Introduction to Quantum Groups

Lecture 6

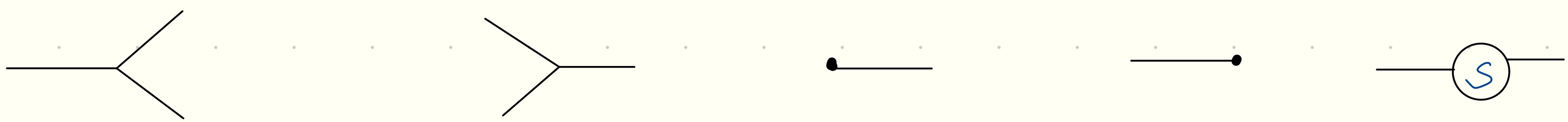
Hopf algebras

Hopf algebra

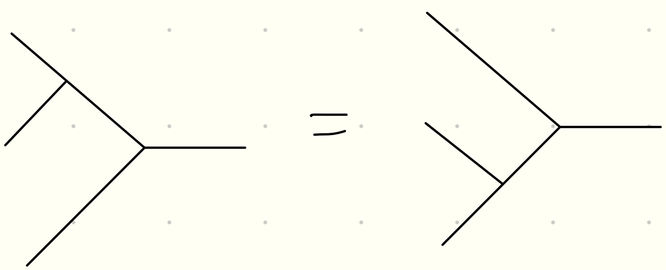
$(A, m, i, \Delta, \varepsilon, S)$

Pictures

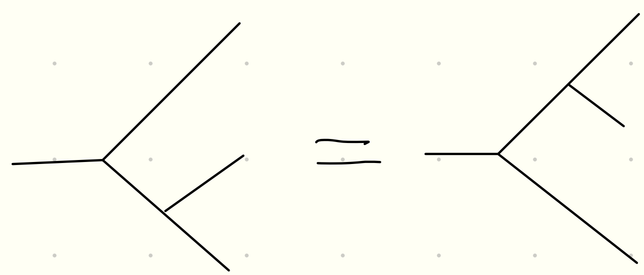
$\Delta : A \to A \otimes A$ $\quad$ $m : A \otimes A \to A$ $\quad$ $i : \mathbb{C} \to A$ $\quad$ $\varepsilon : A \to \mathbb{C}$ $\quad$ $S : A \to A$



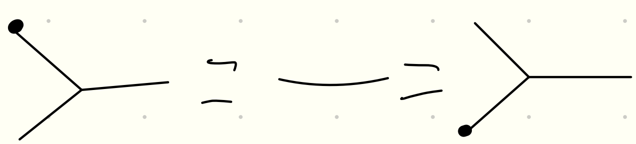
- Assosiat



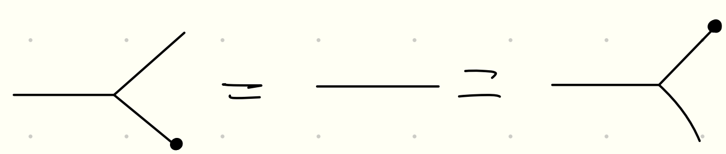
Coassosiat



unit

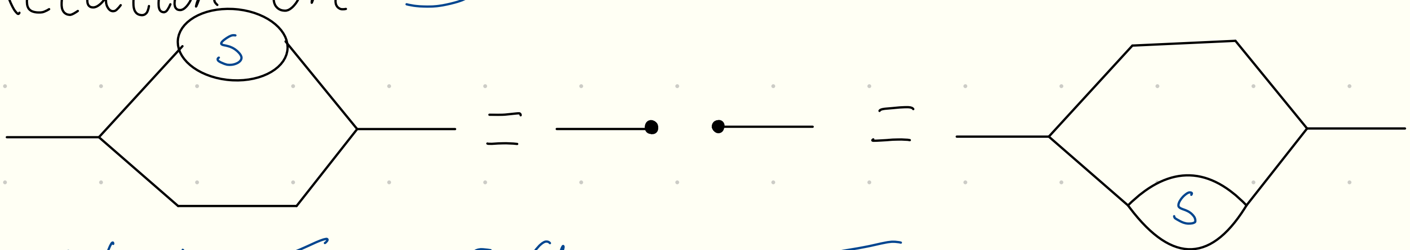


counit



Δ is algebra homomorphism $\Delta(ab)=\Delta(a)\Delta(b)$

Relation on S



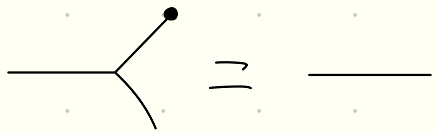
$\Delta(a)=\sum a_{(1)}\otimes a_{(2)}$ $\qquad$ $\sum S(a_{(1)})a_{(2)} = i\,\varepsilon(a) = \sum a_{(1)} S(a_{(2)})$

- $U_\hbar(sl_2)$ $\qquad$ E, H, F

$[H,E]=2E$, $\qquad$ $[H,F]=-2F$, $\qquad$ $[E,F]=\dfrac{e^{\hbar H}-e^{-\hbar H}}{e^{\hbar}-e^{-\hbar}}$

$\Delta(E)=E\otimes e^{\hbar H}+1\otimes E$, $\qquad$ $\Delta H = H\otimes 1+1\otimes H$, $\qquad$ $\Delta F = F\otimes 1 + e^{-\hbar H}\otimes F$

• i, ε, S



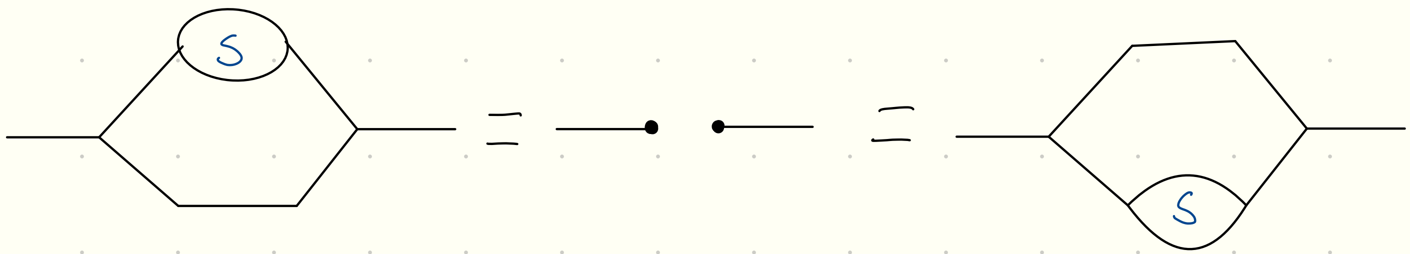
$$(\varepsilon\otimes id)\Delta(a) = \varepsilon_1(\Delta(a)) = a$$

$a=1$ $\quad (\varepsilon\otimes id)(1\otimes 1) = 1$ $\qquad \varepsilon(1)=1$

$a=H$ $\quad \varepsilon(H)+\varepsilon(1)H = H$ $\qquad \varepsilon(H)=0$

$a=E$ $\quad \varepsilon(E)e^{\hbar H}+E = E$ $\qquad \varepsilon(E)=0$

$\varepsilon(F)=0$



$$0 = i\varepsilon(E) = m(S\otimes id)\Delta E = m(S\otimes id)(E\otimes e^{\hbar H} + 1\otimes E)$$
$$= S(E)e^{\hbar H} + E$$

$$0 = i\varepsilon(H) = m(S\otimes 1)\Delta H = m(S\otimes 1)(H\otimes 1 + 1\otimes H) = S(H) + H$$

Hence

$S(E) = -Ee^{-\hbar H}$ $\qquad S(H) = -H$ $\qquad S(F) = -e^{\hbar H}F$

- Compute S^2

$$S^2(E) = -S(e^{-\hbar H})S(E) = e^{\hbar H} E e^{-\hbar H}$$

Remark If A is comm or cocomm then $S^2 = id$

In our case S^2 is conjugation by $e^{\hbar H}$

$$S^2(H) = H = e^{\hbar H} H e^{-\hbar H}$$

$$S^2(F) = -S(F)S(e^{\hbar H}) = e^{\hbar H} F e^{-\hbar H}$$

Problem S is antihom of algebra and coalgebra

Hint Let $(\Delta \otimes id)\Delta(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$ $\quad (\Delta \otimes id)\Delta(b) = \sum b_{(1)} \otimes b_{(2)} \otimes b_{(3)}$

Then $S(ab) = \sum S(a_{(1)} b) a_{(2)} S(a_{(3)})$

$$= \sum S(a_{(1)} b_{(1)}) a_{(2)} b_{(2)} S(b_{(3)}) S(a_{(3)}) = S(b) S(a)$$

Remark Usually we assume that S is bijective

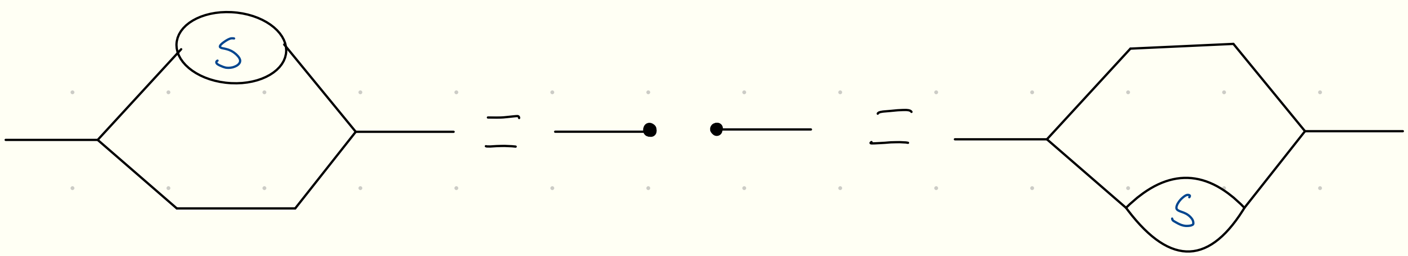
- **Def** $f, g : A \to A$ convolution product $f \bullet g = m(f \otimes g)\Delta$

  $f \bullet g(a) = f(a_{(1)})\, g(a_{(2)})$

  $i\varepsilon : A \to A$ – unit for convolution product

Example $A = \mathbb{C}[G]$ $\qquad (f \bullet g)(x) = \sum\limits_{yz = x} f(y)\, g(z)$

$i\varepsilon(x) = \delta_{x=e}$

- 

Defining property of S is equivalent to

$S \bullet id = i\varepsilon \implies S$ is determined by Δ, m

- Representations of Hopf algebras

$\Delta : A \to A \otimes A$ | $\varepsilon : A \to \mathbb{C}$

$V_1, V_2 - Mod_A \rightsquigarrow V_1 \otimes V_2 \in Mod_A$ | $\mathbb{C} \in Mod\,A$

Rem Tensor product of group reps or Lie algebra reps $\Longleftrightarrow$ $\exists$ of Δ

- $S : A \to A$

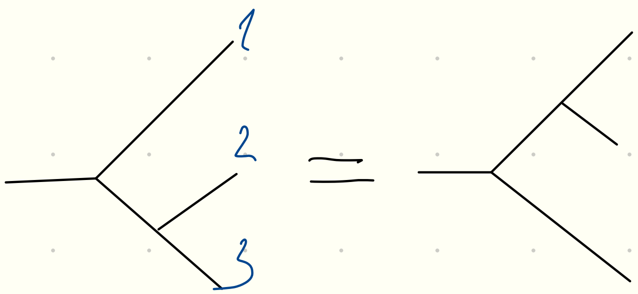
$V \in Mod_A$ $\qquad V^*$ $\qquad \rho_{V^*}(a) = \rho_V(S(a))^*$

$\qquad {}^*V$ $\qquad \rho_{{}^*V}(a) = \rho_V(S^{-1}(a))^*$

${}^*(V^*) = V = ({}^*V)^*$ $\qquad (V^*)^* \overset{?}{=} V$

Rem If S^2 is conjugation by u then ${}^*V \simeq V^*$

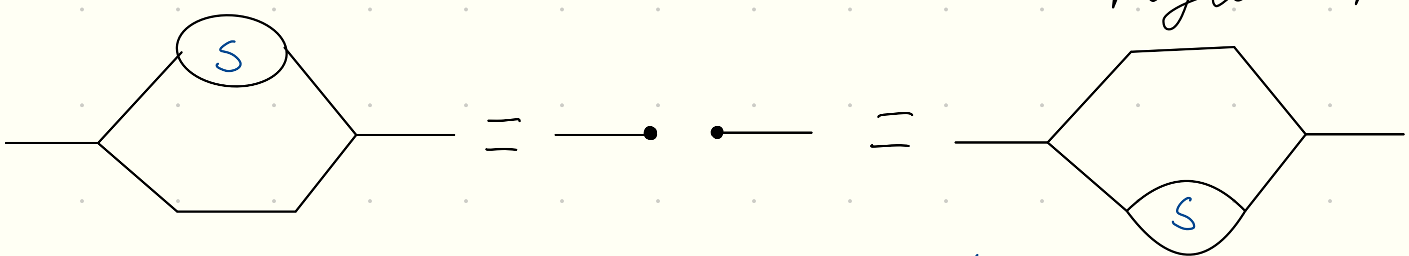
- Properties



monoidal category

$V_1 \otimes (V_2 \otimes V_3) = (V_1 \otimes V_2) \otimes V_3$

rigid monoidal category

- 

$\exists \quad V^* \otimes V \to \mathbb{C} \qquad \mathbb{C} \to V \otimes V^*$

Problem a) Show existance of maps above

b) Show that $(V \otimes W)^* = W^* \otimes V^*$

Hint Use problem above

Reps of $U_\hbar(sl_2)$

- Reps of sl_2 $\quad [h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h$

$$L_\ell = \langle v_\ell, v_{\ell-2}, \dots, v_{-\ell} \rangle$$

$e v_\ell = 0 \quad h v_\ell = \ell v_\ell \qquad v_{\ell-2k} = f^k v_\ell$

$$f v_m = v_{m-2}, \quad h v_m = m v_m, \quad e v_m = \left(\frac{\ell-m}{2}\right)\left(\frac{\ell+m+2}{2}\right) v_{m+2}$$

Another normalization $\quad \tilde{v} = \frac{1}{k!} f^k v_\ell$

$$f \tilde{v}_m = \left(\frac{\ell-m+2}{2}\right) v_{m-2} \quad h\tilde{v}_m = m\tilde{v}_m, \quad e\tilde{v}_m = \left(\frac{\ell+m+2}{2}\right)\tilde{v}_{m+2}$$

- Reps $U_\hbar(sl_2)$

$$[H,E]=2E, \qquad [H,F]=-2F, \qquad [E,F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}$$

$$L_\ell = \langle v_\ell, v_{\ell-2}, \dots, v_{-\ell} \rangle$$

$E v_\ell = 0 \quad H v_\ell = \ell v_\ell \qquad\qquad v_{\ell-2k} = F^k v_\ell$

<u>Problem</u> a) Find formulas for action of E, H, F

b) Define basis $\tilde{v}_m$

<u>Example</u> $L_1 = \mathbb{C}^2 = \langle v_1, v_{-1} \rangle$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

same as before since $\dfrac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}} = H$ for $H = \pm 1$

- Tensor product for $U_\hbar sl_2$.

$$\Delta(E) = E \otimes e^{\hbar H} + 1 \otimes E, \quad \Delta H = H \otimes 1 + 1 \otimes H, \quad \Delta F = F \otimes 1 + e^{-\hbar H} \otimes F$$

$L_1 \otimes L_1 \qquad v_1 \otimes v_1 \qquad v_1 \otimes v_{-1} \qquad v_{-1} \otimes v_1 \qquad v_{-1} \otimes v_{-1}$

$$H \mapsto \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \qquad E \mapsto \begin{pmatrix} 0 & 1 & e^{\hbar} & 0 \\ 0 & 0 & 0 & e^{-\hbar} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad F \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ e^{-\hbar} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & e^{\hbar} & 0 \end{pmatrix}$$

$L_1 \otimes L_1 = L_2 \oplus L_0 \qquad L_0 = \langle e^{\hbar} v_1 \otimes v_{-1} - v_{-1} \otimes v_1 \rangle$

- Δ vs $\Delta^{op} = P \circ \Delta \qquad P = \sigma_{12}$

$\Delta : A \to A \otimes A$ vs $\Delta^{op} : A \to A \otimes A$

$L_1 \otimes_{\Delta} L_1 = L_2 \oplus L_0 \qquad L_1 \otimes_{\Delta^{op}} L_1 = L_2 \oplus L_0$

$L_0 = \langle e^{\hbar} v_1 \otimes v_{-1} - v_{-1} \otimes v_1 \rangle \qquad L_0 = \langle v_1 \otimes v_{-1} - e^{\hbar} v_{-1} \otimes v_1 \rangle$

$$R \qquad \begin{array}{c} L_1 \otimes_\Delta L_1 \\ \big\downarrow \\ L_1 \otimes_{\Delta^{op}} L_1 \end{array} \qquad\qquad R = \begin{pmatrix} e^{\hbar} & 0 & 0 & 0 \\ 0 & 1 & e^{\hbar}-e^{-\hbar} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\hbar} \end{pmatrix}$$

R-matrix $\quad R: V_1 \otimes_\Delta V_2 \to V_2 \otimes_{\Delta^{op}} V_1$

$$\hat{R} = PR : V_1 \otimes_\Delta V_2 \longrightarrow V_2 \otimes_\Delta V_1$$

$$\hat{R} = \begin{pmatrix} e^{\hbar} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & e^{\hbar}-e^{-\hbar} & 0 \\ 0 & 0 & 0 & e^{\hbar} \end{pmatrix} \qquad (\hat{R} - e^{\hbar})(\hat{R} + e^{-\hbar}) = 0$$

Hecke relation

<u>Problem*</u> a) Show directly that $L_1 \otimes L_\ell \simeq L_{\ell+1} \oplus L_{\ell-1}$ $\quad \ell \geq 1$

b) Show that $L_{\ell_1} \otimes L_{\ell_2} \simeq \displaystyle\bigoplus_{\substack{|\ell_1-\ell_2| \leq \ell \leq \ell_1+\ell_2 \\ \ell+\ell_1+\ell_2 \text{ -even}}} L_\ell$

References

- Chary, Pressley A guide to quantum groups
  Ch. 5
- Etingof, Gelaki, Nikshych, Ostrik Tensor categories
  Ch 2,4,5 (some parts of)