Introduction to Quantum Groups

Lecture 1

Poisson algebra and quantization

- Quantum groups − result of quantization
  non-commutative deformation

- Formal deformations $A = A_0 \otimes \mathbb{C}[[\hbar]]$
  $\hbar$ − formal parameter A_0 − comm. algebra

$$a * b = a \cdot b + \hbar \mu_1(a,b) + \hbar^2 \mu_2(a,b) + \ldots$$

$$a, b \in A_0 \qquad \mu_i : A_0 \otimes A_0 \to A_0$$

- Associativity $a*(b*c) = (a*b)*c$ ⟶ quadratic rel.s on μ_i

$$\hbar \qquad a\,\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)\,c = 0$$

$$\ldots$$

- Gauge freedom $a \mapsto a + \hbar v_1(a) + \hbar^2 v_2(a) + \ldots$

$$v_i : A_0 \to A_0$$

- Geometrical Setting

$A_0 = C^\infty(M)$ (or $A = \mathbb{C}[M]$) locally $\mathbb{C}^\infty(\mathbb{R}^n)$

$\mu_1, \mu_2, \dots, \quad \nu_1, \nu_2, \dots$ – diff operators

<u>Prop</u> If μ_1 satisfies

$$a\,\mu_1(b,c) - \mu_1(ab,c) + \mu_1(a,bc) - \mu_1(a,b)c = 0$$

then using ν we can get

$$\mu_1(b,c) = \sum K_{ij}(\partial_i b\, \partial_j c - \partial_j b\, \partial_i c) \qquad K_{ij} \in A_0$$

Example / Explanation / sketch of proof

Consider degree of μ_1

- degree 0: $\mu_1(a,b) = K a \cdot b \rightsquigarrow \nu_1(a) = Ka$

$a \rightsquigarrow a + \hbar K a \rightsquigarrow$ trivial

- degree 1 : $\mu_1(a,b) = K a \partial_i b$

$$K a b \partial_i c - K a b \partial_i c + K a b \partial_i c + K a \partial_i b c - K a \partial_i b c = $$
$$= K a b \partial_i c$$

$\Rightarrow K = 0 \Rightarrow$ no deformation

- degree 2 $\mu_2(a,b)$

$$a \partial_i \partial_j b \quad + \quad \partial_i a \partial_j b + \partial_j a \partial_i b \quad\quad \partial_i a \partial_j b - \partial_j a \partial_i b$$

no deformation trivial ! new deformation

$$\gamma(a) = \partial_i \partial_j a$$

$$a \rightsquigarrow a + \hbar \partial_i \partial_j a$$

- Degree ≥ 3 . . . $\square$

Poisson algebras

- **Def** A_0 - Poisson algebra if $\{\cdot,\cdot\}: A_0 \otimes A_0 \to A_0$

  - $\{a,b\} = -\{b,a\}$ anti-commut
  - $\{a,bc\} = \{a,b\}c + b\{a,c\}$ Leibnitz rule
  - $\{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\} = 0$ Jacobi identity

Def Formal deformation of the Poisson algebra is $A = A_0 \otimes \mathbb{C}[[\hbar]]$ s.t

$$a * b - b * a = 2\hbar \{a,b\} + O(\hbar^2)$$

- **Def** M is Poisson manyfold if $C^\infty(M)$ is Poisson algebra.

In coordinates $\Pi = \sum \Pi_{ij}\, \partial_i \wedge \partial_j \in \Lambda^2 TM$

$$\{f,g\} = \sum \Pi_{ij} (\partial_i f\, \partial_j g - \partial_j f\, \partial_i g)$$

Remark If π_{ij} - invertible $\Rightarrow \omega = \pi^{-1}$ is symplectic form.

Example ① • $A_0 = \mathbb{C}[x_1, .., x_n]$ $\quad \{x_i, x_j\} = \varepsilon_{ij} \in \mathbb{C}$

$\varepsilon_{ij} = -\varepsilon_{ji}$

constant bracket

• Ex. $\{x, p\} = 1 \quad \sim \quad [\hat{x}, \hat{p}] = \hbar$

Moyal product

$f, g \in \mathbb{C}[x, p]$ $\qquad f * g = m \cdot e^{\frac{1}{2}\hbar(\partial_x \otimes \partial_p - \partial_p \otimes \partial_x)} f \otimes g$

$m: A_0 \otimes A_0 \to A_0$ multiplication

$$x * p = m\left(x \otimes p + \frac{1}{2}\hbar(\partial_x \otimes \partial_p - \partial_p \otimes \partial_x)\, x \otimes p + \hbar^2 \ldots\right) = xp + \frac{1}{2}\hbar$$

$$p * x = px - \frac{1}{2}\hbar \qquad [x, p]_* = \hbar$$

Problem Show that $*$ is associative

- For any $\varepsilon_{ij} \sim \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \\ & & & & & & \ddots \; 0 \end{pmatrix}$ ~ product of previous cases

Example ② $A_0 = \mathbb{C}[x_1, \dots, x_n]$ $\{x_i, x_j\} = \varepsilon_{ij}^{k} x_k$ $\varepsilon_{ij}^{k} \in \mathbb{C}$

Linear bracket

$\varepsilon_{ij}^{k} = -\varepsilon_{ji}^{k}$

Jacobi $\rightarrow$ ε_{ij}^{k} – structure constants of Lie algebra $\langle x_1, \dots, x_n \rangle = \mathfrak{g}$

- Geometrically $\mathfrak{g}^*$ – Poisson manifold

A_0 – functions on $\mathfrak{g}^*$

$$A_0 = \mathbb{C}[x_1, \dots, x_n] = S^{\cdot}\mathfrak{g} = T^{\cdot}\mathfrak{g} \big/ x \otimes y - y \otimes x$$

- Quantization $\mathcal{U}(\mathfrak{g}) = T^{\cdot}\mathfrak{g} \big/ x_i \otimes x_j - x_j \otimes x_i - \sum \varepsilon_{ij}^{k} x_k$

$\mathcal{U}(\mathfrak{g})$ - filtred algebra

$\mathcal{U}(\mathfrak{g})_0 \subset \mathcal{U}(\mathfrak{g})_1 \subset \mathcal{U}(\mathfrak{g})_2 \subset \dots$

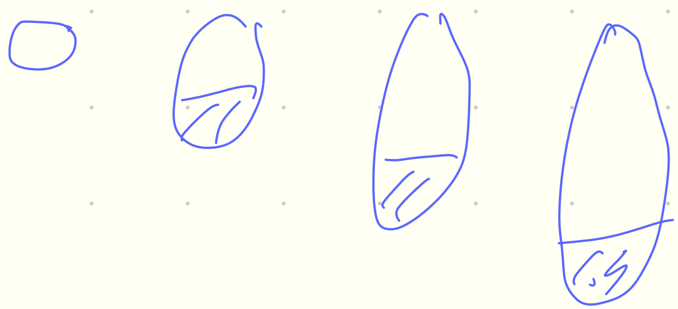
PBW $\quad \mathcal{U}(\mathfrak{g})_n \big/ \mathcal{U}(\mathfrak{g})_{n-1} \simeq S^n(\mathfrak{g})$

$$\bigoplus_n \mathcal{U}(\mathfrak{g})_n \big/ \mathcal{U}(\mathfrak{g})_{n-1} \simeq S^{\cdot}(\mathfrak{g})$$

- Rees algebra $\quad A = \left(\bigoplus_n \hbar^n \mathcal{U}(\mathfrak{g})_n\right)[[\hbar]]$

elements $u_0 + \hbar u_1 + \hbar^2 u_2 + \dots$, $u_i \in \mathcal{U}_i$ and $\exists N$, $u_i \in \mathcal{U}_N$ $\forall i$

$\mathcal{U}_0 \quad \mathcal{U}_1 \hbar \quad \mathcal{U}_2 \hbar^2 \quad \mathcal{U}_3 \hbar^3$



As vector space

$A \simeq A_0 \otimes \mathbb{C}[[\hbar]]$

– generators as $\mathbb{C}[[\hbar]]$ not canonical

$\forall x, y \in \mathfrak{g} \subset \mathcal{U}_1$, $\quad (\hbar x) \cdot (\hbar y) - (\hbar y)(\hbar x) = \hbar(\hbar[x,y])$

$\hbar x, \hbar y \in \hbar\, \mathcal{U}_1 \subset A$

Hochschild cohomology

- $C^n(A,A)$ - linear maps from $A^{\otimes n}$ to A

$d^n : C^n(A,A) \longrightarrow C^{n+1}(A,A)$

$$d^n f(a_1,\dots,a_{n+1}) = a_1 f(a_2,\dots,a_{n+1}) - f(a_1a_2,a_3,\dots,a_{n+1}) + f(a_1,a_2a_3,\dots,a_{n+1}) - \dots$$
$$+ (-1)^n f(a_1,a_2,\dots,a_{n-1},a_na_{n+1}) + (-1)^{n+1} f(a_1,\dots,a_n)a_{n+1}$$

$d^2=0$ we have a complex

$$0 \to C^0 \to \dots \quad C^{n-1} \to C^n \to C^{n+1} \to \dots$$

<u>Def</u> $HH^n(A) = \mathrm{Ker}\, d^n / \mathrm{Im}\, d^{n-1}$ Hochschild cohomology

<u>Rem</u> $HH^n(A) = \mathrm{Ext}_{A\otimes A^{op}}(A,A)$

Examples $HH^0(A) = Z(A)$

$$HH^1(A) = \frac{\text{Derivations } A}{\text{inner derivations}}$$

$$HH^2(A) = \frac{\{m_1 \text{ satisfying } * \}}{\text{gauge equiv}}$$

Th (Hochschild - Kostant - Rozenberg) $A = C^\infty(M)$ then $HH^n(A) = \Lambda^n(TM)$ – polyvector fields

This Th. implies Prop above.

Problem* Show that $HH^2(U(\mathfrak{g})) = 0$ where $\mathfrak{g}$ - simple Lie algebra

Hint PBW filtration $\Rightarrow$ Spectral sequence $E_2 = H(\mathfrak{g}, S(\mathfrak{g}))$
$\Rightarrow H(\mathfrak{g}, \mathbb{C}) \otimes Z(U(\mathfrak{g})) \Rightarrow H^2 = 0$

- **Theorem** (Kontsevich) $A = C^\infty(M)$, $\forall \{\cdot,\cdot\}$ $\exists$ deformation quantization

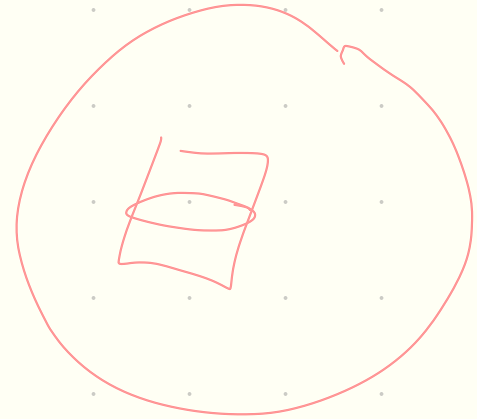
Rem If M is symplectic – known before (Fedosov ...)

- **Example** $M = T^*X$, $\mathrm{Diff}\,X$ – algebra of differential operators on X. $\mathrm{Diff}\,X$ – filtred by degree of operator, $\mathrm{Diff}_0 \subset \mathrm{Diff}_1 \subset \ldots$

A = Rees algebra of $\mathrm{Diff}\,X$ – deform. quantization

Problem* $\exists$ Poisson algebra A_0 s.t no deform. quantization

Symplectic Leaves



- Π - Poisson structure on M

$\forall x \in M \quad \Pi \in \Lambda^2 T_x M \rightsquigarrow \Pi : T_x^* M \to T_x M$

$T_x^\Pi = \operatorname{Im} \Pi \subset T_x M$

Equivalently $T_x^\Pi = \{ V_H \mid H \in C^\infty(M) \}$, here V_H - hamiltonian vector field

Equivalently $\Pi = \sum \Pi^i_{(1)} \otimes \Pi^i_{(2)}$, $T_x^\Pi = \langle \Pi^i_{(1)} \rangle = \langle \Pi^i_{(2)} \rangle$

minimal

- <u>Problem</u> Show that distribution T^Π is integrable

<u>Hint</u> Use Frobenius theorem. It is sufficient to check closedness for vector fields V_H

(Actually Frobenius theorem works only on open subset of fixed $\operatorname{rk} T^\Pi$. More general thms Stefan-Sussmann or Weinstain splitting)

Def Symplectic leaves — submanifolds tangent to T^{Π}.

Equivalently $x \sim y$ if $\exists$ piecewise smooth curve tangent to T^{Π}

Symplectic leaves: equiv. classes for $\sim$

Remark Symplectic leaves are generally not submanifolds e.g. irrational winding $\exists$

Example $M = V$ $\quad \Pi = \sum \varepsilon_{ij}\, \partial_{x_i} \wedge \partial_{x_j}$ constant $\varepsilon_{ij} \in \Lambda^2 V$, $\operatorname{Ker} \varepsilon \subset V^*$. For any $c \in \operatorname{Ker}\varepsilon$, c is Casimir function $\{c, F\} = 0 \quad \forall F \in C^\infty(V)$

Let $U = (\operatorname{Ker}\varepsilon)^\perp \subset V$ Then symplectic leaves are cosets $a + U \subset V$

In coordinates in some basis
$$\varepsilon \sim \begin{array}{c} 1 \\ 2 \\ \vdots \\ 2n \\ 2n+1 \\ \vdots \\ 2n+k \end{array} \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & 0 \\ & & \ddots & & & \\ & & & 0 & 1 & \\ 0 & & & -1 & 0 & \\ & & & & & 0 \ddots 0 \end{pmatrix}$$

Then last k coordinate functions $x^{2n+1}, \dots, x^{2n+k}$ — Casimir functions

$U = \langle e_1, \dots, e_{2n} \rangle$ Symplectic leaves $a + U = \{x \mid x^{2n+i} = a^{2n+i},\ 1 \le i \le k\}$

<u>Example</u> $M = \mathfrak{g}^*$ with Kostant - Kirillov bracket.

$\forall \xi \in \mathfrak{g}$, ξ - linear function on M.

V_ξ - hamilt. vector field i.e $\forall \alpha \in \mathfrak{g}^*$, $\eta \in \mathfrak{g} = T_\alpha^* M$

$$(V_\xi \cdot \eta)(\alpha) = \{\xi, \eta\}(\alpha) = (\alpha, [\xi, \eta]) = (ad_\xi^* \alpha, \eta)$$

Hence $V_\xi = ad_\xi^*$ (vector field for coadjoint action)

Hence Symplectic leaves = coadjoint orbits

References

- Etingof, Schiffmann Lectures on quantum groups Ch 1
- Chary, Pressley A guide to quantum groups Sec 1.1, 1.6
- Kontsevich Deformation quantization of Poisson manifolds
- Calaque, Rossi Lectures on Duflo isomorphisms in Lie algebras and complex geometry Sec 1, 2, 3