Introduction to Quantum Groups

Lecture 4

Dual Poisson Lie groups. Dressing action.

- $(\mathfrak{g}, \delta)$ — Lie bialgebra
$[\cdot,\cdot] : \Lambda^2\mathfrak{g} \to \mathfrak{g}$ $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$

$(\mathfrak{g}^*, \delta_{\mathfrak{g}}^*)$ - dual Lie bialgebra
$\delta^* : \Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$ $\delta_{\mathfrak{g}^*} = [\cdot,\cdot]^* : \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$

<u>Def</u> G^* - corresp connected, (simply connected) P.-L. group

- <u>Example</u> 1 G, $\Pi = 0$ $\delta = 0$
Dual Lie bialgebra $\mathfrak{g}^*$ with zero bracket
Dual P.-L. group $G^* = \mathfrak{g}^*$ Kostant-Kirillov bracket

2 G - simple Lie group, $\mathfrak{g}$ simple algebra.
Standard bialgebra str

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

Recall
2 constr of
standard bialg str

- $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$, $\mathfrak{d}_+ = \{x, x\} \simeq \mathfrak{g}$

  $\mathfrak{d}_- = \{x, y \mid x \in \mathfrak{b}_+, y \in \mathfrak{b}_- , \; pr_+ x = -pr_- y\}$

  $\mathfrak{d}_- \cong \mathfrak{g}^* \subset \mathfrak{b}_+ \oplus \mathfrak{b}_-$

- $\mathfrak{b}_+$ $\quad D(\mathfrak{b}_+) \simeq \mathfrak{g} \oplus \mathfrak{h}$ $\quad (\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{b}_+, \mathfrak{b}_-)$

  $(x, pr_+ x)$ $\quad (y, -pr_- y)$

  $D(\mathfrak{b}_+)/\mathfrak{h} \simeq \mathfrak{g}$

  $r = \sum_{\alpha \in \Delta_+} e_\alpha \wedge e_{-\alpha}$ – quasi triangular

$G^* = \{(X, Y) \mid pr_+ X \, pr_- Y = 1\} \subset B_+ \times B_-$

- $\mathfrak{g}^* \longrightarrow \mathrm{Vect}(G)$

  $\alpha \longmapsto \alpha_e \in \Omega^1(G) \xrightarrow{\Pi} V_e(\alpha) \in \mathrm{Vect}\, G$

  left. inv. 1-form

$$\alpha \mapsto \underbrace{\alpha_e}_{\text{right inv. 1-form}} \in \Omega^1(G) \xrightarrow{\Pi} V_r(\alpha) \in \mathrm{Vect}\, G$$

Th a) $V_e : \mathfrak{g}^* \to \mathrm{Vect}\, G$ Lie alg anti homomorphism

b) $V_e : \mathfrak{g}^* \to \mathrm{Vect}\, G$ Lie alg homomorphism

Problem* Prove a).

Hint • For any Poisson variety define bracket on Ω^1

$$[\alpha, \beta] = d(\Pi(\alpha,\beta)) + i_{\Pi\alpha} d\beta - i_{\Pi\beta} d\alpha$$
$$= -d(\Pi(\alpha,\beta)) + L_{\Pi\alpha}\beta - L_{\Pi\beta}\alpha$$

Here $\Pi\alpha = \sum \Pi^{ij} \alpha_i \partial_j$ -vector field

• Show that for $[dH_1, dH_2] = d\{H_1, H_2\}$

and for any α, β, and $f \in C^\infty$

$$[f\alpha, \beta] = f[\alpha,\beta] - (L_{\Pi\beta} f)\alpha$$

commutator 1-forms — commutator vector fields

Hence $\Pi[\alpha,\beta] = [\Pi\alpha, \Pi\beta]$

- Let $\xi \in Vect(G)$ be left invariant. Then for any $\alpha, \beta \in \mathfrak{g}^*$ $i_\xi[\alpha_e, \beta_e]$ - constant (Use $i_X\alpha_e, i_X\beta_e$ - constants, $L_X\Pi$ - left invariant)

  Hence $[\alpha_e, \beta_e]$ - left invariant

- Show that $[\alpha_e, \beta_e] = [\alpha, \beta]_e$ (sufficient at $T_e^* G$)

Hence V_ℓ, V_r integrates (locally?) to actions of G^* on G

Tangent space to orbit $\iff \langle V_\ell(x)\rangle = \operatorname{Im}\Pi\langle\alpha_e\rangle$
$= \operatorname{Im}\Pi$

Hence G^* orbits — symplectic leaves on G

<u>Example</u> G, $\Pi = 0$ $G^* = \mathfrak{g}^*$

- Action of G^* on G — $\alpha \in \mathfrak{g}^*$, $\alpha_e \in \Omega^1(G) \xrightarrow{\Pi} 0 \in \mathrm{Vect}_G$

  G^* on G acts trivially

  Orbits — points — symplectic leaves

- (Dually) Action of G on G^*

$\xi \in \mathfrak{g} \rightsquigarrow \xi \in \xi_e \in \Omega^1(\mathfrak{g}^*) \xrightarrow{\Pi} ad^*_\xi \in \mathrm{Vect}(\mathfrak{g}^*)$

G acts on $G^* = \mathfrak{g}^*$ by Ad^*

Coadj orbits — sympl leaves on $\mathfrak{g}^*$

- $(\mathfrak{g}, \partial) \rightsquigarrow \mathcal{D}(\mathfrak{g})$ - double

  $\mathfrak{g} \nearrow \quad \nwarrow \mathfrak{g}^*$

  Lie algebra homomorphism

Def $\mathcal{D}(G)$ — Lie group corresp $\mathfrak{g} \oplus \mathfrak{g}^*$

- $G \times G^* \to \mathcal{D}(G)$ (local) diffeomorphism

Example G, $\Pi = 0$ $\quad \mathcal{D}(G) = T^*G = G \ltimes \mathfrak{g}^*$

$\mathcal{D}(\mathfrak{g}) = \mathfrak{g} \ltimes \mathfrak{g}^*$ $\quad$ as manifold $\quad$ as group

- Refactorization $\quad G_+ = G \quad G_- = G^*$

Def For $\forall g_+ \in G_+$, $g_- \in G_-$ represent $g_- g_+ \in \mathcal{D}(G)$ as

$$g_- g_+ =: (g_+)^{g_-} (g_-)^{g_+} \quad \text{where}$$

$$(g_+)^{g_-} \in G_+ \qquad (g_-)^{g_+} \in G_-$$

Rem Works if $G_+ \times G_- \to \mathcal{D}(G)$ is <u>global</u> diffeom

Otherwise — requires care

• <u>Prop</u> a) $\left((g_-)^{g_+}\right)^{h_+} = g_-^{g_+ h_+}$

b) $(g_+^{g_-})^{h_-} = g_+^{h_- g_-}$

<u>Pf</u> $g_+ h_+ g_- = g_-^{g_+ h_+} \cdot x_+$ $\Rightarrow$ $\square$

$\parallel$

$g_+ g_-^{h_+} y_+ = (g_-^{h_+})^{g_+} z_+ y_+$

<u>Th</u> This action G_- on G_+ corresponds

$V_r : G_- \curvearrowright G_+$

<u>Pf</u> See refs $\square$

<u>Example</u> G, $\Pi = 0$ $\quad D(G) = T^*G$

$g \cdot \alpha = Ad_g^* \alpha \cdot g$ $\quad \mathfrak{g}^* \curvearrowright G$ trivially

$G \curvearrowright \mathfrak{g}^*$ coadjoint

- **Corollary** Symplectic leaf through $g \in G$ is an (connected component) image $G^* g \subset D(G)$ under
left orbit

$$pr: \quad D(G) \to D(G)/G^* \simeq G$$
$$G \times G^* \simeq$$

Equivalently symplectic leafs – images of two-sided cosets $G^* g G^* \longrightarrow D(G)/G^* \simeq G$

- **Nontrivial Example** $(\mathfrak{d}, \mathfrak{d}_+, \mathfrak{d}_-)$ $\qquad \mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$

$\mathfrak{d}_+ = \{(x,x)\}$ $\qquad \mathfrak{d}_- = \{(x,y) \mid x \in \mathfrak{b}_+, y \in \mathfrak{b}_-, pr_+ a + pr_- b = 0\}$

$D(G) = G \times G$ $\qquad G_+ = G \xhookrightarrow{diag} G \times G$ $\qquad G_- = B_+ \underset{H}{\times} B_- \subset B_+ \times B_-$

- Bruhat decomposition

$$G = \bigsqcup_{w \in W} B_+ w_+ B_+ = \bigsqcup B_- w_- B_- = \bigsqcup B_- w B_+$$

For GL_n $\quad g = \begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix} w \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$

Since $w_0 B_+ w_0 = B_-$ $\quad w_0 B_- w_0 = B_+$ all Bruhat decomp are equiv.

$$G = \bigsqcup_{w \in W} B_- w B_+ \Rightarrow \quad G = \bigsqcup_{w \in W} w_0 B_- w B_+ = \bigsqcup_{w \in G} B_+ w_0^{-1} w B_+$$

- $\forall w_+, w_- \in W$ let $D_{w_+, w_-} = B_+ w_+ B_+ \times B_- w_- B_- \subset G \times G = D(G)$

We have $D(G) = \bigsqcup D_{w_+, w_-}$ – double $B_+ \times B_-$ cosets

Let $C_{w_+, w_-} = D_{w_+, w_-} \cap G = B_+ w_+ B_+ \cap B_- w_- B_-$

double Bruhat cell

G_- double cosets – subsets of C_{w_+, w_-}

<u>Cor</u> C_{w_+, w_-} – union of symplectic leaves

<u>Prop</u> $\dim C_{w_+, w_-} = \ell(w_+) + \ell(w_-) + \operatorname{rk} G$.

<u>Ex</u> $w_+ = w_- = e$ $\qquad C_{e,e} = B_+ \cap B_- = H \qquad \dim C_{e,e} = \operatorname{rk} G$

$w_+ = w_- = w_0 \qquad C_{w_0, w_0} = B_+ w_0 B_+ \cap B_- w_0 B_- =$

$= w_0 (B_- B_+ \cap B_+ B_-)$ – open subset in G.

$\dim C_{w_0, w_0} = \dim G$

Problem – C_{w_+, w_-} are <u>not</u> symplectic leaves (but Poisson submanifolds)

Main reason $G^* \neq B_+ \times B_+$, instead $B_+ \times B / G^* \simeq H$

<u>Th</u> (Hoffman–Kellendonk–Kutz–Reshetikhin)

$G_- \backslash \mathcal{D}_{w_+, w_-} / G_- \simeq H_{w_-^{-1} w_+}$ where $H_{w_-^{-1}w_+} = \{ h \in H \mid w_-^{-1} w_+ (h) = h \}$

(equivalently $H_{w_-^{-1}w_+}$ – H orbits on H with respect to action $h : x \mapsto h x\, w(h)^{-1}$)

<u>Problem</u> a) Find explicitly double Bruhat cells for $SU(2)$ with standard bracket

b) Find symplectic leaves on $SU(2)$. Find Casimir functions on each double Bruhat cells.

References

- Chary, Pressley A guide to quantum groups
  Sec. 1.5
- Lu, Weinstein Poisson-Lie groups, dressing transformations and Bruhat decompositions
- Weinstein Some remarks on dressing transformations
- Hoffmann, Kellendonk, Kutz, Reshetikhin Factorization dynamics and Coxeter-Toda lattices