

Introduction to Quantum Groups  
Lecture 5  
Quantum groups and algebras. Example  $SL_2$

● Quantize  $G \rightsquigarrow$  Quantize  $\mathbb{C}[G]$

•  $G$  group  $G \times G \rightarrow G$   $G \rightarrow G$   
 $(g, h) \mapsto gh$   $g \mapsto g^{-1}$

• Hence  
coproduct  $\Delta: \mathbb{C}[G] \rightarrow \mathbb{C}[G] \otimes \mathbb{C}[G] (= \mathbb{C}[G \times G])$

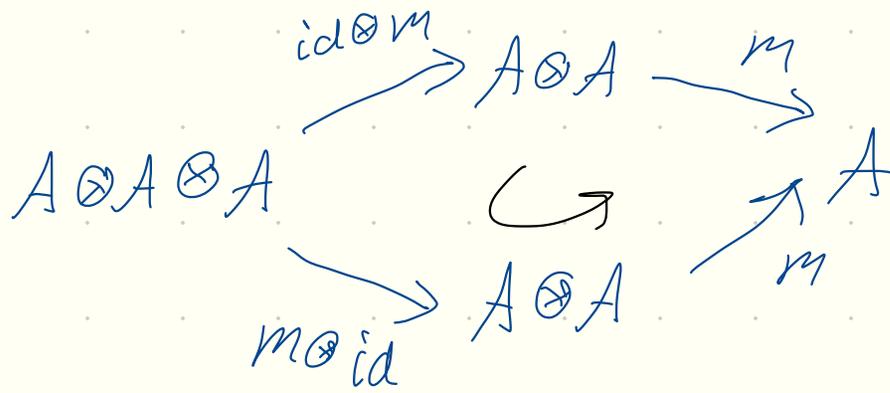
antipode  $S: \mathbb{C}[G] \rightarrow \mathbb{C}[G]$

unit  $- i: \mathbb{C} \rightarrow \mathbb{C}[G]$   $1 \mapsto 1$  counit  $- \epsilon: \mathbb{C}[G] \rightarrow \mathbb{C}$   $f \mapsto f(e)$

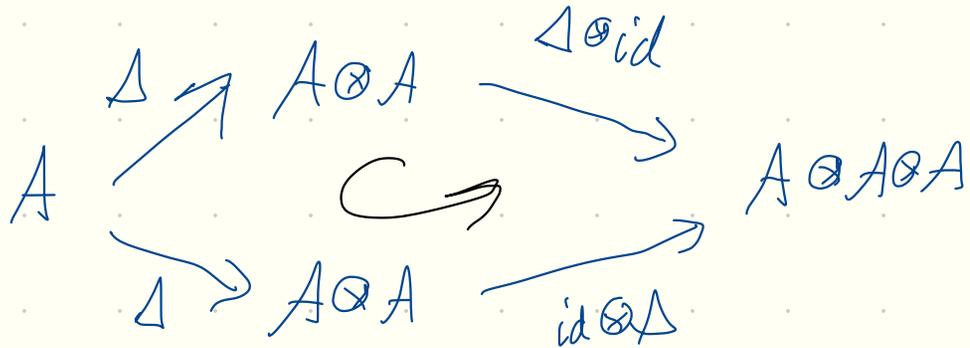
Overall  $(\mathbb{C}[G], m, i, \Delta, \epsilon, S)$  - is Hopf algebra

●  $(A, m, i, \Delta, \epsilon, S)$

associativity

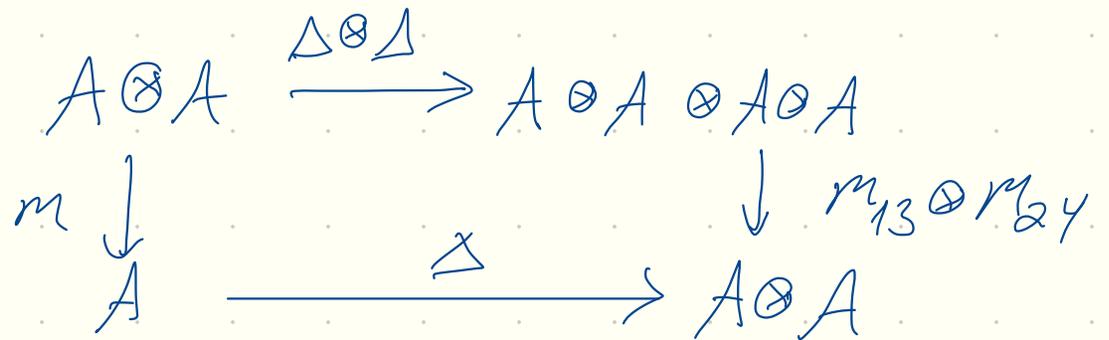


coassociativity



$$\Delta(ab) = \Delta(a) \cdot \Delta(b)$$

$\Delta$  is homomorphism



Axioms with  $i, \epsilon, S$

● Quantization

$$\mathbb{C}[A] = (A_0, m_0, i_0, \Delta_0, \varepsilon_0, S_0)$$



$$(A, m, i, \Delta, \varepsilon, S)$$

$$A = A_0 \llbracket \hbar \rrbracket \quad m, \Delta, \dots \quad \hbar \text{ linear}$$

$$m \equiv m_0, \quad \Delta \equiv \Delta_0, \dots \quad \text{mod } \hbar$$

• For short  $m_0(a, b) = ab$        $\Delta_0(a) = a_{(1)} \otimes a_{(2)}$

$$m(a, b) = a * b = ab + \hbar m_1(a, b) + O(\hbar^2)$$

$$m_1(a, b) - m_1(b, a) = \langle a, b \rangle$$

•  $\Delta(a * b - b * a) = \hbar \Delta(\langle a, b \rangle) + O(\hbar^2)$

||

$$\Delta(a) * \Delta(b) - \Delta(b) * \Delta(a) = (a_{(1)} * b_{(1)}) \otimes (a_{(2)} * b_{(2)}) - (b_{(1)} * a_{(1)}) \otimes (b_{(2)} * a_{(2)}) + O(\hbar^2)$$

$$= \hbar (m_1(a_{(1)}, b_{(1)}) \otimes a_{(2)} b_{(2)} + a_{(1)} b_{(1)} \otimes m_1(a_{(2)}, b_{(2)}) - m_1(b_{(1)}, a_{(1)}) \otimes b_{(2)} a_{(2)} - b_{(1)} a_{(1)} \otimes m_1(b_{(2)}, a_{(2)}) + O(\hbar^2) = \hbar (\langle a_{(1)}, b_{(1)} \rangle \otimes a_{(2)} b_{(2)} + a_{(1)} b_{(1)} \otimes \langle a_{(2)}, b_{(2)} \rangle) + O(\hbar^2)$$

$$= \hbar (\langle a, b \rangle) + O(\hbar^2)$$

Hence  $\Delta_0(\alpha, \beta) = \alpha_{(1)} \beta_{(1)} \otimes \alpha_{(2)} \beta_{(2)} + \alpha_{(1)} \beta_{(1)} \otimes \alpha_{(2)} \beta_{(2)}$

- Compare to P.L. condition

$$\alpha(\psi, \psi)(gh) = \alpha(\psi, \psi)(gh) \Big|_{h\text{-fixed}} + \alpha(\psi, \psi)(gh) \Big|_{g\text{-fixed}}$$

If  $\psi(gh) = \psi_{(1)}(g) \psi_{(2)}(h)$ ,  $\psi(gh) = \psi_{(1)}(g) \psi_{(2)}(h)$

$$\alpha(\psi(gh), \psi(gh)) = \alpha(\psi_{(1)}, \psi_{(1)})(g) \psi_{(2)} \psi_{(2)}(h) + \psi_{(1)} \psi_{(1)}(g) \alpha(\psi_{(2)}, \psi_{(2)})(h)$$

Quantization requires P.L. property

- Trivial Example  $G, \hbar=0 \quad \mathbb{C}[a] \rightarrow \mathbb{C}[a] \otimes \mathbb{C}[\hbar]$

$$G^* = \mathbb{C}[a]^*$$

$$\mathbb{C}[G^*] = S(\mathbb{C}[a]^*)$$

$$\rightarrow U(\mathbb{C}[a]^*_{\hbar})$$

$\hbar$  rescaled comm

●  $U(\mathcal{A})$       $\Delta(a) = a \otimes 1 + 1 \otimes a$ ,      $\epsilon(a) = 0$ ,      $S(a) = -a$ ,      $a \in \mathcal{A}$

• Dual to  $\mathbb{C}[a]$       $U(\mathcal{A}) \otimes \mathbb{C}[a] \rightarrow \mathbb{C}$

$a_i \in \mathcal{A}$       $a_1 \dots a_n \otimes f \mapsto \left. \frac{\partial}{\partial t_1} \dots \frac{\partial}{\partial t_n} f(e^{t_1 a_1} \dots e^{t_n a_n}) \right|_{t_1 = \dots = t_n = 0}$

right invariant differential operator

Properties

$(X, f_1 f_2) = (\Delta X, f_1 \otimes f_2)$   
 $(X_1 X_2, f) = (X_1 \otimes X_2, \Delta(f))$

•  $\mathbb{C}[a]$  - commutative      $\iff$       $U(\mathcal{A})$  - cocommutative

$\Delta: U(\mathcal{A}) \rightarrow S^2 U(\mathcal{A})$

●  $U_{\hbar}(\mathcal{A})$      quantization of  $U(\mathcal{A})$

$\forall a \in \mathcal{A}$       $\delta(a) = \frac{\Delta(a) - \Delta^{op}(a)}{\hbar} \pmod{\hbar}$

Here

$\Delta^{op} = G_{12} \Delta$   
 $G_{12}(b \otimes c) = c \otimes b$

Th  $(\mathfrak{A}, d)$  - bialgebra Lie

PROOF  $\delta(a) = \sum B_i \otimes C_i \in \mathcal{U}(\mathfrak{A}) \otimes \mathcal{U}(\mathfrak{A})$

? Problem

Problem  $B \in \mathfrak{A} \iff B \in \mathcal{U}(\mathfrak{A}), \Delta_0(B) = B \otimes 1 + 1 \otimes B$

$$\sum \Delta_0 B_i \otimes C_i = \frac{1}{\hbar} (\Delta \otimes 1 (\Delta - G_{12} \Delta) a) \pmod{\hbar}$$

$$= \frac{1}{\hbar} ((1 \otimes \Delta - G_{23} 1 \otimes \Delta) \Delta a + G_{23} (\Delta \otimes 1 - G_{12} \Delta \otimes 1) \Delta a) \pmod{\hbar}$$

$$= (1 \otimes d) \Delta_0 a + G_{23} (d \otimes 1) \Delta_0 a = (1 \otimes d + G_{23} d \otimes 1) (a \otimes 1 + 1 \otimes a)$$

$$= \sum 1 \otimes B_i \otimes C_i + B_i \otimes 1 \otimes C_i$$

Hence  $\Delta : \mathfrak{A} \rightarrow \Lambda^2 \mathfrak{A}$

Problem\* CoJacobi + Cocycle for  $\delta$



● Conclusion

Deformation of  $\mathbb{C}[a]$  — P.L. str on  $G$

Deformation of  $\mathcal{U}(\mathfrak{A})$  — L. bialgebra str on  $\mathfrak{A}$

$\hbar$  (Etingof-Kazhdan)  $\exists$  canonical quantization of  
 $\forall$  Lie bialgebras

● Big Example  $U_{\hbar}(\mathfrak{sl}_2)$

•  $[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h$

cobracket  $\Gamma = e \wedge f$

$\delta(e) = e \wedge h, \quad \delta(h) = 0, \quad \delta(f) = f \wedge h$

•  $\mathfrak{h}_+, \mathfrak{h}_-$  - subbialgebras

●  $U_{\hbar}(\mathfrak{h}_+)$   $[H, E] = 2E$   $\Delta H = H \otimes 1 + 1 \otimes H$

$\Delta E = E \otimes g_2(\hbar) + g_1(\hbar) \otimes E$   $g_i(\hbar) = 1 + O(\hbar)$   $i=1,2$

$$(\Delta \otimes 1) \Delta(E) = E \otimes g_2(H) \otimes g_2(H) + g_1(H) \otimes E \otimes g_2(H) + \Delta(g_1(H)) \otimes E$$

$$(1 \otimes \Delta) \Delta(E) = E \otimes \Delta(g_2(H)) + g_1(H) \otimes E \otimes g_2(H) + g_1(H) \otimes g_1(H) \otimes E$$

Hence  $\Delta g_i(H) = g_i(H) \otimes g_i(H) \quad i=1,2$

Problem Let  $g(H) = 1 + o(\hbar) \in \mathcal{U}(\hbar)[[\hbar]]$  is group-like element (i.e.  $\Delta g = g \otimes g$ ). Then  $g(H) = \exp(\alpha \hbar H)$ ,  $\alpha \in \mathbb{C}[[\hbar]]$

Hence  $\Delta E = E \otimes e^{\hbar \alpha_2 H} + e^{\hbar \alpha_1 H} \otimes E$

$\Delta(E) - \Delta^{op}(E) = \hbar \delta(E) + o(\hbar)$  Hence

$$(\alpha_2 - \alpha_1)(E \otimes H - H \otimes E) = \delta(E) = E \otimes H - H \otimes E \Rightarrow$$

$$\alpha_2 - \alpha_1 = 1$$

Rescaling  $\tilde{E} = e^{\hbar \alpha H} E \Rightarrow \Delta(\tilde{E}) = \Delta(e^{\alpha \hbar H}) \Delta E = \tilde{E} \otimes e^{\hbar(\alpha_1 + \alpha)H} + e^{\hbar(\alpha_2 + \alpha)H} \otimes \tilde{E}$

We can take  $\Delta(E) = E \otimes e^{\hbar H} + 1 \otimes E$

- $U_{\hbar}(\mathfrak{sl}_2)$   $H, F$   $[H, F] = -2F$   
 $\Delta H = H \otimes 1 + 1 \otimes H, \quad \Delta F = F \otimes 1 + e^{-\hbar H} \otimes F$

- $U_{\hbar}(\mathfrak{sl}_2)$  generated by  $E, H, F$   
 know  $\Delta(E), \Delta(H), \Delta(F), [H, E], [H, F]$

Problem Show that relation  $[E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}$  agrees with  $\Delta$ .  
Hint Use relations  $E\varphi(H) = \varphi(H-2)E, F\varphi(H) = \varphi(H+2)F$ , for any function  $\varphi$ .

Conclusion: we found  $U_{\hbar}(\mathfrak{sl}_2)$

- Let  $c = fe + \frac{\hbar(\hbar+2)}{4} = ef + \frac{\hbar(\hbar-2)}{4} = \frac{ef + fe}{2} + \frac{\hbar^2}{4}$   
 quadratic Casimir

Problem  $\exists$  homomorphism  $U_{\hbar}(\mathfrak{sl}_2) \rightarrow U(\mathfrak{sl}_2)[[\hbar]]$   
 $E \mapsto e \quad H \mapsto h \quad F \mapsto \varphi(h, c)f$

Hint  $[E, F] \mapsto e \Phi(h, c) f - \Phi(h, c) f e =$

$$= \Phi(h-2, c) e f - \Phi(h, c) f e = \Phi(h-2) \left( c - \frac{\hbar(h-2)}{4} \right) - \Phi(h, c) \left( c - \frac{\hbar(h+2)}{4} \right)$$

$$\Phi(h, c) = \Psi(h, c) \left( c - \frac{\hbar(h+2)}{2} \right)^{-1} (e^{\frac{\hbar}{2}} - e^{-\frac{\hbar}{2}})$$

$$\Psi(h-2, c) - \Psi(h, c) = e^{\frac{\hbar}{2} h} - e^{-\frac{\hbar}{2} h}$$

Also exists inverse map

● Let  $q = e^{\frac{\hbar}{2}}$ ,  $k^{1/2} = e^{\frac{\hbar}{2} h/2}$

•  $KE = q^2 EK$      $KF = q^{-2} FK$      $[E, F] = \frac{k - k^{-1}}{q - q^{-1}}$

•  $\Delta E = E \otimes k + 1 \otimes E$      $\Delta K = K \otimes k$      $\Delta F = F \otimes 1 + k^{-1} \otimes F$

$$U_q(\mathfrak{sl}_2)$$

# References

- Chary, Pressley A guide to quantum groups  
Ch. 5
- Etingof, Schiffmann Lectures on quantum  
groups Ch 9