Introduction to Quantum Groups
Lecture 5
Quantum groups and algebras. Example $\mathfrak{sl}_2$

- Quantize G $\rightsquigarrow$ Quantize $\mathbb{C}[G]$

- G group $\quad G\times G \to G \quad\quad G \to G$

$(g,h) \mapsto gh \quad\quad g \mapsto g^{-1}$

- Hence

coproduct $\Delta: \mathbb{C}[G] \to \mathbb{C}[G]\otimes\mathbb{C}[G]$ $(= \mathbb{C}[G\times G])$

antipode $S: \mathbb{C}[G] \to \mathbb{C}[G]$

unit $- i: \mathbb{C} \to \mathbb{C}[G]$, $1 \mapsto 1$ $\quad$ counit $- \varepsilon: \mathbb{C}[G] \to \mathbb{C}$, $f \mapsto f(e)$

Overall $(\mathbb{C}[G], m, i, \Delta, \varepsilon, S)$ – is Hopf algebra

- $(A, m, i, \Delta, \varepsilon, S)$

assosiativity

$$\begin{tikzcd}
 & A\otimes A \arrow[dr, "m"] & \\
A\otimes A\otimes A \arrow[ur, "id\otimes m"] \arrow[dr, "m\otimes id"'] & \circlearrowright & A \\
 & A\otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

coassosiativity

$$\begin{tikzcd}
 & A\otimes A \arrow[dr, "\Delta\otimes id"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & \circlearrowright & A\otimes A\otimes A \\
 & A\otimes A \arrow[ur, "id\otimes\Delta"'] &
\end{tikzcd}$$

Δ is homomorphism

$$\Delta(ab) = \Delta(a)\cdot\Delta(b)$$

$$\begin{tikzcd}
A\otimes A \arrow[r, "\Delta\otimes\Delta"] \arrow[d, "m"'] & A\otimes A\otimes A\otimes A \arrow[d, "m_{13}\otimes m_{24}"] \\
A \arrow[r, "\Delta"] & A\otimes A
\end{tikzcd}$$

Axioms with i, ε, S

- Quantization $\quad \mathbb{C}[G] = (A_0, m_0, i_0, \Delta_0, \varepsilon_0, S_0)$

$$\downarrow$$

$$(A, m, i, \Delta, \varepsilon, S)$$

$A = A_0[[\hbar]]$ $\quad m, \Delta, \ldots$ $\hbar$ linear

$m \equiv m_0, \quad \Delta \equiv \Delta_0, \ldots \ldots$ mod $\hbar$

- For short $\quad m_0(a,b) = ab \qquad \Delta_0(a) = a_{(1)} \otimes a_{(2)}$

$m(a,b) = a*b = ab + \hbar m_1(a,b) + O(\hbar)$

$m_1(a,b) - m_1(b,a) = \{a,b\}$

- $\Delta(a*b - b*a) = \hbar \Delta_0(\{a,b\}) + O(\hbar^2)$

$$\parallel$$

$\Delta(a)*\Delta(b) - \Delta(b)*\Delta(a) = (a_{(1)}*b_{(1)}) \otimes (a_{(2)}*b_{(2)}) - (b_{(1)}*a_{(1)}) \otimes (b_{(2)}*a_{(2)}) + O(\hbar^2)$

$= \hbar\big(m_1(a_{(1)}, b_{(1)}) \otimes a_{(2)}b_{(2)} + a_{(1)}b_{(1)} \otimes m_1(a_{(2)}, b_{(2)}) - m_1(b_{(1)}, a_{(1)}) \otimes b_{(2)}a_{(2)}$

$- b_{(1)}a_{(1)} \otimes m_1(b_{(2)}, a_{(2)})\big) + O(\hbar^2) = \hbar\big(\{a_{(1)}, b_{(1)}\} \otimes a_{(2)}b_{(2)} + a_{(1)}b_{(1)} \otimes \{a_{(2)}, b_{(2)}\}\big) + O(\hbar^2)$

Hence $\Delta_0\{a,b\} = \{a_{(1)}, b_{(1)}\} \otimes a_{(2)} b_{(2)} + a_{(1)} b_{(1)} \otimes \{a_{(2)}, b_{(2)}\}$

- Compare to P.L. condition

$$\{\varphi,\psi\}(gh) = \{\varphi,\psi\}(gh)\Big|_{h\text{-fixed}} + \{\varphi,\psi\}(gh)\Big|_{g\text{-fixed}}$$

If $\varphi(gh) = \varphi_{(1)}(g)\varphi_{(2)}(h)$, $\psi(gh) = \psi_{(1)}(g)\psi_{(2)}(h)$

$$\{\varphi(gh), \psi(gh)\} = \{\varphi_{(1)}, \psi_{(1)}\}(g)\varphi_{(2)}\psi_{(2)}(h) + \varphi_{(1)}\psi_{(1)}(g)\{\varphi_{(2)}(h), \psi_{(2)}(h)\}.$$

Quantization requires P.L. property

- <u>Trivial Example</u> G, $\Pi = 0$ $\mathbb{C}[G] \rightsquigarrow \mathbb{C}[G] \otimes \mathbb{C}[[\hbar]]$

$G^* = \mathfrak{g}^*$ $\mathbb{C}[G^*] = S(\mathfrak{g}) \rightsquigarrow \mathcal{U}(\mathfrak{g}_\hbar)$

$\hbar$ rescalled comm

- $U(\mathfrak{g})$ $\quad \Delta(a) = a\otimes 1 + 1\otimes a, \quad \varepsilon(a)=0, \quad S(a)=-a, \quad a\in\mathfrak{g}$

- Dual to $\mathbb{C}[G]$ $\quad U(\mathfrak{g})\otimes\mathbb{C}[G]\to\mathbb{C}$

$a_i\in\mathfrak{g}$ $\qquad a_1\cdots a_n\otimes f \mapsto \frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_n} f\left(e^{t_1 a_1}\cdots e^{t_n a_n}\right)\Big|_{t_1=\ldots=t_n=0}$

right invariant differential operator

Properties $\quad (X, f_1 f_2) = (\Delta X, f_1\otimes f_2)$

$(X_1 X_2, f) = (X_1\otimes X_2, \Delta(f))$

- $\mathbb{C}[G]$ - commutative $\longleftrightarrow$ $U(\mathfrak{g})$ — cocommutative

$\Delta: U(\mathfrak{g})\to S^2 U(\mathfrak{g})$

- $U_\hbar(\mathfrak{g})$ quantization of $U(\mathfrak{g})$

Here $\Delta^{op} = \sigma_{12}\Delta$

$\sigma_{12}(b\otimes c) = c\otimes b$

$\forall a\in\mathfrak{g} \qquad \delta(a) = \dfrac{\Delta(a)-\Delta^{op}(a)}{\hbar} \mod \hbar$

Th $(\mathfrak{g}, \delta)$ - bialgebra Lie

Proof $\delta(a) = \sum B_i \otimes C_i \subset \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$

? Problem

Problem $B \in \mathfrak{g} \iff B \in \mathcal{U}(\mathfrak{g})$, $\Delta_0(B) = B \otimes 1 + 1 \otimes B$

$$\sum \Delta_0 B_i \otimes C_i = \frac{1}{\hbar}\left(\Delta \otimes 1 (\Delta - \sigma_{12}\Delta) a\right) \mod \hbar$$

$$= \frac{1}{\hbar}\left(\left(1 \otimes \Delta - \sigma_{23}\, 1 \otimes \Delta\right)\Delta a + \sigma_{23}\left(\Delta \otimes 1 - \sigma_{12}\, \Delta \otimes 1\right)\Delta a\right) \mod \hbar$$

$$= (1 \otimes \delta)\Delta_0 a + \sigma_{23}(\delta \otimes 1)\Delta_0 a = (1 \otimes \delta + \sigma_{23}\, \delta \otimes 1)(a \otimes 1 + 1 \otimes a)$$

$$= \sum 1 \otimes B_i \otimes C_i + B_i \otimes 1 \otimes C_i$$

Hence $\Delta : \mathfrak{g} \to \Lambda^2 \mathfrak{g}$

Problem* CoJacobi + cocycle for δ $\square$

- Conclusion

Deformation of $\mathbb{C}[G]$ — P.L. str on G

Deformation of $\mathcal{U}(\mathfrak{g})$ — L. bialgebra str on $\mathfrak{g}$

Th (Etingof - Kazhdon) $\exists$ canonical quantization of $\forall$ Lie bialgebra

- **Big Example** $U_\hbar(sl_2)$

  - $[h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h$

    cobracket $\quad r = e \wedge f$

    $\delta(e) = e\wedge h, \quad \delta(h)=0 \quad \delta(f) = f\wedge h$

  - $\mathfrak{b}_+, \ \mathfrak{b}_-$ - subbialgebras

- $U_\hbar(\mathfrak{b}_+) \qquad [H,E]=2E \qquad \Delta H = H\otimes 1 + 1 \otimes H$

$\Delta E = E \otimes g_2(H) + g_1(H)\otimes E \qquad g_i(H) = 1 + O(\hbar) \qquad i=1,2$

$$(\Delta\otimes 1)\Delta(E) = E\otimes g_2(H)\otimes g_2(H) + g_1(H)\otimes E\otimes g_2(H) + \Delta(g_1(H))\otimes E$$

$$\parallel$$

$$(1\otimes\Delta)\Delta(E) = E\otimes\Delta(g_2(H)) + g_1(H)\otimes E\otimes g_2(H) + g_1(H)\otimes g_1(H)\otimes E$$

Hence $\Delta g_i(H) = g_i(H)\otimes g_i(H) \quad i=1,2$

- Problem Let $g(H) = 1 + O(\hbar) \in \mathcal{U}(\mathfrak{h})[[\hbar]]$ is group-like element (i.e. $\Delta g = g\otimes g$). Then $g(H) = \exp(\alpha\hbar H)$, $\alpha\in\mathbb{C}[[\hbar]]$

Hence $\Delta E = E\otimes e^{\hbar\alpha_2 H} + e^{\hbar\alpha_1 H}\otimes E$

- $\Delta(E) - \Delta^{op}(E) = \hbar\,\delta(E) + o(\hbar)$ Hence $(\alpha_2-\alpha_1)(E\otimes H - H\otimes E) = \delta(E) = E\otimes H - H\otimes E \Rightarrow$
  $\alpha_2 - \alpha_1 = 1$

- Rescaling $\tilde{E} = e^{\hbar\alpha H}E \Rightarrow \Delta(\tilde{E}) = \Delta(e^{\alpha\hbar H})\Delta E = \tilde{E}\otimes e^{\hbar(\alpha_1+\alpha)H} + e^{\hbar(\alpha_2+\alpha)H}\otimes\tilde{E}$

We can take $\Delta(E) = E\otimes e^{\hbar H} + 1\otimes E$

- $\mathcal{U}_h(\mathfrak{b}_-)$ H, F $[H,F]=-2F$
$\Delta H = H\otimes 1 + 1\otimes H$, $\Delta F = F\otimes 1 + e^{-\hbar H}\otimes F$

- $\mathcal{U}_\hbar(\mathfrak{sl}_2)$ generated by E, H, F
Know $\Delta(E)$, $\Delta(H)$, $\Delta(F)$, $[H,E]$, $[H,F]$

Problem Show that relation $[E,F]=\dfrac{e^{\hbar H}-e^{-\hbar H}}{e^{\hbar}-e^{-\hbar}}$ agrees with Δ.
Hint Use relations $E\varphi(H)=\varphi(H-2)E$, $F\varphi(H)=\varphi(H+2)F$, for any function φ.

Conclusion: we found $\mathcal{U}_\hbar(\mathfrak{sl}_2)$

- Let $c = fe + \dfrac{h(h+2)}{4} = ef + \dfrac{h(h-2)}{4} = \dfrac{ef+fe}{2} + \dfrac{h^2}{4}$
quadratic Casimir

Problem $\exists$ homomorphism $\mathcal{U}_\hbar(\mathfrak{sl}_2) \to \mathcal{U}(\mathfrak{sl}_2)[[\hbar]]$
$E \mapsto e$ $H \mapsto h$ $F \mapsto \varphi(h,c)f$

Hint $[E,F] \mapsto e\,\varphi(h,c) f - \varphi(h,c) fe =$

$$= \varphi(h-2,c) ef - \varphi(h,c) fe = \varphi(h-2)\left(c - \frac{h(h-2)}{4}\right) - \varphi(h,c)\left(c - \frac{h(h+2)}{4}\right)$$

$$\varphi(h,c) = \Psi(h,c)\left(c - \frac{h(h+2)}{2}\right)^{-1}\left(e^{\hbar} - e^{-\hbar}\right)$$

$$\Psi(h-2,c) - \Psi(h,c) = e^{\hbar h} - e^{-\hbar h}$$

Also exists inverse map

- Let $q = e^{\hbar}$, $K^{1/2} = e^{\hbar H/2}$
- $KE = q^2 EK$ $\quad KF = q^{-2} FK$ $\quad [E,F] = \dfrac{K - K^{-1}}{q - q^{-1}}$
- $\Delta E = E \otimes K + 1 \otimes E$ $\quad \Delta K = K \otimes K$ $\quad \Delta F = F \otimes 1 + K^{-1} \otimes F$

$\mathcal{U}_q(sl_2)$

References

- Chary, Pressley A guide to quantum groups
  Ch. 5
- Etingof, Schiffmann Lectures on quantum groups Ch 9