Introduction to Quantum Groups

Lecture 8

Drinfeld-Jimbo quantum groups

Kac-Moody Lie algebras

Φ - root system $\Pi = \{\alpha_1 \dots \alpha_r\}$ simple roots

Usually assume $(\alpha,\alpha)=2$ $\quad a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ – Cartan matrix

- <u>Def</u> $\mathfrak{g}$ - Lie algebra with generators $F_1, \dots, F_r$ $H_1 \dots H_r$ $E_1, \dots, E_r$

and relations $[H_i, E_j] = a_{ij} E_j$, $[H_i, F_j] = -a_{ij} F_j$

$[H_i, H_j] = 0$ $\quad [E_i, F_j] = \delta_{ij} H_i$

Serre relations $\quad \mathrm{ad}_{E_i}^{1-a_{ij}} E_j = 0 \quad \mathrm{ad}_{F_i}^{1-a_{ij}} F_j = 0$

- It is convenient to rewrite Serre relations as

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} E_i^k E_j E_i^{1-a_{ij}-k} = 0$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} F_i^k F_j F_i^{1-a_{ij}-k} = 0$$

- Basic example $\mathfrak{g} = \mathfrak{sl}_{r+1}$

$F_i = E_{i+1,i}$ | $H_i = E_{ii} - E_{i+1\,i+1}$ | $E_i = E_{i\,i+1}$

$$\begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&-1 \end{pmatrix} \quad \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}$$

- $[E_1,[E_1,E_2]] = 0$

  $E_1^2 E_2 - 2E_1E_2E_1 + E_2E_1^2 = 0$

- $[E_2,[E_2,E_1]] = 0$

- $[E_1, E_3] = 0$

  $E_1E_3 - E_3E_1 = 0$

Def $U_q(\mathfrak{g})$ is an algebra generated by $F_1, \dots, F_r$, $K_1^{\pm1}, \dots, K_r^{\pm1}$, $E_1, \dots, E_r$ $\quad \alpha_i \in \Pi$

and relations $K_i K_i^{-1} = 1 = K_i^{-1} K_i \qquad K_i K_j = K_j K_i$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j \qquad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

Serre relations
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q E_i^k E_j E_i^{1-a_{ij}-k} = 0$$

$$i \neq j$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q F_i^k F_j F_i^{1-a_{ij}-k} = 0$$

- **Remarks** 1

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!} \qquad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

$$[n]_q! = [n]_q \cdot [n-1]_q \cdot \ldots \cdot [1]_q$$

q Pascal triangle

$$\begin{matrix} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & q+q^{-1} & & 1 & \\ 1 & & q^2+1+q^{-2} & & q^2+1+q^{-2} & & 1 \end{matrix}$$

$$[0]_q! = 1$$

② Here we assumed Φ is simply laced

$a_{ij}=0 \qquad 0=\begin{bmatrix}1\\0\end{bmatrix}_q E_iE_j-\begin{bmatrix}1\\1\end{bmatrix}_q E_jE_i = E_iE_j-E_jE_i$

$a_{ij}=-1 \qquad \begin{bmatrix}2\\0\end{bmatrix}_q=\begin{bmatrix}2\\2\end{bmatrix}_q=1 \qquad \begin{bmatrix}2\\1\end{bmatrix}_q=q+q^{-1}$

$$E_i^2E_j-(q+q^{-1})E_iE_jE_i+E_jE_i^2$$

③ This generalize $U_q(sl_2)$. Moreover $\forall i$ $\exists$ homomorphism

$\phi_i: U_q(sl_2)\to U_q(\mathfrak{g})$

$(F,K,E)\longmapsto(F_i,K_i,E_i)$

$\mathbb{C}[[\hbar]]$-algebra

④ One can define $U_\hbar(\mathfrak{g})$ with generators F_i,H_i,E_i with relations — // — +

$$[H_i,E_j]=a_{ij}E_j \qquad [H_i,F_j]=-a_{ij}F_j \qquad [E_i,F_j]=\delta_{ij}\frac{e^{\hbar H_i}-e^{-\hbar H_i}}{e^{\hbar}-e^{-\hbar}}$$

- Let $\widetilde{\mathcal{U}}_q(\mathfrak{g})$ be an algebra with generators F_i, H_i, E_i, $\alpha_i \in \Pi$ and relations <u>without</u> Serre

For shortness $\widetilde{\mathcal{U}}_q(\mathfrak{g}), \mathcal{U}_q(\mathfrak{g}) \rightsquigarrow \widetilde{\mathcal{U}}, \mathcal{U}$

Let $\mathcal{U}^+, \widetilde{\mathcal{U}}^+$ subalgebras $\mathcal{U}, \widetilde{\mathcal{U}}$ generated by E_i

Let $\mathcal{U}^-, \widetilde{\mathcal{U}}^-$ subalgebras $\mathcal{U}, \widetilde{\mathcal{U}}$ generated by F_i

Let $\mathcal{U}^0, \widetilde{\mathcal{U}}^0$ subalgebras $\mathcal{U}, \widetilde{\mathcal{U}}$ generated by $K_i^{\pm 1}$

- For any $\lambda \in Q$ – root lattice

$\lambda = \sum m_i \alpha_i$ $\quad m_i \in \mathbb{Z}$ Let $K_\lambda := \prod K_i^{m_i}$

$$K_\lambda E_i K_\lambda^{-1} = q^{(\lambda, \alpha_i)} E_i \qquad K_\lambda F_i K_\lambda^{-1} = q^{-(\alpha_i, \lambda)} F_i$$

Sometimes it is convenient to extend $\mathcal{U}, \widetilde{\mathcal{U}}$ adding K_λ, $\lambda \in P$ - weight lattice

For $\mathcal{U}_\hbar(\mathfrak{g})$ – no such issue.

- Algebras $\mathcal{U}, \tilde{\mathcal{U}}$ are Q - graded.
  $\deg E_i = \alpha_i$, $\deg K_i = 0$ $\deg F_i = -\alpha_i$

- <u>Prop</u> There is a Hopf algebra str on $\tilde{\mathcal{U}}$ given by

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \qquad \Delta K_i = K_i \otimes K_i \qquad \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i$$
$$\varepsilon(E_i) = 0 \qquad \varepsilon(K_i) = 1 \qquad \varepsilon(F_i) = 0$$
$$S(E_i) = -E_i K_i^{-1} \qquad S(K_i) = K_i^{-1} \qquad S(F_i) = -K_i F_i$$

<u>Remark</u> Determined by property: $\phi_i : \mathcal{U}_q(sl_2) \to \tilde{\mathcal{U}}$ is homom. of Hopf algebras

<u>Proof</u> $[\Delta(E_i), \Delta(F_j)] = [E_i \otimes K_i + 1 \otimes E_i, F_j \otimes 1 + K_j^{-1} \otimes F_j] = [E_i \otimes K_i, K_j^{-1} \otimes F_j]$
$i \neq j$
$$= E_i K_j^{-1} \otimes K_i F_j - K_j^{-1} E_i \otimes F_j K_i = (q^{a_{ij} - a_{ij}} - 1) K_j^{-1} E_i \otimes F_j K_i = 0 \quad \square$$

Rk Let $X \in \tilde{\mathcal{U}}_\lambda$, $Y \in \tilde{\mathcal{U}}_\mu$ $ad_{q,X} Y = XY - q^{(\lambda,\mu)} YX$

Serre relations $ad_{q,E_i}^{1-a_{ij}} E_j = 0$ $ad_{q,F_i}^{1-a_{ij}} F_j = 0$

Def Adjoint action Hopf algebra A on itself
$Ad_a b := a_{(1)} b S(a_{(2)})$

Example ① $A = \mathbb{C}[G]$ $Ad_g h = g h g^{-1}$
② $A = \mathcal{U}(\mathfrak{g})$ $Ad_x y = xy - yx$

Problem Show that Serre relations are equivalent to

$$(Ad^{coop}_{E_i})^{1-a_{ij}} E_j = 0 \qquad (Ad_{F_i})^{1-a_{ij}} F_j = 0$$

adjoint action for
$\Delta \mapsto \Delta^{op}$ (this leads to $S \mapsto S^{-1}$)

Problem here $\forall u \in \tilde{\mathcal{U}} \quad S^2(u) = K_{2\rho}\, u\, K_{2\rho}^{-1}$

$2\rho = \sum_{\alpha \in \Phi_+} \alpha \;, \quad (2\rho, \alpha_i) = 2 \quad \forall \alpha_i \in \Pi$

Prop Let $u_{ij}^+ , u_{ij}^- \in \tilde{\mathcal{U}}$ s.t.

$$u_{ij}^+ = \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q E_i^k E_j E_i^{1-a_{ij}-k} \qquad u_{ij}^+ \in \tilde{\mathcal{U}}_{\alpha_j + (1-a_{ij})\alpha_i}$$

$$u_{ij}^- = \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q F_i^k F_j F_i^{1-a_{ij}-k} \qquad u_{ij}^+ \in \tilde{\mathcal{U}}_{-\alpha_j - (1-a_{ij})\alpha_i}$$

Then $\Delta u_{ij}^+ = u_{ij}^+ \otimes K_{\deg u_{ij}^+} + 1 \otimes u_{ij}^+ \qquad \Delta u_{ij}^- = u_{ij}^- \otimes 1 + K_{\deg u_{ij}^-} \otimes u_{ij}^-$

"Lie" algebra like elements

C.f $\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \qquad \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i$

Pf $\Delta(u_{12}^+) = \Delta(E_1^2E_2 - (q+q^{-1})E_1E_2E_1 + E_2E_1^2) =$

$$(E_1\otimes K_1 + 1\otimes E_1)(E_1\otimes K_1 + 1\otimes E_1)(E_2\otimes K_2 + 1\otimes E_2)$$
$$- (q+q^{-1})(E_1\otimes K_1 + 1\otimes E_1)(E_2\otimes K_2 + 1\otimes E_2)(E_1\otimes K_1 + 1\otimes E_1)$$
$$+ (E_2\otimes K_2 + 1\otimes E_2)(E_1\otimes K_1 + 1\otimes E_1)(E_1\otimes K_1 + 1\otimes E_1)$$

$$= u_{12}^+\otimes K_1^2K_2 + E_1^2\otimes E_2K_1^2(q^{-2} - q^{-1}(q+q^{-1}) + 1)$$
$$+ E_1E_2\otimes E_1K_1K_2\left((1+q^2) - (q+q^{-1})q\right) + E_2E_1\otimes E_1K_1K_2\left(-(q+q^{-1}) + q^{-1}(1+q^2)\right)$$
$$+ E_2\otimes E_1^2K_2(1 - (q+q^{-1})q^{-1} + q^{-2}) + E_1\otimes E_1E_2K_1\left((q+q^{-1}) - (q+q^{-1})\right)$$
$$+ E_1\otimes E_2E_1(-q(q+q^{-1}) + (1+q^2)) + 1\otimes u_{12}^+ = u_{12}^+\otimes K_1^2K_2 + 1\otimes u_{12}^+$$

computation for u_{12}^- — similar $\square$

Th $\exists$ Hopf algebra str. on $\mathcal{U}$ given by the same formulas

Pf By Prop above ideal generated by u_{ij}^+, u_{ij}^- is also Hopf idea

$$\Delta(a u_{ij}^+ b) = \Delta(a)(u_{ij}^+ \otimes K_{\deg u_{ij}} + 1 \otimes u_{ij})\Delta(b) \in I \otimes U + U \otimes I \quad \square$$

- For any sequence $I = (\beta_1, \dots, \beta_\ell)$ of simple roots let $E_I = E_{\beta_1} \cdot \ldots \cdot E_{\beta_\ell}$ $\quad F_I = F_{\beta_1} \cdot \ldots \cdot F_{\beta_\ell}$ $\quad wt(I) = \sum_{i=1}^{\ell} \beta_i$
In particular $E_\phi = F_\phi = 1$

$E_i \leftrightarrow E_{\alpha_i}$
$F_i \leftrightarrow F_{\alpha_i}$

<u>Lemma</u> $$\Delta(E_I) = \sum_{A,B} c_{A,B}^I(q) E_A \otimes E_B K_{wt(A)}$$

$$\Delta(F_I) = \sum_{A,B} \tilde{c}_{A,B}^I(q) F_A K_{-wt(B)} \otimes F_B$$

$wt(A) + wt(B) < wt(I)$

$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \qquad \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i$

$$c_{A\,\phi}^I = \delta_{A,I}, \quad c_{\phi,B}^I = \delta_{I,B}.$$

Prop (Universal) Verma module $\widetilde{M}_c$ over $\widetilde{U}$ has a basis v_I with action

$$F_i v_I = v_{(\alpha_i, I)} \qquad K_i v_I = c_i\, q^{-(\alpha_i,\, wt\, I)} v_I$$

$$E_i v_I = \sum_{\substack{1 \le j \le \ell \\ \beta_j = \alpha_i}} \frac{c_i q^{-(\alpha_i, \mu_j)} - c_i^{-1} q^{(\alpha_i, \mu_j)}}{q - q^{-1}} v_{\beta_1, \dots, \beta_{j-1}, \hat{\beta}_j, \beta_{j+1}, \dots, \beta_\ell} \qquad \text{here } \mu_j = \sum_{p=j+1}^{\ell} \beta_p$$

$$F_I v_\emptyset = v_I$$

In other words, we have highest weight vector $v_\emptyset$ s.t.

$$E_i v_\emptyset = 0, \quad K_i v_\emptyset = c_i v_\emptyset, \quad F_I v_\emptyset = v_I$$

Similarly we define $\widetilde{M}^\vee_{c^\vee}$ with $F_i v^\vee_\emptyset = 0$, $K_i v^\vee_\emptyset = c^\vee_i v^\vee_\emptyset$, $E_I v^\vee_\emptyset = v^\vee_I$

Theorem The elements $F_I K_\mu E_J$ form a basis in $\widetilde{U}$

Corollary $\hat{U}^- \otimes \hat{U}^0 \otimes \hat{U}^+ \to \widetilde{U}$ $\quad u_1 \otimes u_2 \otimes u_3 \mapsto u_1 u_2 u_3$ is isomorphism

Pf Assume that there is a relation $X = \sum a_{I,\mu,J} F_I K_\mu E_J = 0$

Take I_0 with maximal $wt(I_0)$ s.t $\exists\, a_{I_0,\mu,J} \neq 0$

Consider action on $\tilde{M}^\vee_{C^\vee} \otimes \tilde{M}_C$. Consider $\Delta(X) v^\vee_\phi \otimes v_\phi = 0$

Recall $\Delta E_J = \sum c^J_{A,B} E_A \otimes E_B K_{wt(A)}$

For $B \neq \phi$ $E_B v_\phi = 0$. Hence

$$0 = \Delta(X) v^\vee_\phi \otimes v_\phi = \sum_{I,\mu,J} a_{I,\mu,J} \Delta(F_I) \Delta(K_\mu) \left(c^J_{A,B} E_A v^\vee_\phi \otimes E_B K_{wt(A)} v_\phi \right)$$

$$= \sum_{I,\mu,J} c^{wt(J)} a_{I,\mu,J} \Delta(F_I) K_\mu \otimes K_\mu\, v^\vee_J \otimes v_\phi$$

$$= \sum_{I,\mu,J} c^{wt(J)} c^{wt(\mu)} (c^\vee)^{wt\mu} q^{(\mu, wt J)} a_{I,\mu,J} \Delta(F_I) \left(v^\vee_J \otimes v_\phi \right)$$

Recall $\Delta(F_I) = \sum_{A,B} \tilde{c}^I_{A,B}(q) F_A K_{-wt(B)} \otimes F_B$. Hence, the only terms of the form $* \otimes v_{I_0}$ come from $I = I_0$, $B = I_0$, $A = \phi$

Therefore

$$0=\sum_{\mu,J} c^{wt(J)}\, c^{wt(\mu)} (c^\vee)^{wt\,\mu}\, q^{(\mu, wt\, J)}\, a_{I_0,\mu,J}\; K_{-wt(I_0)} \otimes F_{I_0} (v_J^\vee \otimes v_\emptyset)$$

$$=\sum_{\mu,J} c^{wt(J)}\, c^{wt(\mu)} (c^\vee)^{wt\,\mu}\, q^{(\mu, wt\, J)}\, q^{-(wt(I_0), wt(J))}\, a_{I_0,\mu,J}\, (c^\vee)^{wt\, I_0}\, v_J^\vee \otimes v_{I_0}$$

For given I_0 vectors $v_J^\vee \otimes v_{I_0}$ are linearly independent

Hence $\sum_\mu c^{wt(\mu)} (c^\vee)^{wt(\mu)}\, q^{(\mu, wt\, J)}\, a_{I_0,\mu,J} = 0$

Since $c, c^\vee$ are arbitrary we have $a_{I_0,\mu,J}=0$

<u>Contradiction</u> $\blacksquare$

<u>Problem</u> $[F_k, u_{ij}^+]=0$ (similarly $[E_k, u_{ij}^-]=0$)

Theorem a) The multiplication

$$\mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+ \to \mathcal{U}$$

is isomorphism of vector spaces

b) The algebra $\mathcal{U}^+$ is isomorphic to the algebra with generators $E_1, \dots, E_r$ and Serre relations

c) — // — $\mathcal{U}^-$ — // — $F_1, \dots, F_r$ — // —

d) K_μ, $\mu \in Q$ form a basis in $\mathcal{U}^0$

Pf Let $I \subset \tilde{\mathcal{U}}$ ideal generated by u^+_{ij}, u^-_{ij},

$I^+ \subset \tilde{\mathcal{U}}^+$ — // — u^+_{ij},

$I^- \subset \tilde{\mathcal{U}}^-$ — // — u^-_{ij},

Due to Prob. above

$$F_I K_\mu E_J u^+_{ij} F_{I'} K_{\mu'} E_{J'} = F_I K_\mu E_J F_{I'} K_{\mu'} q^{-(\mu', wt(u^+_{ij}))} u^+_{ij} E_{J'} \in \tilde{\mathcal{U}}^- \otimes \mathcal{U}^0 \otimes I^+$$

$$F_I K_\mu E_J u^-_{ij} F_{I'} K_{\mu'} E_{J'} = F_I u^-_{ij} q^{-(\mu, wt(u^-_{ij}))} K_\mu E_J F_{I'} K_{\mu'} E_{J'} \in I^- \otimes \mathcal{U} \otimes \mathcal{U}^+$$

Therefore $I = \tilde{U}^- \otimes \tilde{U}^0 \otimes I^+ + I^- \otimes \tilde{U}^0 \otimes \tilde{U}^+$

Hence
$$I \cap \tilde{U}^+ = I \cap (1 \otimes 1 \otimes \tilde{U}^+) = 1 \otimes 1 \otimes I^+ \Rightarrow b)$$
$$I \cap \tilde{U}^- = I \cap (\tilde{U}^- \otimes 1 \otimes 1) = I^- \otimes 1 \otimes 1 \Rightarrow c)$$
$$I \cap \tilde{U}^0 = I \cap (1 \otimes \tilde{U}^0 \otimes 1) = 0 \Rightarrow d)$$

$$U \simeq \tilde{U}/I = \frac{\tilde{U}^- \otimes \tilde{U}^0 \otimes \tilde{U}^+}{\tilde{U}^- \otimes \tilde{U}^0 \otimes I^+ + I^- \otimes \tilde{U}^0 \otimes \tilde{U}^+} = \tilde{U}^-/I^- \otimes U^0 \otimes \tilde{U}^+/I^+$$
$$= U^- \otimes U^0 \otimes U^+ \quad \square$$

<u>Corollary</u> $\phi_i : U_q(sl_2) \to U_q(\mathfrak{g})$ is embedding.

<u>Pf</u> Relations in U^+ has weights $wt(u_{je}) = \alpha_j + \alpha_e$, $\Rightarrow$ no relations of the weight $n\alpha_i$ $\Rightarrow$ $E_i^n \neq 0$. $\square$

References

- Jantzen Lectures on Quantum groups Ch. 4