Introduction to Quantum Groups

Lecture 3

Classical r matrices

- Lie Bialgebra $(\mathfrak{g}, \delta)$ $\delta : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$
  - coJacobi
  - cocycle $\delta([a,b]) = ad_a \delta(b) - ad_b \delta(a)$

<u>Def</u> $(\mathfrak{g}, \delta)$ is coboundary if $\exists r \in \Lambda^2 \mathfrak{g}$ s.t.

$$\delta(a) = ad_a r \qquad (= [a \otimes 1 + 1 \otimes a, r])$$

<u>Rem</u> If $\mathfrak{g}$ is s/s, then $H^1(\mathfrak{g}, \#) = 0$. Hence $\exists r$

- <u>Th</u> $\mathfrak{g}$ - Lie alg. $r \in \mathfrak{g} \otimes \mathfrak{g}$, $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, $\delta(a) = ad_a r$

ⓐ δ maps to $\Lambda^2 \mathfrak{g}$ <=> $r_{12} + r_{21} \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$

ⓑ δ satisfies coJacoby <=>

$$C[[r,r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$$

Here $r = \sum x_i \otimes y_i$ $\quad r_{12} = \sum x_i \otimes y_i \otimes 1$, $\quad r_{13} = \sum x_i \otimes 1 \otimes y_i$

$r_{23} = \sum 1 \otimes x_i \otimes y_i$

Problem Prove ⓐ

- Assume $\exists$ non-degen. invar. scalar prod. on $\mathfrak{g}$
  Hence $\exists$ canonical invariant $\Lambda^3\mathfrak{g} \to \mathbb{C}$

$$a_1 \wedge a_2 \wedge a_3 \mapsto ([a_1, a_2], a_3)$$

(Invariance $\forall b, a_1, a_2, a_3$

$$[b, a_1] \wedge a_2 \wedge a_3 + a_1 \wedge [b, a_2] \wedge a_3 + a_1 \wedge a_2 \wedge [b, a_3] \mapsto$$

$$\mapsto ([[b, a_1], a_2], a_3) + ([[a_1, [b, a_2]], a_3) + ([a_1, a_2], [b, a_3]) = 0$$
)

- In other terms $[a_i, a_j] = \sum c_{ij}^k a_k \rightsquigarrow c^{ijk} \in (\Lambda^3\mathfrak{g})^{\mathfrak{g}}$

- Classical Yang-Baxter equation

CYBE $[[r,r]]=0$

Modified classical Yang-Baxter equation

MCYBE $[[r,r]]=\varepsilon C$

- Let $r = r^S + r^A$ $r^S \in S^2\mathfrak{g}$, $r^A \in \Lambda^2\mathfrak{g}$

$ad_a r \in \Lambda^2\mathfrak{g}$ $\forall a$ $\iff$ $r^S \in (S^2\mathfrak{g})^{\mathfrak{g}}$

Assume $r^S = 2\Omega = \sum a_i \otimes a^i \in S^2\mathfrak{g}$ → $Id \in \mathfrak{g}\otimes\mathfrak{g}^*$ tensor Casimir

dual bases

<u>Problem</u> (a) $\delta_r = \delta_{r^A}$, where $\delta_r(a) = ad_a r$

(b) $[[r,r]] = [[r^A, r^A]] + 2^2 C$

Hence $\begin{pmatrix} r^A \in \Lambda^2\mathfrak{g} \\ \text{MCYBE} \end{pmatrix} \longleftrightarrow \begin{pmatrix} r \in \mathfrak{g}\otimes\mathfrak{g} \\ \text{CYBE} \end{pmatrix}$

- Remark If $\mathfrak{g}$ – simple then $S^2\mathfrak{g} = \langle \Omega \rangle$
  $(\Lambda^3\mathfrak{g})^{\mathfrak{g}} = H_3(\mathfrak{g}, \mathbb{C}) = \langle c \rangle$

- Using scalar product $r \in \mathfrak{g}\otimes\mathfrak{g} \rightsquigarrow \hat{r} : \mathfrak{g} \to \mathfrak{g}$
  If $r \in \Lambda^2\mathfrak{g}$ and MCYBE then
  $[\hat{r}a, \hat{r}b] - \hat{r}[\hat{r}a, b] - \hat{r}[a, \hat{r}b] = \varepsilon[a, b]$

- Example $\mathfrak{g} = sl_2 = \langle e, h, f \rangle$
  $r \in \Lambda^2\mathfrak{g} \simeq$ adjoint rep sl_2
  different $r \Longleftrightarrow$ adjoint orbits for sl_2

(i) $\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ $r = \lambda e \wedge f$ (ii) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $r = e \wedge h$ (iii) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $r = 0$

CYBE: $\Lambda^2 sl_2 \to \Lambda^3 sl_2$ $= \mathbb{C} = (\Lambda^3 sl_2)^{sl_2}$

$r \mapsto [[r,r]]$

Hence

- MCYBE always satisfied
- $[[r,r]] \sim \det$

(i) $\delta(e) = \lambda e \wedge h$ $\delta(h) = 0$ $\delta(f) = \lambda f \wedge h$

(ii) $\delta(e) = 0$, $\delta(h) = 2 e \wedge h$ $\delta(f) = 2 e \wedge f$

(iii) $\delta = 0$

Note $sl_2^* \neq sl_2$ as Lie algebra

- Let $(\mathfrak{g}, \delta)$ – bialg Lie

Consider Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$

$\mathfrak{g}_+ = \mathfrak{g}$, $\mathfrak{g}_- = \mathfrak{g}^*$, $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ — Lie algebras

Let $\mathfrak{g} = \langle a_i \rangle$, $\mathfrak{g}^* = \langle a^i \rangle$ dual bases

Recall commutation relations in $\mathfrak{q}$

$[a_i, a_j] = \sum_k C_{ij}^k a_k \quad [a^i, a^j] = \sum_k \tilde{C}_k^{ij} a^k \quad [a_i, a^j] = \sum_k \tilde{C}_i^{jk} a_k + C_{ki}^j a^k$

Prop $r = \sum a_i \wedge a^i \in \Lambda^2 \mathfrak{q}$ satisfies MCYBE

Hence we got Lie Bialgebra str on $\mathfrak{q} = \mathfrak{g} \oplus \mathfrak{g}^*$

This is classical (Drinfeld) double. $\mathfrak{q} = \mathcal{D}(\mathfrak{g})$

- Equivalently one can take $r = \sum a_i \otimes a^i \in \mathfrak{q} \otimes \mathfrak{q}$
  r is not antisymm but satisfies CYBE

Proof $[[r,r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] =$

$$= \sum_{i,j,k} C_{ij}^k a_k \otimes a^i \otimes a^j - \tilde{C}_j^{ik} a_i \otimes a_k \otimes a^j - C_{kj}^i a_i \otimes a^k \otimes a^j + \tilde{C}_k^{ij} a_i \otimes a_j \otimes a^k$$

$= 0$

Problem $\mathfrak{g} \hookrightarrow \mathcal{D}(\mathfrak{g})$ and $(\mathfrak{g}^*)^{coop} \hookrightarrow \mathcal{D}(\mathfrak{g})$ imbedding of Lie Bialgebras.

Here $(\mathfrak{g}, \delta)^{coop} = (\mathfrak{g}, -\delta)$ – permutation of factors in coalgebra homomorphism.

This determines coalgebra structure.

- Let $\mathfrak{g}$ – simple Lie algebra

$$(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-) = (\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{b}_+, \mathfrak{b}_-)$$

$\mathfrak{b}_+ = (a, pr_+ a)$, $\mathfrak{b}_- = (a, -pr_-(a))$

$pr_\pm \, \mathfrak{b}_\pm \Rightarrow \mathfrak{h}$

$$[(x_1, y_1), (x_2, y_2)] = [x_1, x_2] - [y_1, y_2]$$

$x_1, x_2 \in \mathfrak{g}$
$y_1, y_2 \in \mathfrak{h}$

Hence $\mathfrak{g} \oplus \mathfrak{h} = \mathcal{D}(\Delta_+)$

$\exists\, r \in \Lambda^2(\mathfrak{g} \oplus \mathfrak{h})$ $\quad r = \sum_{\alpha \in \Delta_+} e_\alpha \wedge e_{-\alpha} + \sum h_i \wedge h^i$

$h_i \wedge h^i = (h_i, h_i) \wedge (h^i, -h^i)$

- $0 \oplus \mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{h}$ — subalgebra, Lie ideal

  $\delta(0 \oplus \mathfrak{h}) = 0$

Hence we can define Lie bialgebra str on

$\mathfrak{g} \oplus \mathfrak{h} / 0 \oplus \mathfrak{h} \simeq \mathfrak{g}$

- This bialgebra is coboundary

  $r = \sum_{\alpha \in \Delta_+} e_\alpha \wedge e_{-\alpha} \in \Lambda^2 \mathfrak{g}$

  Standard structure

Problem ⓐ Find $\delta(h_i)$, $\delta(e_\alpha)$, $\delta(e_{-\alpha})$ α - simple
ⓑ Find Lie algebra $\mathfrak{g}^*$
ⓒ Show that $r = \sum h_i \otimes h^i + 2\sum e_\alpha \otimes e_{-\alpha}$ defines the same δ and satisfies CYBE

- $(\mathfrak{g}, \delta)$ - coboundary Lie bialgebra $\delta(a) = \mathrm{ad}_a r$
  G - corresp. connected P-L group π - ?

- Define $\pi_e(g) = r - (\mathrm{Ad}_{g^{-1}})\, r$
  we have $\pi_e(e) = 0$, $(d\pi_e)\big|_{g=e} a = \mathrm{ad}_a r = \delta(a)$

$\pi_e(gh) = \pi_e(h) + (\mathrm{Ad}_{h^{-1}})\, \pi_e(g)$ — multiplicativity

$\boxed{\pi(g) = (\lambda_g)_* r - (\rho_g)_* r} = (\lambda_g)_* (\pi_e(g))$ — Sklyanin bracket

$\Downarrow$

$\pi(gh) = (\lambda_g)_* \pi(h) + (\rho_h)_* \pi(g)$

Problem Show that Sklyanin bracket defines Poisson structure. (If r satisfies MCYBE)

Problem + Multiplicativity $\Rightarrow$ P.L. property

- Let G - matrix group g_i^j - coordinates
  corresp vector fields $\partial_{i'}^{j'} g_i^j = \delta_i^{j'} \delta_{i'}^j$
  Let $r = \sum r_{i_1 i_2}^{j_1 j_2} \partial_{j_1}^{i_1} \wedge \partial_{j_2}^{i_2}$ Then

$(\lambda g)_* r = \sum g_{i_1}^{k_1} g_{i_2}^{k_2} r_{k_1 k_2}^{j_1 j_2} \partial_{j_1}^{i_1} \wedge \partial_{j_2}^{i_2}$ $\quad$ $(\rho g)_* r = \sum g_{k_1}^{j_1} g_{k_2}^{j_2} r_{i_1 i_2}^{k_1 k_2} \partial_{j_1}^{i_1} \wedge \partial_{j_2}^{i_2}$

Hence $\{g_{i_1}^{j_1}, g_{i_2}^{j_2}\} = g_{i_1}^{k_1} g_{i_2}^{k_2} r_{k_1 k_2}^{j_1 j_2} - g_{k_1}^{j_1} g_{k_2}^{j_2} r_{i_1 i_2}^{k_1 k_2}$

In matrix terms $\{g \overset{\otimes}{,} g\} = [g \otimes g, r]$

- **Hint for Problem***

Def $K \in \Lambda^k T G$ is multiplicative if

$$K(gh) = (\lambda_g)_* K(h) + (\rho_h)_* K(g)$$

- K is multiplicative $\iff$ $K(e) = 0$, and for any left invariant vector field X and right invariant Y we have $L_X L_Y K = 0$

- K is mult, $d_e K = 0 \implies K = 0$

- Π is mult $\implies$ Schouten bracket $[\Pi, \Pi]$ is mult

- r satisfies MCYBE $\implies d_e [\Pi, \Pi] = 0$

$$\implies [\Pi, \Pi] = 0$$

References

- Etingof, Schiffmann Lectures on quantum groups Ch 3
- Chary, Pressley A guide to quantum groups Ch 2
- Lu, Weinstein Poisson-Lie groups, dressing transformations and Bruhat decompositions