Introduction to Quantum Groups

Lecture 2

Poisson-Lie groups and Lie Bialgebras

- More Poisson geometry

(M, Π_M), (N, Π_N) — Poisson manifolds

Def $(M \times N, \Pi_M + \Pi_N)$ — product of Poisson manifolds

Def $\varphi: M \to N$ is Poisson map if $\varphi_* \Pi_M = \Pi_N$

Def $M \subset N$ is Poisson submanifold if $\Pi_N|_M \in \Lambda^2 T_M$

Example Symplectic leaves are Poisson submanifolds

- Def G — Poisson-Lie group if
  - G — Lie group
  - G — Poisson manifold
  - $G \times G \to G$ Poisson map

more explicitly $\forall \varphi, \psi \in C^\infty(G)$

$$\{\varphi, \psi\}(gh) = \{\varphi, \psi\}(gh)\Big|_{h \text{ fixed}} + \{\varphi, \psi\}(gh)\Big|_{g \text{ fixed}} \qquad (*)$$

Remark $i : G \to G$ $\quad g \mapsto g^{-1}$ is not Poisson

One can show $\{\varphi \circ i, \psi \circ i\} = -\{\varphi, \psi\} \circ i$

- Example 1 $\quad \mathfrak{g}$ Lie algebra $\quad G = \mathfrak{g}^*$, w.r.t "+"

linear Poisson bracket $\{x_i, x_j\} = \sum c_{ij}^k x_k$

$\mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^* \qquad x_i = x_i' + x_i''$

$$\sum c_{ij}^k x_k = \{x_i, x_j\} = \{x_i' + x_i'', x_j' + x_j''\} = \sum c_{ij}^k x_k' + \sum c_{ij}^k x_k''$$

c.f $(*)$ above

- Example $G \subset GL_N$ matrix group

$$\{L_1 \overset{\otimes}{,} L_2\} = [r, L_1 \otimes L_2] \qquad r \in Mat_n \otimes Mat_n$$

- L – matrix of coordinate functions (e.g. $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$)

- $\{L_1 \overset{\otimes}{,} L_2\}$ matrix consist of brackets of coordinate functions

  (e.g. $\{L_1 \overset{\otimes}{,} L_2\} = \begin{pmatrix} \{a,a\} & \{a,b\} \\ \{a,c\} & \{a,d\} \\ \{c,a\} & \{c,b\} \\ \{c,c\} & \{c,d\} \end{pmatrix}$)

- r – classical r-matrix

$\{\cdot,\cdot\}$ skew symmetry + Jacoby $\Rightarrow$ conditions on r

$G \times G \to G \qquad L = L' L'' \qquad (L_1' L_1'' \otimes L_2' L_2'') = (L_1' \otimes L_2')(L_1'' \otimes L_2'')$

$$[r, L_1' L_1'' \overset{*}{\otimes} L_2' L_2''] = [r, L_1' \otimes L_2'](L_1'' \otimes L_2'') + L_1' \otimes L_2' [r, L_1'' \otimes L_2'']$$

c.f. $\{\varphi, \psi\}(gh) = \{\varphi, \psi\}(gh)\big|_{h \text{ fixed}} + \{\varphi, \psi\}(gh)\big|_{g \text{ fixed}} \quad (*)$

- Def $H \subset G$ is Poisson-Lie subgroup if H is subgroup and Poisson submanifold

- Problem a) Let $H \subset G$ is P.-L. subgroup. Show that $C^\infty(G)^H$ is Poisson subalgebra
b) $H \subset G$ subgroup and satisfies $r|_H \subset TH \otimes TG + TG \otimes TH$. Show that $C^\infty(G)^H$ is Poisson subalgebra

Remark Geometrically G/H P. manifold

- **Problem** $G = GL_2$, $r = h \otimes h + e \otimes f$, $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Compute brackets of a, b, c, d.

Check skew-comm. Check P.-L. property.

- Recall $\{\varphi, \psi\}(gh) = \{\varphi, \psi\}(gh)\big|_{h \text{ fixed}} + \{\varphi, \psi\}(gh)\big|_{h \text{ fixed}}$

In terms of Π

$$\Pi(gh) = (\rho_h)_* \Pi(g) + (\lambda_g)_* \Pi(h)$$

Here $\rho_h : G \to G \quad x \mapsto xh$ right multiplication

$\lambda_h : G \to G \quad x \mapsto hx$ left multiplication

- **Remark** If $g = h = e$ $\quad \Pi(e) = \Pi(e) + \Pi(e)$

Hence $\Pi(e) = 0$, P.-L. group is not symplectic

- $I = \{\varphi \in C^\infty(G) \mid \varphi(e) = 0\}$

$\varphi, \psi \in C^\infty(G) \Rightarrow \{\varphi, \psi\} \in I$ since $\pi(e) = 0$

$\varphi \in I^2, \psi \in C^\infty(G) \Rightarrow \{\varphi, \psi\} \in I^2$

- $\delta^* = \{\cdot, \cdot\}$ $\quad \Lambda^2\left(I/I^2\right) \to I/I^2$

$I/I^2 = T_e^* G = \mathfrak{g}^*$ we get $\delta^*: \Lambda^2 \mathfrak{g}^* \to \mathfrak{g}^*$

$d\varphi \wedge d\psi \mapsto d\{\varphi, \psi\}$

we get Lie algebra structure on $\mathfrak{g}^*$

Jacoby for $\{\cdot, \cdot\} \Rightarrow$ Jacoby for $\delta^* \Rightarrow$ CoJacoby for δ

- Dually we get $\delta: \mathfrak{g} \to \Lambda^2 \mathfrak{g}$

Another way to define: Consider map $G \to \Lambda^2 \mathfrak{g}$

$g \mapsto (\lambda_g)_* \pi(g^{-1}) =: \pi_e(g)$ Differential is $\delta: \mathfrak{g} \to \Lambda^2 \mathfrak{g}$

- **Prop** δ satisfies cocycle condition

$$\delta([a,b]) = ad_a(\delta(b)) - ad_b(\delta(a))$$

ad' action $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$

Pf $g = exp(ta)$, $h = exp(tb)$

$$t^2\delta([a,b]) = \Pi_e(gh) - \Pi_e(hg) = (\lambda_{gh})_* \Pi(h^{-1}g^{-1}) - (\lambda_{hg})_* \Pi(g^{-1}h^{-1}) =$$

$$= (\lambda_g)_* \Pi(g^{-1}) + (Ad_g)_* (\lambda_h)_* \Pi(h^{-1}) - (\lambda_h)_* \Pi(h^{-1}) - (Ad_h)_* (\lambda_g)_* \Pi(g^{-1}) =$$

$$= ((Ad_g)_* - id)\cdot \Pi_e(h) - ((Ad_h)_* - id)\cdot \Pi_e(g) = t^2(ad_a \delta(b) - ad_b(\delta(a))).$$ $\square$

- **Def** A Lie bialgebra $(\mathfrak{g}, [\cdot,\cdot], \delta)$

$\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$

$(\mathfrak{g}, [\cdot,\cdot])$ Lie algebra

- δ satisfies coJacobi relation
- δ satisfies cocycle condition

- **Th** (a) G is P.L. group. Then $\mathfrak{g}$ is Lie bialgebra.
(b) If $\mathfrak{g}$ is Lie bialgebra, then $\exists!$ connected, symply connected P.L. group G s.t. $\mathrm{Lie}\,G = \mathfrak{g}$

Pf (a) above (b) - see refs ∎

Rem Notion of Lie bialgebra is self-dual.

Rem (a) $H \subset G$ is P.-L. subgroup $\Rightarrow$ $\mathfrak{h} \subset \mathfrak{g}$ subbialgebra
(b) $H \subset G$ subgroup s.t. $\Pi|_H \subset TH \otimes TG + TG \otimes TH$ $\Rightarrow$ $\mathfrak{h} \subset \mathfrak{g}$, $\mathfrak{h}^{\perp} \subset \mathfrak{g}^*$ subalgebras

- **Def** Manin triple is $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$
  - $\mathfrak{g}$ is Lie algebra with non-degenerate symmetric invariant form $(\cdot\,;\cdot)$
  - $\mathfrak{g}_+$, $\mathfrak{g}_-$ – Lie subalgebras

- $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a vector space (not as Lie algebra)
- $\mathfrak{g}_+, \mathfrak{g}_-$ – are isotropic w.r.t. $(\cdot\,;\cdot)$

Rem Isotropic : $(\cdot\,;\cdot)$ $\mathfrak{g}_+ \xrightarrow{\sim} \mathfrak{g}_-^*$

- Problem For $\forall$ fin. dim $\mathfrak{g}$ $\exists$ bijection

$$\left\{ \begin{matrix} \text{Lie bialgebra} \\ \text{str on } \mathfrak{g} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Manin triples} \\ \text{with } \mathfrak{g}_+ \simeq \mathfrak{g} \end{matrix} \right\}$$

Hint $\mathfrak{g}$ Lie bialgebra $\Rightarrow$ $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ has Lie algebra structure

$[a_i, a^j] = \tilde{c}_i^{jk} a_k - c_{ik}^j a^k$ $\{a_i\}$ – basis $\mathfrak{g}$ $\{a^j\}$ – basis $\mathfrak{g}^*$

$c, \tilde{c}$ – structure constants in $\mathfrak{g}, \mathfrak{g}^*$

Example 1 $\mathfrak{g}$ – simple Lie algebra

$\mathfrak{g} = \underbrace{n_- \oplus h}_{b_- =} \overbrace{\oplus\, n_+}^{= b_+}$ (with $h \oplus n_+ = b_+$ and $n_- \oplus h = b_-$) $\qquad D = \mathfrak{g} \oplus h \qquad pr_\pm : b_\pm \to h$

$\mathfrak{g}_+ = \{ (a, b) \mid a \in b_+,\ b \in h,\ pr_+ a = b \} \simeq b_+$

$\mathfrak{g}_- = \{ (a, b) \mid a \in b_-,\ b \in h,\ pr_+ a = -b \} \simeq b_-$

Form given $\big((a_1, b_1), (a_2, b_2)\big) = (a_1, a_2) - (b_1, b_2)$

Example 2 G – Lie group with trivial P.B.

$D = \mathfrak{g} \oplus \mathfrak{g}^*$ $\qquad [a, \alpha] = ad^*_a \alpha \qquad a \in \mathfrak{g}$, $\alpha \in \mathfrak{g}^*$

Example 3 $\mathfrak{g}$ simple Lie algebra

$D = \mathfrak{g} \oplus \mathfrak{g}, \quad D_+ = \{ (a, a) \mid a \in \mathfrak{g} \} \simeq \mathfrak{g}$

$$\mathfrak{g}_- = \{(a,b) \mid a\in\mathfrak{h}_+,\ b\in\mathfrak{h}_-,\ pr_+ a + pr_- b = 0\} \subset \mathfrak{h}_+\oplus\mathfrak{h}_-$$

$$((a_1,b_1),(a_2,b_2)) = (a_1,a_2) - (b_1,b_2) \text{ - form on } \mathfrak{g}$$

$$((a_1,b_1),(a_2,b_2)) = (a_1,a_2)-(b_1,b_2) = (pr_+ a_1, pr_+ a_2) - (pr_- b_1, pr_- b_2) = 0$$

since $a_1, a_2 \in \mathfrak{h}_+$, $b_1, b_2 \in \mathfrak{h}_-$

- **Example 4** $\mathfrak{g}$ simple Lie algebra

$$\mathfrak{g} = \mathfrak{g}[t^{\pm}] \qquad \mathfrak{g}_+ = \mathfrak{g}[t], \quad \mathfrak{g}_- = \mathfrak{g}[t^{-1}]t^{-1}$$

$$(f,g) = \mathrm{Res}_{t=0}(f,g)\,dt \text{ - form on } \mathfrak{g}$$

<u>Problem</u> Find Lie bialgebra structure (i.e. δ) in examples 1 and 3 for $\mathfrak{g} = sl_2$

References

- Etingof, Schiffmann Lectures on quantum groups Ch 2
- Chary, Pressley A guide to quantum groups Sec 1.2, 1.3