Introduction to Quantum Groups

Lecture 9

Finite dimensional Representations $\mathcal{U}_q(\mathfrak{sl}_2)$

Today $\mathfrak{g}$ - f.d. s/s Lie algebra.

Def $U_q(\mathfrak{g})$ is an algebra generated by $F_1, \ldots, F_r$, $K_1^{\pm 1}, \ldots, K_r^{\pm 1}$, $E_1, \ldots, E_r$ $\quad \alpha_i \in \Pi$

and relations $K_i K_i^{-1} = 1 = K_i^{-1} K_i \qquad K_i K_j = K_j K_i$

$K_i E_j K_i^{-1} = q^{a_{ij}} E_j \qquad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

Serre relations
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q E_i^k E_j E_i^{1-a_{ij}-k} = 0 \qquad i \neq j$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q F_i^k F_j F_i^{1-a_{ij}-k} = 0$$

Q 1-dim reps of $U = U_q(\mathfrak{g})$?

Prop Any 1-dim rep $\mathcal{U}$ is isomorphic to $\mathbb{C}_\sigma$, where $\sigma : \Pi \to \{\pm 1\}$

$$\rho_{\mathbb{C}_\sigma}(E_i) = \rho_{\mathbb{C}_\sigma}(F_i) = 0, \quad \rho_{\mathbb{C}_\sigma}(K_i) = \sigma(\alpha_i)$$

Rk For $\mathcal{U}_\hbar(\mathfrak{g})$ no σ, only one 1dim rep.

- Let V be $\mathcal{U}$ module. Let

$$V_{(\lambda,\sigma)} = \{v \in V \mid K_i v = \sigma_i q^{(\alpha_i,\lambda)} v\}$$

Prop V - f.d rep. $V = \bigoplus_{\lambda,\sigma} V_{(\lambda,\sigma)}$ and

$$E_i V_{(\lambda,\sigma)} \subset V_{(\lambda+\alpha_i,\sigma)}, \quad F_i V_{(\lambda,\sigma)} \subset V_{(\lambda-\alpha_i,\sigma)}$$

Corollary $V_\sigma = \bigoplus_\lambda V_{(\lambda,\sigma)}$, then $V = \bigoplus_\sigma V_{(\sigma)}$ - direct sum of reps

Def V is of type σ is $V = V_{(\sigma)}$

$\mathrm{Rep}_{\mathcal{U}}^{\text{type } \sigma}$ — corresponding category

- **Prop** For $\forall\,\sigma$ and V rep of type τ we have $V\otimes\mathbb{C}_\sigma$ - rep. of type $\sigma\cdot\tau$

Problem Show that $V\otimes\mathbb{C}_\sigma\otimes\mathbb{C}_\sigma \simeq V$ for any f.d. V

Hence For any σ. $\mathrm{Rep}^{\text{type } I}_{U_q\mathfrak{g}}$ equivalent to $\mathrm{Rep}^{\text{type } \sigma}_{U_q\mathfrak{g}}$

- Below - only reps of type I
  Below $q \neq \sqrt{1}$

- Rem Only type I for $U_\hbar(\mathfrak{g})$ - $\mathbb{C}[[\hbar]]$ algebra with generators E_i, F_i, H_i, $q=e^\hbar$, $K_i=e^{\hbar H_i}$

- **Prop** V - f.d U module. Then $\exists\, v\in V$ s.t. $E_i v=0$ $\forall i$, $K_i v = q^{(\lambda,\alpha_i)} v$, where $\lambda\in P_+$ - dominant, i.e. $(\lambda,\alpha_i)\in\mathbb{Z}_{\geq 0}$ $\forall i$

Pf $\exists$ of h.w vector — take "maximal" λ in weight decomp. Dominance — reduces to sl_2 case

$$E v_\lambda = 0, \quad K v_\lambda = q^\lambda v_\lambda, \quad v_\lambda, \ F v_\lambda, \ F^2 v_\lambda, \ \ldots$$

f.d $\Rightarrow$ $\exists\, n>0$ s.t $F^n v_\lambda = 0$, $F^{n-1} v_\lambda \neq 0$. then

$$EF^n v_\lambda = \sum_{k=0}^{n-1} F^{n-k-1} \frac{K-K^{-1}}{q-q^{-1}} F^k v_\lambda = \sum_{k=0}^{n-1} \frac{q^{\lambda-2k} - q^{2k-\lambda}}{q-q^{-1}} F^{n-1} v_\lambda$$

$$= \frac{q^{\lambda+1} - q^{\lambda-2n+1} - q^{2n-1-\lambda} - q^{-\lambda-1}}{(q-q^{-1})^2} F^{n-1} v_\lambda = \frac{q^n - q^{-n}}{q-q^{-1}} \, \frac{q^{\lambda-n+1} - q^{n-1+\lambda}}{q-q^{-1}} F^{n-1} v_\lambda = [\lambda-n+1][n] F^{n-1} v_\lambda$$

$\Rightarrow \lambda = n-1 \in \mathbb{Z}_{\geq 0} \Rightarrow (\lambda, \alpha_i) \in \mathbb{Z}_0$ for arbitrary α_i $\square$

<u>Def</u> Let $\lambda \in P_+$. M_λ — Verma module over $\mathcal{U}$

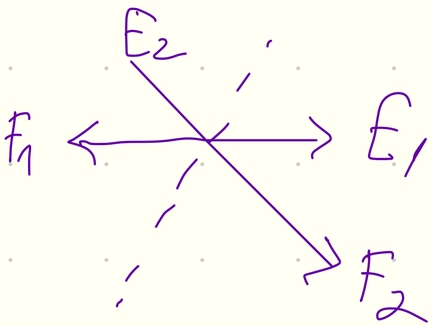
L_λ — irreducible quotient

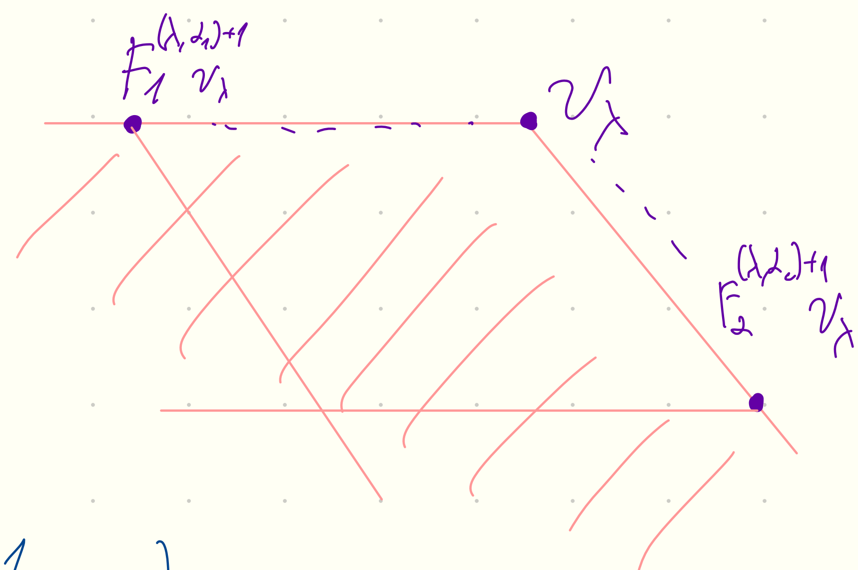
(As in $q=1$ case, $I_\lambda = \sum(\text{non-triv submodules of } M_\lambda)$
I_λ - sub.mod. $v_\lambda \notin I_\lambda$, $L_\lambda = M_\lambda / I_\lambda$)

<u>Lemma</u> $\forall i$, $F_i^{(\lambda,\alpha_i)+1} v_\lambda \in M_\lambda$ is singular vector
i.e $E_j (F_i^{(\lambda,\alpha_i)+1} v_\lambda) = 0$

<u>Pf</u> If $j \neq i$ trivial, since $[E_j, F_i] = 0$. If $j = i$ reduces to sl_2 computation above. $\square$

<u>Figure</u>





<u>Def</u> $L_\lambda = M_\lambda \Big/ \sum_i U(F_i^{(\lambda,\alpha_i)+1} v_\lambda)$

- **Prop** $\forall F_i$ act on $\tilde{L}_\lambda$ locally nilpotently
(i.e. $\forall v \in L_\lambda$, $\forall i$ $\exists N$ s.t. $F_i^N v = 0$)

Pf Induction by $\deg v$
Base $v = v_\lambda$

Assume loc. nilp for v. Consider $\tilde{v} = F_j v$

- loc. nilp for F_j on $\tilde{v}$ triv.
- If $a_{ij} = 0$, then $[F_i, F_j] = 0$ => loc. nilp for F_i on $\tilde{v}$
- If $a_{ij} = -1$. Assume $F_i^N v = 0$.

$F_i^M F_j v = C_1 F_j F_i^M v + C_2 F_i F_j F_i^{M-1} v$ Hence $F_i^{N+1} F_j v = 0$

Using Serre relations $F_i^2 F_j = (q - q^{-1}) F_i F_j F_i - F_j F_i^2$ $\square$

Prop a) Character of $\tilde{L}_\lambda$ is W invariant
b) $\dim \tilde{L}_\lambda < \infty$

Here $V = \bigoplus V_\lambda$ $\quad ch V = \sum \dim V_\lambda \, x^\lambda$
W acts on $x^\lambda \mapsto x^{w(\lambda)}$

Equivalently $ch V \sim Tr K_\mu|_V = \sum_\lambda \dim V_\lambda \, q^{(\lambda,\mu)}$

W acts on $K_\mu \Leftrightarrow$ acts on λ.

__Pf__ a) Suff. to check s_i invariance for $\forall i$
By previous prop $\tilde{L}_\lambda$ is a sum of f.d. $U_q(sl_2)$ modules, for them s_i invariance is known.

b) For $\mu \in P_+$ $\dim (L_\lambda)_{(\mu)} < \infty$.
$\forall \mu \in P$ $\exists w \in W$ s.t $w(\mu) \in P_+ \Rightarrow (\dim L_\lambda)_{(\mu)} = (\dim L_\lambda)_{(w(\mu))}$ $\square$

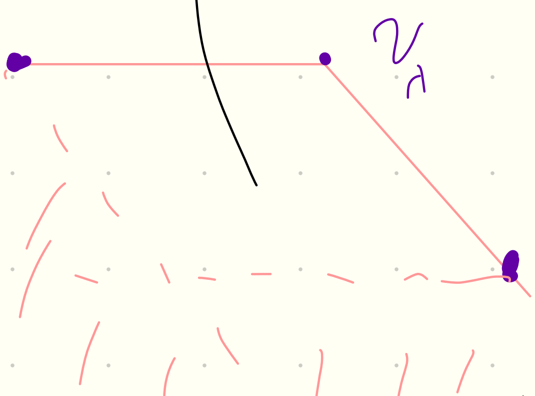
Th Let $u \in \mathcal{U}$ s.t u annihilates all f.d. $\mathcal{U}$-mod. Then $u = 0$.

Idea of the proof Consider action of u on $\tilde{L}_\lambda \otimes \tilde{L}_\lambda$ $\lambda >> 0$. Consider $\Delta(u)\, v_{w_0(\lambda)} \otimes v_\lambda$,

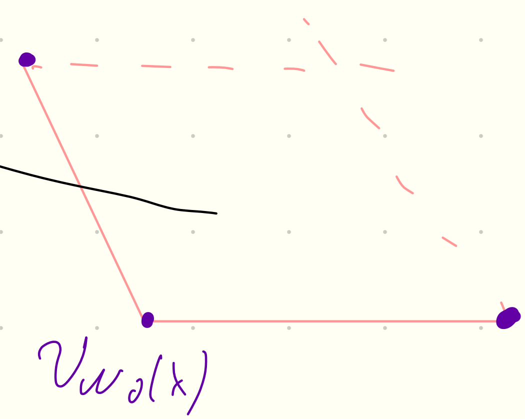
here $v_\lambda \in \tilde{L}_\lambda$ is h.w. vector

$v_{w_0(\lambda)} \in \tilde{L}_\lambda$ is l.w. vector (We used W inv. of character)

"Near" v_λ the module $\tilde{L}_\lambda$ is like $\mathcal{U}^-$



"Near" $v_{w_0(\lambda)}$ the module $\tilde{L}_\lambda$ is like $\mathcal{U}^+$



Then as in Th. last lecture $\square$

Th Let q be generic, $\lambda \in P_+$. Then $\tilde{L}(\lambda) = L(\lambda)$ and $\mathrm{ch}\, L(\lambda)$ is given by Weyl's character formula.

Idea of the proof Recall that $L_\lambda = M_\lambda / I_\lambda$, $\tilde{L}_\lambda = M_\lambda / \sum_i \mathcal{U}(F_i^{(\lambda,\alpha_i)+1} v_\lambda)$

We have $I_\lambda \supset \sum \mathcal{U}(F_i^{(\lambda,\alpha_i)+1} v_\lambda)$.

But for $q=1$ we have equality $\Rightarrow$ for generic q equality

I.e $q=1$ is generic point.

For $q=1$ Weyl formula $\Rightarrow$ for generic q Weyl formula. $\square$

Remark Sufficient to assume $q \neq \sqrt{1}$.

Corollary Size of $\mathcal{U}_q^-(\mathfrak{g})$ is the same as $\mathcal{U}(\mathfrak{n}^-)$

— // — $\mathcal{U}_q^+(\mathfrak{g})$ — // — $\mathcal{U}(\mathfrak{n}^+)$

Pf Characters of L_λ for $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$ coinside $\Rightarrow$ q.e.d. $\square$

Th Let q be generic, V f.d. $\mathcal{U}$-module. Then V is semisimple, i.e. $V = \bigoplus V_k$, V_k - irreps.

Pf Sufficient to check that there no Ext^1, i.e.

any sequence of the form $0 \to L_\lambda \to M \to L_\mu \to 0$ splits

If $\lambda \not> \mu$ (in dominance order) then $v \in M_{(\mu)}$ is h.w vector
Since M is f.d. $F_i^{(\lambda_i,\mu)+1} v = 0 \ \forall i$ and we have splitting map $L_\mu \to M$

If $\lambda > \mu$ consider dual sequence $0 \to L_\mu^* \to M^* \to L_\lambda^* \to 0$
Equivalently $0 \to L_{w_0(\mu)} \to M^* \to L_{w_0(\lambda)} \to 0$

$w_0(\lambda) > w_0(\mu) \Rightarrow$ this sequence splits $\Rightarrow$ initial sequence splits ∎

Problem ⓐ Describe $\mathbb{C}^n = L_{\omega_1}$ as representation of $U_q(sl_n)$

ⓑ Check formula for intertwiner $\tilde{R} : \mathbb{C}^n \otimes_\Delta \mathbb{C}^n \to \mathbb{C}^n \otimes_\Delta \mathbb{C}^n$

$$\tilde{R} = \sum_i q\, E_{ii} \otimes E_{ii} + \sum_{i<j} \left(E_{ij} \otimes E_{ji} + E_{ji} \otimes E_{ij} + (q - q^{-1}) E_{jj} \otimes E_{ii} \right)$$

ⓒ* Hecke algebra H_N for sl_N is an algebra with generators $T_1, \dots, T_{N-1}$ and relations

quadratic relations — braid relations

$$(T_i - q)(T_i + q^{-1}) = 0 \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad |i-j| > 1$$

Construct an action of H_N on $(\mathbb{C}^n)^{\otimes N}$ commuting with $U_q(sl_n)$

q- Schur-Weyl duality

Hint ⓐ The formulas are like for $q=1$, namely

$\mathbb{C}^n = \langle \xi_1, \xi_2, \dots, \xi_n \rangle \qquad E_i \xi_j = \delta_{i,j-1} \xi_{j-1} \qquad F_i \xi_j = \delta_{i,j} \xi_{j+1}$

ⓒ $T_i \to \tilde{R}_{i,i+1}$. Braid relations follows from QYBE.

Problem* ⓐ For $\mathfrak{g}=\mathfrak{sl}_n$ constract representation L_{ϖ_k}, where ϖ_k is k-th fundamental weight. This is q-analog of $\Lambda^k \mathbb{C}_n$

ⓑ For $\mathfrak{g}=\mathfrak{sl}_n$ constract representation $L_{k\varpi_1}$. This is q-analog of $S^k\mathbb{C}^n$

Hint One way: define bases $\xi_{i_1}\wedge_q \xi_{i_2}\wedge_q \cdots \wedge_q \xi_{i_k}$, $i_1<\dots<i_k$ and $\xi_{j_1}\cdot_q \xi_{j_2}\cdot \dots \cdot \xi_{j_k}$, $j_1\leq j_2\leq\dots\leq j_k$ and define action of E_i, K_i, F_i in this bases

Another way: realize $\Lambda^n_q\mathbb{C}^n=(\mathbb{C}^n)^{\otimes k}/Ker(\tilde{R}_{i\,i+1}-q)$ and $S^k\mathbb{C}^n=(\mathbb{C}^n)^{\otimes k}/\sum Ker(\tilde{R}_{i,i+1}+q^{-1})$

References

- Jantzen Lectures on Quantum groups Ch. 5