Introduction to Quantum Groups

Lecture 7

Quantum R matrices

- **Hopf algebras** $(A, m, i, \Delta, \varepsilon, S)$

- Mod_A $\quad \otimes \quad \mathbb{C} \quad V^* \quad {}^*V$

- **Want** $V \otimes W \simeq W \otimes V$ $\qquad \tilde{R}_{V,W} = P R_{V,W}$

$\Delta^{op} = P\Delta$ $\qquad V \otimes_{\Delta} W \simeq V \otimes_{\Delta^{op}} W$ $\qquad R_{V,W}$

Conditions

$$
\begin{tikzcd}
(V_1 \otimes V_2) \otimes V_3 \arrow[rr, "\tilde{R}_{V_1 \otimes V_2, V_3}"] \arrow[dr, "Id \otimes \tilde{R}_{V_2,V_3}"'] & \arrow[d, phantom, "\circlearrowright"] & V_3 \otimes (V_1 \otimes V_2) \\
& V_1 \otimes V_3 \otimes V_2 \arrow[ur, "\tilde{R}_{V_1,V_3} \otimes Id"'] &
\end{tikzcd}
$$

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V_1 \otimes V_2 \otimes V_3 \arrow[rr, "\tilde{R}_{V_1, V_2 \otimes V_3}"] \arrow[dr, "\tilde{R}_{V_1,V_2} \otimes Id"'] & \arrow[d, phantom, "\circlearrowright"] & V_2 \otimes V_3 \otimes V_1 \\
& V_2 \otimes V_1 \otimes V_3 \arrow[ur, "Id \otimes \tilde{R}_{V_1,V_3}"'] &
\end{tikzcd}
$$

Want

● **Def** Quasitriangular str. on Hopf algebra A is invertible $R \in A \otimes A$ s.t.

- $R\,\Delta(x) = \Delta^{op}(x)\,R \quad \forall x \in A$
- $(\Delta \otimes \mathrm{Id})(R) = R_{13} R_{23}, \quad (\mathrm{Id} \otimes \Delta) R = R_{13} R_{12}$

(Notations $R = \sum a_i \otimes b_i \qquad R_{13} = \sum a_i \otimes 1 \otimes b_i$

$R_{23} = \sum 1 \otimes a_i \otimes b_i \qquad (\Delta \otimes \mathrm{Id})(R) = \sum \Delta(a_i) \otimes b_i$)

R is called universal R matrix

$\forall V, W \in \mathrm{Mod}_A \qquad (\pi_V \otimes \pi_W) R = R_{V,W} : V \otimes_{\Delta} W \to V \otimes_{\Delta^{op}} W$

second condition $\Longleftrightarrow$ triangles above

- $\underline{\text{Th}}$ Universal R matrix R satisfies

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \qquad \text{QYBE}$$

- $\underline{\text{Remark}}$ $\tilde{R} = PR$

Left side $\quad G_{12}\tilde{R}_{12}G_{13}\tilde{R}_{13}G_{23}\hat{R}_{23} = G_{12}G_{13}G_{23}\tilde{R}_{23}\hat{R}_{12}\hat{R}_{23}$

Right side $\quad G_{23}\tilde{R}_{23}G_{13}\tilde{R}_{13}G_{12}\hat{R}_{12} = G_{23}G_{13}G_{12}\tilde{R}_{12}\tilde{R}_{23}\hat{R}_{12}$

$$\tilde{R}_{23}\tilde{R}_{12}\hat{R}_{23} = \tilde{R}_{12}\tilde{R}_{23}\hat{R}_{12} \quad - \text{Braid relation}$$

- $\underline{\text{Remark}}$ In terms of Reps

$$\begin{tikzcd}
 & V_1\otimes V_3\otimes V_2 \arrow[r] & V_3\otimes V_1\otimes V_2 \arrow[dr] & \\
V_1\otimes V_2\otimes V_3 \arrow[ur] \arrow[dr] & & & V_3\otimes V_2\otimes V_1 \\
 & V_2\otimes V_1\otimes V_3 \arrow[r] & V_2\otimes V_3\otimes V_1 \arrow[ur] &
\end{tikzcd}$$

- **Proof** of QYBE $R = \sum a_i \otimes b_i = \sum c_i \otimes d_i$

$R\Delta(x) = \Delta^{op}(x) R \qquad (\Delta\otimes Id)(R) = R_{13}R_{23}, \qquad (Id\otimes\Delta)R = R_{13}R_{12}$

$$R_{12}R_{13}R_{23} = R_{12}(\Delta\otimes Id)R = \left(\sum a_i\otimes b_i\otimes 1\right)\left(\sum \Delta(c_i)\otimes d_i\right)$$
$$= \left(\sum \Delta^{op}(c_i)\otimes d_i\right)\left(\sum a_i\otimes b_i\otimes 1\right) = (\Delta^{op}\otimes Id)R\, R_{12} =$$
$$= G_{12}(\Delta\otimes Id)R\, G_{12}R_{12} = G_{12}R_{13}R_{23}G_{12}R_{12} = R_{23}R_{13}R_{12} \qquad \square$$

- **Prop** Let $U_\hbar(g)$ be QUE, with $R = 1 + \hbar r \pmod{\hbar^2}$

Then $r \in g\otimes g$ and $\delta(x) = ad_x r$

Proof $(\Delta\otimes Id)(R) = R_{13}R_{23}, \quad \Delta = \Delta_0 + \hbar(\ldots)$

$(\Delta_0\otimes Id)r = r_{13} + r_{23}$ If $r = \sum \alpha_i\otimes\beta_i$, then

$\sum \Delta_0(\alpha_i)\otimes\beta_i = \sum(\alpha_i\otimes 1 + 1\otimes\alpha_i)\otimes\beta_i$. Hence $\Delta_0(\alpha_i) = \alpha_i\otimes 1 + 1\otimes\alpha_i$

Hence $\alpha_i \in g$. Similarly $\beta_i \in g \Rightarrow r \in g\otimes g$

$$R\,\Delta(x) = \Delta^{op}(x)\,R \Rightarrow r\,\Delta_0(x) + \frac{\Delta(x)-\Delta_0(x)}{\hbar} = \frac{\Delta^{op}(x)-\Delta_0(x)}{\hbar} + \Delta_0(x)\,r \Rightarrow$$

$$\Rightarrow \delta(x) = \frac{\Delta(x)-\Delta^{op}(x)}{\hbar} = ad_x\, r \qquad \square$$

<u>Rem</u> QYBE R $\Rightarrow$ CYBE r

- $R\,\Delta(x) = \Delta^{op}(x)\,R \Rightarrow R_{21}\,\Delta^{op}(x) = \Delta(x)\,R_{21} \Rightarrow$
  $(R_{21})^{-1}\Delta(x) = \Delta^{op}(R_{21})^{-1} \Rightarrow R_{21}^{-1}$ intertwines Δ and Δ^{op}

<u>Def</u> R is unitary if $R R_{21} = Id$

<u>Rem</u> Classically unitary $\Rightarrow r + r_{21} = 0 \Rightarrow r \in \Lambda^2\mathfrak{g}$

<u>Def</u> R is triangular if R is quasitriangular and unitary.

- $\mathcal{U}_\hbar(sl_2)$ $\quad E, H, F$

$$[H,E]=2E, \quad [H,F]=-2F, \quad [E,F]=\frac{e^{\hbar H}-e^{-\hbar H}}{e^{\hbar}-e^{-\hbar}}$$

$$\Delta(E)=E\otimes e^{\hbar H}+1\otimes E, \quad \Delta H = H\otimes 1+1\otimes H, \quad \Delta F = F\otimes 1 + e^{-\hbar H}\otimes F$$

<u>Th</u> $\quad R = e^{\frac{1}{2}\hbar H\otimes H}\sum_{n=0}^{\infty} q^{\binom{n}{2}}\frac{(q-q^{-1})^n}{[n]_q!}E^n\otimes F^n$

Notations $q=e^{\hbar}$, $\binom{n}{2}=\frac{n(n-1)}{2}$, $[n]_q=\frac{q^n-q^{-n}}{q-q^{-1}}$, $[n]_q!=[n]_q\cdot[n-1]_q\cdot\ldots\cdot[1]_q$

Can be proven by direct computation
More conceptual proof Drinfeld double (later)

<u>Problem</u> Show that $\Delta^{op}(E)R = R\Delta(E)$

Hint We want $(E\otimes 1 + e^{\hbar H}\otimes E)\, e^{\frac{1}{2}\hbar H\otimes H} \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E^n \otimes F^n -$

$$- e^{\frac{1}{2}\hbar H\otimes H} \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E^n\otimes F^n (E\otimes e^{\hbar H} + 1\otimes E) = 0$$

$$\sum (\cdots) E^{n+1}\otimes F^n$$

Use

- $(E\otimes 1) e^{\frac{1}{2}\hbar H\otimes H} = e^{\frac{1}{2}\hbar H\otimes H} E\otimes e^{-\hbar H}$
- $[E, F^{n+1}] = \frac{[n+1]_q}{q-q^{-1}} \left(e^{\hbar(H+n)} - e^{-\hbar(H+n)}\right) F^n$

- **Example** $R = e^{\frac{1}{2}\hbar H\otimes H} \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(q-q^{-1})^n}{[n]_q!} E^n \otimes F^n$

$L_1 = \mathbb{C}^2$

$$H \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- $L_1 \otimes L_1$ $\quad v_1\otimes v_1, \quad v_1\otimes v_{-1} \quad v_{-1}\otimes v_1 \quad v_{-1}\otimes v_{-1}$

$$e^{\frac{1}{2}\hbar(H\otimes H)} \mapsto \begin{pmatrix} q^{1/2} & & & 0 \\ & q^{-1/2} & & \\ & & q^{-1/2} & \\ 0 & & & q^{1/2} \end{pmatrix} \qquad E\otimes F \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E^2\otimes F^2 \mapsto 0$$

$$R \mapsto q^{-1/2} \begin{pmatrix} q & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & q \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (q-q^{-1}) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

Remark R – infinite series $\sum_{k=0}^{\infty} A_k \hbar^k$

- Recall, duality $V \rightsquigarrow V^*$ $\rho_{V^*}(a) = \rho_V(S(a))^*$

$$V^* \otimes V \to \mathbb{C} \qquad \mathbb{C} \to V \otimes V^*$$

- $\exists R \rightsquigarrow V \to V \otimes V^* \otimes V^{**} \xrightarrow{PR\otimes id} V^* \otimes V \otimes V^{**} \to V^{**}$

In basis $V = \langle e_j \rangle$ $V^* = \langle e^j \rangle$ $V^{**} = \langle e_j \rangle$ $R = \sum a_i \otimes b_i$

$$e_k \mapsto \sum e_k \otimes e^j \otimes e_j \mapsto \sum b_i e^j \otimes a_i e_k \otimes e_j \mapsto \sum (S(b_i) a_i)^j_k \, e_j$$

Let $u = \sum S(b_i) a_i$ intertwines $V \to V^{**}$
(or ${}^*V \to V^*$)

• **Th** a) $\forall x \in A \qquad S^2(x) = u x u^{-1}$

b) $u^{-1} = \sum S^{-1}(d_i)\, c_i \qquad\qquad R^{-1} = \sum c_i \otimes d_i$

Proof a) $(\Delta \otimes Id)\Delta(x) = x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \qquad \Delta x = x_{(1)} \otimes x_{(2)}$

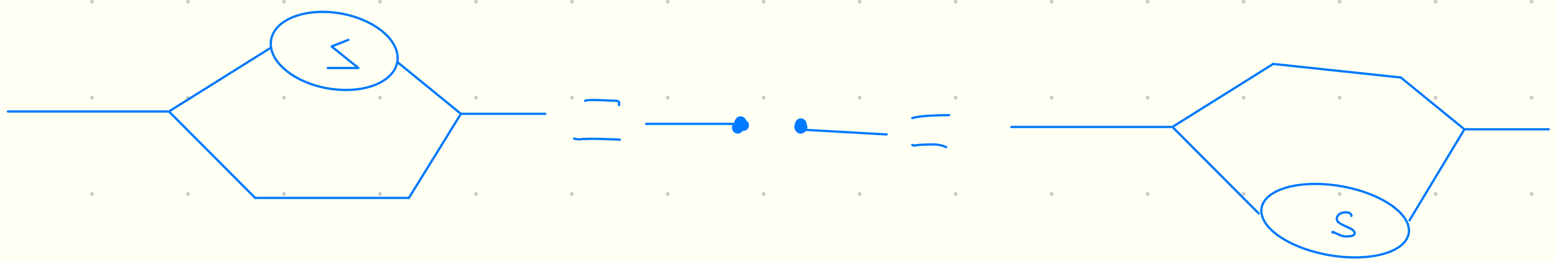
$R\Delta(x) = \Delta^{op}(x) R \implies \sum a_i x_{(1)} \otimes b_i x_{(2)} \otimes x_{(3)} = \sum x_{(2)} a_i \otimes x_{(1)} b_i \otimes x_{(3)}$

$$\sum S^2(x_{(3)})\, S(b_i x_{(2)})\, a_i x_{(1)} = \sum S^2(x_{(3)})\, S(x_{(1)} b_i)\, x_{(2)} a_i$$

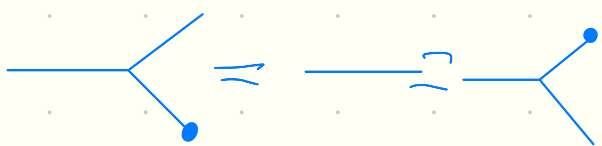
$$\Vert \qquad\qquad\qquad\qquad \Vert$$

$$\sum S(x_{(2)} S(x_{(3)}))\, S(b_i)\, a_i x_{(1)} \qquad\qquad \sum S^2(x_{(3)})\, S(b_i)\, S(x_{(1)}) x_{(2)} a_i$$

$$\Vert \qquad\qquad\qquad\qquad \Vert$$



$$\sum i\varepsilon(x_{(2)})\, u\, x_{(1)} \qquad\qquad \sum S^2(x_{(2)})\, S(b_i)\, i\varepsilon(x_{(1)})\, a_i$$



$$\Vert \qquad\qquad\qquad\qquad \Vert$$

$$u x \qquad\qquad\qquad\qquad S^2(x)\, u$$

$S^2(x) = u x u^{-1}$ $\quad u = \sum S(b_i) a_i$ $\quad R^{-1} = \sum c_i \otimes d_i$

b) $\sum u S^{-1}(d_j) c_j = \sum S(d_j) u c_j = \sum S(d_j) S(b_i) a_i c_j =$

$= \sum S(b_i d_j) a_i c_j = 1$ since

$R \cdot R^{-1} = \sum a_i c_j \otimes b_i d_j = 1$ $\square$

- Some central elements in A

- R_{21}^{-1} intertwines Δ and Δ^{op} $\Rightarrow$

$v = \sum S(c_i) d_i$ $\quad v x v^{-1} = S^2(x) \Rightarrow$

$u v^{-1}$ – central

$u\ S^{-1}(\sum S(c_j) d_j) = 1$

$u\ S^{-1}(v) = 1$

<u>Equivalently</u> $u S(w)$ – central

<u>Proof</u> $1 = u \sum S^{-1}(d_j) c_j = u S^{-1}(\sum S(c_j) d_j) \Rightarrow u S^{-1}(v) = 1 \Rightarrow$

$\Rightarrow v^{-1} = S(w)$

• $A = U_\hbar(sl_2)$ $\quad S^2$ is conj by $e^{\hbar H}$

Hence $e^{-\hbar H}u$ is central

Recall $Z(U(sl_2)) = \mathbb{C}[c]$ $\quad c = fe + \frac{h(h+2)}{4}$

Problem a) Show that $C_\hbar = FE + \frac{e^{\frac{\hbar}{2}(H+1)} + e^{-\frac{\hbar}{2}(H+1)}}{(e^{\frac{\hbar}{2}} - e^{-\frac{\hbar}{2}})^2}$ is central in $U_\hbar(sl_2)$

b) Find action of $C_\hbar$ and $e^{-\hbar H}u$ on L_m

c) Let $\varphi_\hbar^{-1} : U_{sl_2}[[\hbar]] \to U_\hbar(sl_2)$ isomorphism. Find $\varphi_\hbar^{-1}(c)$ and $\varphi_\hbar^{-1}(e^{\hbar c})$ on L_m, relate to elements above.

References

- Chary, Pressley A guide to quantum groups
  Ch. 5

- Drinfeld Almost cocommutative Hopf algebras