Introduction to Quantum Groups

Lecture 10

Drinfeld double

- Reminder (Classical double)

$(\mathfrak{g}, \delta)$ Lie bialgebra

$\Updownarrow$

$(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ Manin triple $\mathfrak{g}_+ = \mathfrak{g}$, $\mathfrak{g}_- = \mathfrak{g}^*$, $\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}^*$ as vector space

Th $\mathfrak{g}$ has str of cobound. Lie bialg given by $r \in \Lambda^2 \mathfrak{g}$ $r = \sum a_i \wedge a^i$, where a_i, a^i dual bases in $\mathfrak{g}$ and $\mathfrak{g}^*$

- Manin triple $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}, \mathfrak{g}_+ \oplus \mathfrak{g}_-)$
  $\mathfrak{g} = D(\mathfrak{g})$ (classical) Drinfeld double

Rk $\mathfrak{g}$, $(\mathfrak{g}^*)^{coop}$ - subbialgebras in $D(\mathfrak{g})$

• Def Given bialgebras A^-, A^+ a bialgebra pairing $(\cdot,\cdot): A^- \otimes A^+ \to \mathbb{C}$ s.t.

$$(a\cdot a', b) = (a \otimes a', \Delta(b)) = (a, b_{(1)})(a', b_{(2)})$$
$$(a, b\cdot b') = (\Delta^{op}(a), b\otimes b') = (a_{(2)}, b)(a_{(1)}, b')$$

Below pairing is <u>nondegenerate</u>

Def $A = A^- \otimes A$ s.t $A^- \otimes 1$ and $1 \otimes A^+$ are subbialgebras with the produst $*$ s.t

$$(a_{(1)}, b_{(1)})\, a_{(2)} * b_{(2)} = b_{(1)} * a_{(1)}\, (a_{(2)}, b_{(2)})$$

$a \in A_-$
$b \in A_+$

is called Drinfeld double of A^+

Problem This equivale to $a \in A^-$, $b \in A^+$

$$b * a = \langle a_{(1)}, b_{(1)}\rangle\, a_2 * b_{(2)} \langle a_{(3)}, S^{-1}(b_{(3)})\rangle$$

Th a) Drinfeld double is a Hopf algebra
b) The element $R=\sum(1\otimes e^i)\otimes(e_i\otimes 1)\in\mathcal{D}(A)\otimes\mathcal{D}(A)$ is universal R matrix e_i, e^i - dual bases in A^-, A^+

- R matrix determines product

$(a\cdot a', b) = (a\otimes a', \Delta(b)) = (a, b_{(1)})(a', b_{(2)})$
$(a, b\cdot b') = (\Delta^{op}(a), b\otimes b') = (a_{(2)}, b)(a_{(1)}, b')$

$e_i e_j = c_{ij}^k e_k$ $\quad\Delta e^k = c_{ij}^k e^i\otimes e^j$

$e^i e^j = d_k^{ij} e^k$ $\quad\Delta e_k = d_k^{ij} e_j\otimes e_i$

$$R\Delta(e_k\otimes 1) = \sum(1\otimes e^i)\otimes(e_i\otimes 1)\, d_k^{pq}(e_q\otimes 1)\otimes(e_p\otimes 1) = \sum c_{ip}^j d_k^{pq}\, e^i * e_q\otimes e_j$$
$$\parallel$$
$$\Delta^{op}(e_k\otimes 1)R = \sum d_k^{pq}(e_p\otimes 1)\otimes(e_q\otimes 1)(1\otimes e^i)\otimes(e_i\otimes 1) = \sum d_k^{pq} c_{qi}^j\, e_p * e^i\otimes e_j$$

$$\sum_{i,p,q} c_{ip}^j d_k^{pq}\, e^i * e_q = \sum d_k^{pq} c_{qi}^j\, e_p * e^i \quad \forall k, j$$

$e_{k(2)}\ e^j_{(2)}$

Use $\Delta e^j = c_{ip}^j e^i\otimes e^p$, $\Delta e_q = d_k^{pq} e_q\otimes e_p$ and $\Delta e^j = c_{qi}^j e^q\otimes e^j$

$$e^j_{(1)} e_{k(1)}(e_{k(2)}, e^j_{(2)}) = (e_{k(1)}, e^j_{(1)}) e_{k(2)} e^j_{(2)}$$

$$\Longleftrightarrow \quad (a_{(1)}, b_{(1)})\, a_{(2)} * b_{(2)} = b_{(1)} * a_{(1)}\, (a_{(2)}, b_{(2)})$$

- Let $U_\hbar(\mathfrak{b}^+)$ be algebra with generators E_i, H_i, $i=1,\dots,r$ and relations

$$[H_i, E_j] = a_{ij} E_j \qquad [H_i, H_j] = 0$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q E_i^k E_j E_i^{1-a_{ij}-k} = 0 \qquad q = e^{\hbar}$$

- $\Delta(E_i) = E_i \otimes e^{\hbar H_i} + 1 \otimes E_i \qquad \Delta H_i = H_i \otimes 1 + 1 \otimes H_i$

  $\varepsilon(E_i) = 0 \qquad \varepsilon(H_i) = 0$

  $S(E_i) = -E_i e^{-\hbar H_i} \qquad S(H_i) = -H_i$

- $U_\hbar(\mathfrak{b}^-)$ generated by H_i, F_i $i = 1, \dots, r$

$$\Delta(F_i) = F_i \otimes 1 + e^{-\hbar H_i} \otimes F_i$$

- Th (Drinfeld) $\exists!$ nondegenerate pairing $U_\hbar(\mathfrak{b}^-)\otimes U_\hbar(\mathfrak{b}^+)\to\mathbb{C}$ s.t.

$$(e^{\hbar H_i}, e^{\hbar H_j}) = e^{\hbar (H_i, H_j)} \qquad (e^{\hbar H_i}, E_j) = (F_i, e^{\hbar H_j}) = 0 \qquad (F_i, E_j) = \delta_{i,j}\frac{1}{e^{\hbar}-e^{-\hbar}}$$

- Pairing $(\cdot,\cdot)$ defined on generators $\to$ uniqueness

- <u>Example</u>

$$(a\cdot a', b) = (a\otimes a', \Delta(b)) = (a, b_{(1)})(a', b_{(2)})$$
$$(a, b\cdot b') = (\Delta^{op}(a), b\otimes b') = (a_{(2)}, b)(a_{(1)}, b')$$
$$\Delta F_i = F_i\otimes 1 + e^{-\hbar H_i}\otimes F_i$$

$$(e^{\hbar H_\mu}, e^{\hbar H_\lambda}) = e^{\hbar (H_\mu, H_\lambda)}$$

$$(F_i, e^{\hbar H_\lambda}E_j) = (1, e^{\hbar H_\lambda})(F_i, E_j) + (F_i, e^{\hbar H_\lambda})(e^{-\hbar H_i}, E_j) = \delta_{i,j}\frac{1}{e^{\hbar}-e^{-\hbar}}$$

$$(e^{\hbar H_\mu}F_i, e^{\hbar H_\lambda}E_j) = (e^{\hbar H_\mu}, e^{\hbar H_\lambda}E_j)(F_i, e^{\hbar(H_\lambda+H_j)}) + (e^{\hbar H_\mu}, e^{\hbar H_\lambda})(F_i, E_j) = \delta_{i,j}\frac{q^{\hbar(H_\lambda, H_\mu)}}{e^{\hbar}-e^{-\hbar}}$$

Remark Root lattice grading $(\mathcal{U}_\hbar(\mathfrak{b}^-)_{(\mu)}, \mathcal{U}_\hbar(\mathfrak{b}^+)_\lambda) = 0$ if $\lambda+\mu \neq 0$

• **Remark** $\mathcal{U}(\mathfrak{b}^+)$ and $\mathcal{U}(\mathfrak{b}^-)$ are not dual as Hopf alg.
But $\mathfrak{b}^+$ and $\mathfrak{b}^-$ are dual as bialgebras

• Commutation relations

$$(a_{(1)}, b_{(1)})\, a_{(2)} * b_{(2)} = b_{(1)} * a_{(1)}\, (a_{(2)}, b_{(2)})$$

$b \mapsto E_i$
$a \mapsto F_j$
$\Delta E_i = E_i \otimes K_{i,+} + 1 \otimes E_i$
$\Delta F_j = F_j \otimes 1 + K_{j,-}^{-1} \otimes F$

$$(F_j, E_i) K_{i,+} + (K_{j,-}^{-1}, 1) F_j E_i = E_i F_j (1, K_{i,+}) + K_{j,-}^{-} (F_j, E_i)$$

$$[E_i, F_j] = \delta_{i,j} \frac{K_{i,+} - K_{i,-}^{-1}}{q - q^{-1}}$$

Similarly $b \mapsto E_i$, $a \mapsto K_{j,-}$

$$(K_{j-}, 1) K_{j-} E_i = E_i K_{j-} (K_{j-}, K_{i+}) \Rightarrow K_{j-} E_i K_{j-}^{-1} = q^{a_{ij}} E_i$$

$q \mapsto K_{i,+} \quad a \mapsto F_j$

$$(K_{j,-}^{-1}, K_{i,+}) F_j K_{i,+} = K_{i,+} F_j (1, K_{i,+}) \Rightarrow K_{i,+} F_j K_{i,+}^{-1} = q^{-a_{ij}} F_j$$

Hence $K_{\ell,+} K_{\ell,-}^{-1}$ - central elements

$$\mathcal{U}_\hbar(\mathfrak{g}) = D(\mathcal{U}_\hbar(\mathfrak{h}^+)) \big/ \left(e^{\hbar(H_{\ell+} - H_{\ell,-})} - 1\right)$$

Pf of thm ① Construct a pairing $\hat{\mathcal{U}}_{\leq 0} \otimes \hat{\mathcal{U}}_{\geq 0} \to \mathbb{C}$

$\forall i$ define $\psi_i \in \mathcal{U}_\hbar(\mathfrak{h})^* \subset \hat{\mathcal{U}}_{\geq 0}^*$ $\quad \psi_i(e^{\hbar H_j}) = e^{\hbar a_{ij}} = e^{\hbar (H_i, H_j)}$

$X_i \in \left(\hat{\mathcal{U}}_{\geq 0}^*\right)_{-\alpha_i}$ $\quad X_i(e^{\hbar H_x} E_i) = \dfrac{1}{q - q^{-1}}$

Want ψ_i, X_i satisfy $\hat{\mathcal{U}}_{\leq 0}$ relations and coalgebra

- $\Delta\psi_i(e^{\hbar H_\lambda}\otimes e^{\hbar H_\mu}) = \psi_i(e^{\hbar(H_\lambda+H_\mu)}) = e^{\hbar(H_i, H_\lambda+H_\mu)} = (\psi_i\otimes\psi_i)(e^{\hbar H_\lambda}\otimes e^{\hbar H_\mu})$

Hence $\Delta\psi_i = \psi_i\otimes\psi_i$

$\Delta X_i(e^{\hbar H_\lambda}E_i\otimes e^{\hbar H_\mu} + e^{\hbar H_\lambda}\otimes e^{\hbar H_\mu}E_i) = X_i(e^{\hbar H_\lambda}E_i + E_i e^{\hbar H_\mu}) =$

$= X_i(E_i) + q^{-(H_i,H_\mu)}X_i(E_i) = (X_i\otimes 1 + \psi_i^{-1}\otimes X_i)(E_i\otimes e^{\hbar H_\mu} + e^{\hbar H_\lambda}\otimes E_i)$

Hence $\Delta X_i = X_i\otimes 1 + \psi_i^{-1}\otimes X_i$

- Similarly $\psi_i X_j = e^{-\hbar a_{ij}} X_j\psi_i$

② $u_{ij}^+ \in \tilde{\mathcal{U}}_{\geq 0}$ belong to kernel of $(\cdot , \cdot)$

Indeed $(e^{\hbar H_\lambda}F_I, u_{ij}^+) =$ $\qquad I = (i_1, i_2, \ldots)$

$= (e^{\hbar H}F_{i_1}, (u_{ij}^+)_{(1)})(F_{I'}, (u_{ij}^+)_{(2)}) = 0$ $\qquad I' = (i_2, \ldots)$

Since $\Delta u^+_{ij} = u^+_{ij} \otimes K_{\deg u^+_{i,j}} + 1 \otimes u^+_{ij}$

Hence we have $(\cdot , \cdot)\ \ \mathcal{U}_{\leq 0} \otimes \mathcal{U}_{\geq 0} \to \mathbb{C}$

(3) Remains to check that there is no kernel on $\mathcal{U}$

For $X \in \mathcal{U}^+$ let $\forall i\ \ r_i, r_i' : \mathcal{U}^+ \to \mathcal{U}^+$

$$\Delta X = X \otimes K_{\deg X} + \sum_i r_i(X) \otimes E_i K_{\deg X - \alpha_i} + \dots + \sum_i E_i \otimes r_i'(X) K_i + 1 \otimes X$$

<u>Prop</u> a) $r_i(E_j) = \delta_{ij} = r'(E_j)$

b) $r_i(XX') = r_i(X) X' + X r_i(X') q^{(\alpha_i, \deg X)}$

c) $r_i'(XX') = q^{(\alpha_i, \deg X')} r_i'(X) X' + X r_i'(X')$

<u>Pf</u> $\Delta X \Delta X' = \left(X \otimes K_{\deg X} + \sum r_i(X) \otimes E_i K_{\deg X - \alpha_i} + \dots + \sum_i E_i \otimes r_i'(X) K_i + 1 \otimes X \right)$

$\left(X' \otimes K_{\deg X} + \sum r_i(X') \otimes E_i K_{\deg X - \alpha_i} + \dots + \sum_i E_i \otimes r_i'(X') K_i + 1 \otimes X' \right) \square$

Prop $[X, F_i] = \frac{1}{q-q^{-1}}\left(r_i'(X)K_i - K_i^{-1} r_i(X)\right)$

Pf By induction

$$[XX', F_i] = [X, F_i]X' + X[X', F_i] =$$

$$= \frac{1}{q-q^{-1}}\left(\left(r_i'(X)K_i - K_i^{-1}r_i(X)\right)X' + X\left(r_i'(X')K_i - K_i^{-1}r_i(X')\right)\right)$$

$$= \frac{1}{q-q^{-1}}\left(\left(r_i'(X)X'q^{(\alpha_i, \deg X')} + X r_i'(X')\right)K_i - K_i^{-1}\left(r_i(X)X' + q^{(\alpha_i, \deg X)} X r_i(X')\right)\right) \qquad \square$$

Remark For u^+_{ij} we know $r_\ell(u^+_{ij}) = r_\ell'(u^+_{ij}) = 0 \Rightarrow [F_\ell, u^+_{ij}] = 0$

Remark Secretly, Prop. follows from

$$(a_{(1)}, b_{(1)})\, a_{(2)} * b_{(2)} = b_{(1)} * a_{(1)}\, (a_{(2)}, b_{(2)})$$

- **Lemma** For any $X \in \mathcal{U}^+$ $\exists i$ s.t. $r_i(X) \neq 0$ or $r_i'(X) \neq 0$

Indeed, otherwise $[X, F_i] = 0$ $\forall i \Rightarrow X = 0$ on any L_λ

Hence $X = 0$.

Problem Show that any $x \in \mathcal{U}^+$ do not belong to Ker ω

Hint Induction by λ, $x \in \mathcal{U}^+_\lambda$

$\square$

- Universal R matrix for $\mathcal{U}_\hbar sl_2$

Problem $(F^n, E^m) = \frac{[n]_q!}{(q-q^{-1})^n} q^{-\binom{n}{2}} \delta_{n,m}$

- $e^{2\alpha\beta\hbar} = \left(e^{\alpha\hbar H_-}, e^{\beta\hbar H_+}\right) = \sum_{k,l} \frac{\alpha^k \beta^l \hbar^{k+l}}{k!\, l!} \left(H_-^k, H_+^l\right)$

$\parallel$

$\sum \frac{2^n \alpha^n \beta^n}{n!}$

Hence $(H_-^k, H_+^l) = \delta_{k,l} \frac{2^k k!}{\hbar^k}$ in particular $(H_-, H_-) = \frac{2}{\hbar}$

Therefore $(H_-^k F^n, H_+^l E^m) = \delta_{k,l}\, \delta_{m,n} \frac{2^k k!}{\hbar^k} \frac{[n]_q}{(q-q^{-1})^n} q^{-\binom{n}{2}}$

Hence the universal R matrix for $D(\mathcal{U}_q(b^+))$ has form

$R = \sum (1 \otimes e^i) \otimes (e_i \otimes 1) \in \mathcal{D}(A) \otimes \mathcal{D}(A)$

$$R = \sum_k \frac{\hbar^k}{2^k k!} H_+^k \otimes H_-^k \sum_n \frac{(q-q^{-1})^n}{[n]_q!} q^{\binom{n}{2}} E^n \otimes F^n$$

$$= e^{\frac{\hbar}{2} H_+ \otimes H_-} \sum_n \frac{(q-q^{-1})^n}{[n]_q!} q^{\binom{n}{2}} E^n \otimes F^n$$

- For $\mathcal{U}_q(sl_2)$ $\quad R = e^{\frac{\hbar}{2} H \otimes H} \sum_n \frac{(q-q^{-1})^n}{[n]_q!} q^{\binom{n}{2}} E^n \otimes F^n$

as we had in Lecture 07

- Some central elements

Let $\lambda : \mathcal{U}_q(\mathfrak{g}) \to \mathbb{C}$ s.t. $\lambda(xy) = \lambda(y\, S^2(x))$

<u>Example</u> V - $\mathcal{U}_q(\mathfrak{g})$ module, take $\lambda_V(x) = \mathrm{Tr}|_V (x K_{2\rho})$

Pf $\lambda_V(xy) = \mathrm{Tr}|_V(xyK_{2\rho}) = \mathrm{Tr}|_V(yK_{2\rho}\, x K_{2\rho}^{-1} K_{2\rho}) = \lambda_V(y\, S^2(x))$

- **Prop** $(id\otimes\lambda)(R_{21}R_{12})$ – central element $U_q(\mathfrak{g})$

Pf If $\Delta(x)=\sum y_i\otimes z_i$, then $\sum (1\otimes S^{-1}(z_i))\Delta(y_i) = x\otimes 1$

$$x\,(id\otimes\lambda)[R_{21}R_{12}] = (id\otimes\lambda)\big[(x\otimes1)\,R_{21}R_{12}\big] = (id\otimes\lambda)\Big[\sum(1\otimes S^{-1}(z_i))\Delta(y_i)\,R_{21}R_{12}\Big]$$

prop of λ
$$= (id\otimes\lambda)\Big[\sum\Delta(y_i)\,R_{21}R_{12}\,(1\otimes S(z_i))\Big] \underset{\text{R-mat. prop.}}{=} (id\otimes\lambda)\Big[R_{21}R_{12}\sum\Delta(y_i)(1\otimes S(z_i))\Big]$$

$$=(id\otimes\lambda)[R_{21}R_{12}\,(x\otimes 1)] = (id\otimes\lambda)[R_{21}R_{12}]\,x \qquad \square$$

Corollary $C_V = (id\otimes\lambda_V)(R_{21}R_{12})$ – central element

Problem* For $\mathfrak{g}=\mathfrak{sl}_2$, $V=\mathbb{C}^2$ compute C_V

- **Prop** $V\mapsto C_V$ is algebra homomorphism $K(U_q(\mathfrak{g})\text{-mod})\to Z(U_q(\mathfrak{g}))$

(i.e. ⓐ $0\to V'\to V\to V''\to 0 \rightsquigarrow C_V = C_{V'}+C_{V''}$ ⓑ $V=V'\otimes V''\rightsquigarrow C_V=C_{V'}C_{V''}$

Pf (a) triv. (b):

$$(id\otimes\lambda_V)[R_{21}R_{12}] = (id\otimes\lambda_{V'}\otimes\lambda_{V''})(id\otimes\Delta)[R_{21}R_{12}]$$

$$= (id\otimes\lambda_{V'}\otimes\lambda_{V''})[R_{21}R_{31}R_{13}R_{12}] = (id\otimes\lambda_V)[R_{21}(C_{V''}\otimes 1)R_{12}] = C_{V'}C_{V''}$$

since $C_{V''}$ is central

$\square$

References

- Jantzen Lectures on Quantum groups
- Chary, Pressley A guide to quantum groups
  Sec 4.2 6.5
- Tanisaki Killing forms, Harish-Chandra isomorphisms and universal R-matrices for quantum algebras.