23}^{-1} = R_{23}^{-1} R_{12} R_{13}$ $\quad \rho_{u_1} \otimes id \otimes \rho_{u_2}$

$$LHS = f(a_1/a_2)\, R(a_1/a_2)\, q^{d_1\otimes k} L_1^-(a_1)\, L_2^+(a_2)\, q^{-d_2\otimes k}$$
$$= q^{d_1\otimes k} f\left(\frac{a_1}{K a_2}\right) R\left(\frac{a_1}{K a_2}\right) L_1^-(a_1) L_2^+(a_2)\, q^{-d_2\otimes k}$$

$$RHS = L_2^+(a_2)\, q^{-d_2\otimes k}\, q^{d_1\otimes k} L_1^-(a_1)\, f(a_1/a_2)\, R(a_1/a_2)$$

$$= q^{d_1\otimes k} L_2^+(a_2)\, L_1^-(a_1)\, f\!\left(k\frac{a_1}{a_2}\right) R\!\left(k\frac{a_1}{a_2}\right) q^{-d_2\otimes k}$$

Hence $f\!\left(k^{-1}\frac{a_1}{a_2}\right) R\!\left(k^{-1}\frac{a_1}{a_2}\right) L_1^-(a_1)\, L_2^+(a_2) = L_2^+(a_2)\, L_1^-(a_1)\, f\!\left(k\frac{a_1}{a_2}\right) R\!\left(k\frac{a_1}{a_2}\right)$

we get additional function f since $\widehat{sl}_2$ not $\widehat{gl}_2$

- <u>Problem</u> Show that

$$L^+(a) = \begin{pmatrix} 1 & 0 \\ (q^{-1}-q)\sum_{r\geq 0} E_{\alpha+r\delta}\, a^{-r} & 1 \end{pmatrix} \exp\left(\sum_{r>0} \frac{[r](q^{-1}-q)}{[2r]} h_r \begin{pmatrix} (aq^{-1})^r & 0 \\ 0 & -(aq)^{-r} \end{pmatrix}\right) \begin{pmatrix} e^{-\frac{H_1}{2}\hbar} & \\ & e^{\frac{H_1}{2}\hbar} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \sum_{r>0} (q^{-1}-q) \sum K_1 E_{-\alpha+r\delta}\, a^{-r} \\ 0 & 1 \end{pmatrix}$$

Coproduct

- From universal R matrix

$$(\Delta \otimes id) R = R_{13} R_{23} \quad \text{hence} \quad (\Delta \otimes id) R^{-1} = R_{23}^{-1} R_{13}^{-1}$$

- Using $L^+(a) q^{-k \otimes d} = (id \otimes \rho_a) R^{-1}$

$$\Delta L^+(a) q^{-(k_1+k_2)\otimes d} = L_2^+(a) q^{-k_2 \otimes d} L_1^+(a) q^{-k_1 \otimes d} = L_2^+(a) L_1^+(K_{(2)} a) q^{-(k_1+k_2) d}$$

Hence $\Delta L^+(a) = L_2^+(a) L_1^+(K_{(2)} a)$

In modes $\Delta \ell^+_{ij}(a) = \ell^+_{kj}(a K_2^{-1}) \otimes \ell_{ik}(a)$

Similarly $\Delta L^- = L_2^-(a K_{(1)}^{-1}) L_1^-(a)$

$$\Delta \ell^-_{ij}(a) = \ell^-_{kj}(a) \otimes \ell^-_{ik}(a K_{(1)}^{-1})$$

- **Problem** Show that Δ defined above agrees with RLL relations.

- **Problem** Using formula for $L^+(a)$ show that $\Delta E_1 = E_1 \otimes K_1 + 1 \otimes E_1$, $\Delta E_0 = E_0 \otimes K_0 + 1 \otimes E_0$.

$$\text{RLL}\begin{cases} \text{algebra} & \text{new Drinfeld} \\ \text{coalgebra} & \text{Drinfeld-Jimbo} \end{cases}$$

References

- Ding Frenkel Isomorphism of two realizations of quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$