Affine quantum groups

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Problems for the course Skoltech, fall 2021. There are mistakes here, if you find some please write to mbersht@gmail.com
Definitions and hints are in slides and references.

1 Quantum groups

Problem 1.1 (till 04.10.2021). Let $\mathfrak{g} = \mathfrak{sl}_3$, $V = \mathbb{C}^3$ a) Compute first and second term in E, F of $\overline{\mathcal{R}}$ using pairing. b) Compute $R_{V,V}$ and R_{V,V^*}. c) Find eigenvalues of $\widetilde{R}_{V,V}$.

Problem 1.2. Let $\mathfrak{g} = \mathfrak{sl}_3$, Compute first and second term in E, F of $\overline{\mathcal{R}}$ using factorization formula.

Problem 1.3. For $\mathfrak{g} = \mathfrak{sl}_3$ and $V = \mathbb{C}^3$ compute C_V, C_{V^*}.

Problem 1.4. a) For $\mathfrak{g} = \mathfrak{sl}_3$ relate l_{ij} generators in RTT realization and Cartan-Weyl elements. b)* The same for $\mathfrak{g} = \mathfrak{sl}_{n+1}$

Problem 1.5. Show that $\langle RL_1^+ L_2^+ - L_2^+ L_1^+ R, - \rangle = 0$.

2 Affine algebras, affine Weyl group

Problem 2.1 (till 11.10.2021). Show that generators $E_0, \ldots, E_r, H_0, \ldots, H_r, F_0, \ldots, F_r$ satisfies relations of Kac-Moody algebra.

Problem 2.2. a) If $s_i \lambda = \lambda$ (equivaliently $(\alpha_i^\vee, \lambda) = 0$) then $T_i Y^\lambda = Y^\lambda T_i$.
b)* If $(\alpha_i^\vee, \lambda) = 1$, then $T_i Y^\lambda T_i = Y^{s_i(\lambda)}$.

Problem 2.3. For $\Phi = A_n$ the elements $Y_i = T_{i-1}^{-1} \cdot \cdots \cdot T_1^{-1} \tau T_{n-1} \cdot \ldots T_i$. Here $\tau \in \Omega$ is such that $\tau T_i \tau^{-1} = T_{i+1}$, $\tau T_n \tau^{-1} = T_0$.
a) For $n = 1, 2$ find formulas for Y^{ϖ_i} and show above.
b)* Show above for any n.

Problem 2.4. * Show that $P \in W^{ae}$ is the subgroup of elements with finitely many conjugates.

3 Representations of affine algebra

4 Poincaré-Birkhoff-Witt basis for $U_q(\widehat{sl}_2)$

Problem 4.1 (till 01.11.2021). *Show that*

$$E_{\alpha+(n+1)\delta}E_{\alpha+m\delta} - q^2 E_{\alpha+n\delta}E_{\alpha+(m+1)\delta} + E_{\alpha+m\delta}E_{\alpha+(m+1)\delta} - q^2 E_{\alpha+m\delta}E_{\alpha+(n+1)\delta} = 0$$

Problem 4.2. * *Show that*

$$e^+(z)e^+(w)(z - q^2 w) + e^+(w)e^+(z)(w - q^2 z) = (1 - q^2)(ze^+(w)^2 - we^+(z)^2)$$

5 New Drinfeld realization for $U_q(\widehat{sl}_2)$

Problem 5.1. *Finish the proof of*

$$[X^+(z), X^-(w)] = \frac{1}{q - q^{-1}}\left(\psi^+(z)\delta(\frac{Kw}{z}) - \psi^-(z)\delta(\frac{w}{Kz})\right)$$

Problem 5.2 (till 01.11.2021). *Show that for $r > 0$*

$$[h_r, X^+(w)] = \frac{[2r]_q}{r}w^r X^\pm(w), \quad [h_r, X^+(w)] = -K^r\frac{[2r]_q}{r}w^r X^\pm(w).$$

Problem 5.3. *Show that*

$$\tau\Phi(h_r) = h_{-r}, \quad \tau\Phi(\Psi^+(z)) = K\Psi^-(z^{-1}), \quad \tau\Phi(\Psi^-(z)) = K^{-1}\Psi^+(z^{-1}).$$

Problem 5.4. *Show the existence of evaluation homorphism*

$$\mathrm{ev}_u\colon U_q(\widehat{\mathfrak{sl}}_2) \to U_q(\mathfrak{sl}_2), \quad E_1 \mapsto E,\ F_1 \mapsto F,\ E_0 \mapsto uF,\ F_0 \mapsto u^{-1}E.$$

Problem 5.5. *Find formula for the action of loop generators $X^+[n], X^-[n], h_r, h_{-r}$ for evaluation representation $\mathbb{C}^2(u)$*

Problem 5.6 (till 01.11.2021). *Check formula for R matrix acting on the tensor product $\mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)$*

6 Factorization of $\mathcal{R}$-matrix

Problem 6.1. *Prove any of the formulas for $\Delta(E_{-\alpha+n\delta})$, $\Delta(F_{\alpha-n\delta})$, $\Delta(F_{-\alpha-n\delta})$.*

Problem 6.2. *Show that*

$$\Delta(E_{\alpha+n\delta}) = 1 \otimes E_{\alpha+n\delta} + E_{\alpha+n\delta} \otimes K_{\alpha+n\delta} + (q - q^{-1})\sum_{p=1}^{n} E_{\alpha+(n-p)\delta} \otimes K_{\alpha+(n-p)\delta}E_{p\delta}$$
$$+ \textit{very low terms}$$

where very low terms *are of the form $a \otimes b$, where a contains at least two terms of the form $E_{\alpha+p\delta}$.*

Problem 6.3. *a) Find Gauss decomposition for $R = R^- R^0 R^+$ matrix acting on the tensor product $\mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)$.*
  b) Show that $\mathcal{R}^-_{\mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)} = R^-$, $\mathcal{R}^+_{\mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)} = R^+$.
  c)* Show that $\mathcal{R}^0_{\mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)} = f(u_1/u_2)R^0$, for some function f.

Problem 6.4. *Check that Drinfeld coproduct preserves some (a couple) of relations in new realization.*

7 RLL **relaization**

Problem 7.1. *a) Deduce from RLL relations $\mathcal{K}^+_1$, $\mathcal{F}^+$ relations.*
  b) Deduce from RLL relations $\mathcal{F}^+$, $\mathcal{F}^+$ relations.

Problem 7.2. *Deduce from Yang-Baxter equation that*

$$R(a_1/a_3)L^+_1(a_1)L^+_2(a_2) = L^+_2(a_2)L^+_1(a_1)R(a_1/a_3)$$

Problem 7.3. *Show that for $U_q(\widehat{\mathfrak{sl}}_2)$ we have*

$$L^+(a) = \begin{pmatrix} 1 & 0 \\ (q^{-1}-q)\sum_{r\geq 0} E_{\alpha+r\delta}a^{-r} & 0 \end{pmatrix}$$
$$\exp\left(\sum_{r>0} \frac{[r](q^{-1}-q)}{[2r]}h_r \begin{pmatrix} (aq^{-1})^r & 0 \\ 0 & -(aq)^r \end{pmatrix}\right)$$
$$\begin{pmatrix} 1 & (q^{-1}-q)\sum_{r>0} K_1 E_{-\alpha+r\delta}a^{-r} \\ 0 & 1 \end{pmatrix}.$$

Problem 7.4. *Show that Δ defined by*

$$\Delta(L^+(z)) = L^+_2(a)L^+_1(K_{(2)}a), \quad \Delta(L^-(z)) = L^-_2(K^{-1}_{(1)}a)L^-_1(a),$$

agrees with RLL relations.

Problem 7.5. *Using the formula for L^+ show that $\Delta E_1 = E_1 \otimes K_1 + 1 \otimes E_1$, $\Delta E_0 = E_0 \otimes K_0 + 1 \otimes E_0$.*

8 Finite dimensional representations of $U_q(\widehat{\mathfrak{sl}}_2)$

Problem 8.1. *Show that $\mathbb{C}^2(u_1) \otimes \mathbb{C}^2(u_2)$ is irreducible unless $u_1/u_2 = q^{\pm 2}$. In last case show that*

$$0 \to \mathbb{C} \to \mathbb{C}^2(u) \otimes \mathbb{C}^2(uq^2) \to \mathbb{C}^3(uq) \to 0$$
$$0 \to \mathbb{C}^3(uq) \to \mathbb{C}^2(uq^2) \otimes \mathbb{C}^2(u) \to \mathbb{C} \to 0$$

Problem 8.2. *$V(u)^* \simeq V(q^2 u)$ for any finite dimensional evaluation representation $V(u)$.*

Problem 8.3. *Any irreducible (type I) representation of $U_q(\widehat{\mathfrak{sl}}_2)$ of dimension 2 is isomorphic to $\mathbb{C}^2(u)$ for some u.*

Problem 8.4. *Let V_λ be highest weight representation of $U_q(\mathfrak{sl}_2)$ with highest weight vector ξ_λ such that*

$$K\xi_\lambda = q^\lambda \xi_\lambda, \quad E\xi_\lambda = 0.$$

Then, for evaluation representation $V_\lambda(u)$ of $U_q(\widehat{\mathfrak{sl}}_2)$ show that

$$\Psi^+(z)\xi_\lambda = \Psi^=(z)\xi_\lambda = \phi_\lambda(u,z)\xi_\lambda, \quad \phi_\lambda(u,z) = q^\lambda \frac{z + uq^{-\lambda-2}}{z + uq^{\lambda-2}}.$$

Problem 8.5. *Any finite multiset in $\mathbb{C}^*$ can be uniquely presented as a union of string pairwise in general position*

9 q-characters for $U_q(\widehat{\mathfrak{sl}}_2)$

Problem 9.1. *Assume that all l-weight spaces in V are one dimensional, $\zeta \in V_{(\phi)}$, $\zeta' \in V_{(\phi')}$, are such that $\langle\zeta'|X^-[n]\zeta\rangle = a^n\langle\zeta'|X^-[0]\zeta\rangle$.*
a) Show that $\langle\zeta|X^+[n]\zeta'\rangle = a^n\langle\zeta|X^+[0]\zeta'\rangle$.
b)* Show that $\langle\zeta'|X^-[0]\zeta\rangle\langle\zeta|X^+[0]\zeta'\rangle = \frac{1}{a(q-q^{-1})}\operatorname{Res}_{z=a}\phi(z)$.

Problem 9.2. ** Find irreducible finite dimensional representation of $U_q(\widehat{\mathfrak{sl}}_2)$ whose q-character contains more the one dominant term.*

Problem 9.3. *For strings S_1, S_2 in special position let $S_3 = S_1 \cup C_2$, $S_4 = S_1 \cap S_2$, $\bar{S}_4$ is S_4 with two nearest neighbors and $S_3 \setminus \bar{S}_4 = S_5 \sqcup S_6$. Then we have equality in K_0*

$$V(S_1) \otimes V(S_2) = V(S_3) \otimes V(S_4) + V(S_5) \otimes V(S_6)$$

Problem 9.4. ** Let $S = \{a, aq^{-2}, \ldots, aq^{-2l}\}$. The classes $V(S')$ in $K_0(\operatorname{Rep}_{f.d.})$, for strings $S' \subset S$ satisfy relations of cluster algebra of type A_l, the only frozen variable is $V(S)$.*

10 q-characters in general

Problem 10.1. *For $\mathfrak{g} = \mathfrak{sl}_3$ find q-characters and graphs corresponding to two different 8 dimensional irreducible representations.*

Problem 10.2. *For $\mathfrak{g} = \mathfrak{so}_8$ find q-characters and graphs corresponding to fundamental representations.*

Problem 10.3. ** For $\mathfrak{g} = \mathfrak{sl}_2$ show that $\operatorname{Ker} S = \mathbb{C}[Y(a) + Y^{-1}(aq^2)]$.*

11 Schur-Weyl duality

Problem 11.1. *Show that formulas*

$$E_i e_h = \delta_{i \equiv h+1} e_{h+1}, \quad F_i e_h = \delta_{i \equiv h} e_{h-1}, \quad K_i e_h = q^{\delta_{i \equiv h} - \delta_{i \equiv h+1}} e_h$$

defines representation of $U_q(\widehat{\mathfrak{sl}}_n)$

Problem 11.2. *Show formula* $T_i = s_i^Y \tilde{R}_{i,i+1} + \frac{q-q^{-1}}{Y_1/Y_{i+1}-1}(s_i^Y - 1)$.

Problem 11.3. *Check formula* $T(e_h \otimes e_g)$ *(or* $T(e_g \otimes e_h)$*) for* $g > h$, $s > 0$,

Problem 11.4. *Check commutativity of* E_i *and* T *on* $e_h \otimes e_g$ *for* $g = h + nk + s$, $k \geq 0$, $s > 0$, $N = 2$.

Problem 11.5. * a) *For* $N = 2$ M_{a_1,a_2} *is irreducible if and only if* $a_1/a_2 \neq q^2, q^{-2}$.
b) *If* $a_i/a_j \neq q^2$, $\forall 1 \leq i, j \leq N$, *then* $M_{\vec{a}}$ *is irreducible.*

12 Semi-infinite construction

Problem 12.1. *Show that* $e_g \wedge e_h = -q e_h \wedge e_g$ *for* $0 < g - h < n$.

Problem 12.2. *Let* $i_1, \ldots, i_N \in \mathbb{Z}$ *such that* $\sum_{k=1}^N (i_k + m - k) > 0$ *and* $i_k + m \leq N$. *Then* $e_{i_1} \wedge \cdots \wedge e_{i_N} = 0$.

Problem 12.3. * *Show that*
a) $K_i |\lambda\rangle_{N,m} = q^{|\lambda|_{i-1} - 2|\lambda|_i + |\lambda|_{i+1} + \delta_{m \equiv i} - \delta_{m+N \equiv i}} |\lambda\rangle_{N,m}$, *where* $|\lambda|_j = \#\{\square \in \lambda | c(\square) = j\}$.
b) $K_i |\lambda\rangle_{N,m} = q^{\#\{\square \in \mathrm{Add}(\lambda)|c(\square)=i\} - \#\{\square \in \mathrm{Rem}(\lambda)|c(\square)=i\} - \delta_{m+N \equiv i}} |\lambda\rangle_{N,m}$

Problem 12.4. $\forall k, \exists B_k$ *acting on* $\mathcal{F}_m$ *s.t.* $\lim_{N \to \infty} \varphi_{\infty,N} b_k |\lambda\rangle_{N,m} = B_k |\lambda\rangle_m$.

Problem 12.5. *Compute* $B_1 B_{-1} |\varnothing\rangle_m$

13 Bosonization, vertex operators

Problem 13.1. a) *Check* X^-, X^- *relation.*
b)* *Check* $[X^+, X^-]$ *relation.*
c) *For* $L_{0,1}$ *check* $X^-[0]|0\rangle = X^+[-1]^2|0\rangle = 0$.

Problem 13.2. * *Check intertwining property of* Φ^D *with* $X^+(w)$.

Problem 13.3. *Find dependence of* $\Psi_+^{*,D}$ *and* $\Psi_-^{*,D}$ *on* a_r, a_{-r}

Problem 13.4. *Find normalization* $f(u)$ *of the* R *matrix on relations of vertex operators.*