# Introduction to Quantum Groups

## Lecture 11

RLL realization

- Before $\mathcal{U}(\mathfrak{g}) \rightsquigarrow \mathcal{U}_q(\mathfrak{g})$. Quantize $\mathbb{C}[G]$?

- $\mathbb{C}[GL_n] = \mathbb{C}[t_{ij}, \det^{-1}]$

coproduct

$\Delta t_{ij} = \sum t_{ik} \otimes t_{kj}$ (dual to $(AB)_{ij} = \sum A_{ik} B_{kj}$)

product

Sklyanin bracket $\{T_1 \otimes T_2\} = [r, T_1 \otimes T_2]$

$R = 1 + \hbar r + \dots \leftarrow R T_1 T_2 - T_2 T_1 R = 0$

- $G \rightsquigarrow G^*$ Poisson-Lie dual group

$G^* = \{(L^+, L^-) \in B_+ \times B_- \mid p_+ L_+ \cdot p_- L_- = 1\}$ – recall

$p_\pm : B_\pm \to T$

$\{L_1^+, L_2^+\} = [r, L_1^+ \otimes L_2^+]$ $\{L_1^-, L_2^-\} = [r, L_1^- \otimes L_2^-]$

since $B_+, B_- \subset G$ P.L. subgroups.

• **Def** $\mathcal{U}(R)$ – assos algebra with unit generated by $e^+_{ij}, e^-_{ji}$ $\quad 1\le i\le j\le n$

$$L^+ = \begin{pmatrix} e^+_{11} & e^+_{12} & & e^+_{1n} \\ & \ddots & & \\ & & \ddots & e_{n-1\,n} \\ 0 & & & e^+_{nn} \end{pmatrix} \qquad L^- = \begin{pmatrix} e^-_{11} & & & \\ e^-_{21} & e^-_{22} & & 0 \\ \vdots & & \ddots & \\ e^-_{n1} & \dots & & e^-_{nn} \end{pmatrix}$$

with relations $\quad e^+_{ii} e^-_{ii} = 1$

$$R L_1^+ L_2^+ = L_2^+ L_1^+ R \qquad R L_1^- L_2^- = L_2^- L_1^- R \qquad R L_1^+ L_2^- = L_2^- L_1^+ R$$

here $\quad L_1^{\pm} = L^{\pm}\otimes 1 \quad L_2^{\pm} = 1\otimes L^{\pm}$

• Coproduct $\quad \Delta(L^{\pm}) = L^{\pm}\otimes L^{\pm} \qquad \left(\Delta e^{\pm}_{ij} = \sum e^{\pm}_{ik}\otimes e^{\pm}_{kj}\right)$

$S(L^{\pm}) = (L^{\pm})^{-1}$ (note, it is well defined)

$$R = \sum_i q\, E_{ii}\otimes E_{ii} + \sum_{i<j}\left(E_{ii}\otimes E_{jj} + E_{jj}\otimes E_{ii} + (q-q^{-1})\, E_{ij}\otimes E_{ji}\right)$$

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

R matrix for $\mathbb{C}^n\otimes\mathbb{C}^n$ $\quad U_q(sl_n)$

<u>Th</u> (Ding–Frenkel) Hopf algebras $\mathcal{U}(R)$ and $\mathcal{U}_q(\mathfrak{sl}_n)^{coop}$ are isomorphic $(m \leftrightarrow m \quad \Delta \leftrightarrow \Delta^{op})$

- Here $\mathcal{U}_q(\mathfrak{sl}_n)$ generated by $E_1, \dots, E_{n-1}, K_1^{\pm 1}, \dots, K_n^{\pm 1}, F_1, \dots, F_{n-1}$

Relations (different from $\mathfrak{sl}_n$)

$$i \begin{pmatrix} & 1 & \\ & \otimes & \\ & K_i \otimes & \end{pmatrix}$$

- $[E_i, F_j] = \delta_{ij} \dfrac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}}$

- $K_i E_j = q^{\delta_{ij}} q^{-\delta_{i\,j+1}} E_j K_i$
- $\Delta E_i = E_i \otimes K_i K_{i+1}^{-1} + 1 \otimes E_i$

- $K_i F_j = q^{-\delta_{ij}} q^{\delta_{i\,j+1}} F_j E_i$
- $\Delta F_i = F_i \otimes 1 + K_i^{-1} K_{i+1} \otimes F_i$

<u>Rk</u> $\mathcal{U}(R)$ has $n^2$ generators, many quadratic relations
$\mathcal{U}(\mathfrak{sl}_n)$ has $3n-2$ generators
+ Serre relations

- **Remark** $\mathcal{U}(R)$ is (a quotient of) Drinfeld double

Let $\mathcal{U}^+(R)$ – subalgebra generated by $\ell^+_{ij}$ (Corresp to $\mathcal{U}_{\leq 0}$ secretly)

$R L_1^+ L_2^+ = L_2^+ L_1^+ R \qquad \Delta L_1^+ = L_1^+ \otimes L_1^+$

$\mathcal{U}^-(R)$ ———— // ———— $\ell^-_{ji}$

$R L_1^- L_2^- = L_2^- L_1^- R \qquad \Delta L_1^- = L_1^- \otimes L_1^-$

- Pairing $\langle L_1^+, L_2^- \rangle = R_{12} \qquad \langle \ell^+_{ij}, \ell^-_{i'j'} \rangle = R^{jj'}_{ii'}$

$(a_{(1)}, b_{(1)})\, a_{(2)} * b_{(2)} = b_{(1)} * a_{(1)}\, (a_{(2)}, b_{(2)}) \qquad a \mapsto L^+ \quad b \mapsto L^-$

$R_{12} L_1^+ L_2^- = L_2^- L_1^+ R_{12}$

- Need to check that pairing is well defined.

Problem Show that $\langle R L_1^+ L_2^+ - L_2^+ L_1^+ R, - \rangle = 0$

Hint $\langle R L_1^+ L_2^+ - L_2^+ L_1^+ R, L_3^- \rangle \in \underset{1}{Mat_n} \otimes \underset{2}{Mat_n} \otimes \underset{3}{Mat_n}$

$$\langle R_{12} L_1^+ \otimes L_2^+ - L_2^+ \otimes L_1^+ R_{12}, L_3^- \otimes L_3^- \rangle$$

$$R_{12} R_{13} R_{23} - R_{23} R_{13} R_{12} = 0 \qquad \text{QYBE}$$

● PF Homomorphism $U_q(\mathfrak{gl}_n) \to U(R)^{coop}$

$$L^+ = \begin{pmatrix} K_1 & 0 & & 0 \\ 0 & \ddots & & 0 \\ & & \ddots & \\ 0 & & 0 & K_n \end{pmatrix} \begin{pmatrix} 1 & (q-q^{-1})F_1 & & * \\ 0 & \ddots & \ddots & \\ & & \ddots & (q-q^{-1})F_{n-1} \\ 0 & & 0 & 1 \end{pmatrix} \qquad L^- = \begin{pmatrix} 1 & & & \\ (q^{-1}-q)E_1 & \ddots & & 0 \\ & \ddots & \ddots & \\ * & & (q^{-1}-q)E_{n-1} & 1 \end{pmatrix} \begin{pmatrix} K_1^{-1} & 0 & & 0 \\ 0 & \ddots & & \\ & & \ddots & 0 \\ 0 & & 0 & K_n^{-1} \end{pmatrix}$$

$$K_i \mapsto \ell^+_{ii}, \quad K_i^{-1} \mapsto \ell^-_{ii}, \quad (q-q^{-1})K_iF_i \mapsto \ell^+_{i\,i+1} \quad (q^{-1}-q)E_iK_i^{-1} \mapsto \ell_{i+1,i}$$

coproduct

$$\Delta^{op} \ell^+_{i\,i+1} = \ell_{i\,i+1} \otimes \ell_{ii} + \ell_{i+1\,i+1} \otimes \ell_{i\,i+1}$$

$$\Delta K_iF_i = K_iF_i \otimes K_i + K_{i+1} \otimes K_i F_i$$

$$\Delta F_i = F_i \otimes 1 + K_i^{-1}K_{i+1} \otimes F_i$$

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$$\Delta^{op} \ell^-_{i+1,i} = \ell^-_{i+1\,i} \otimes \ell^-_{i+1\,i+1} + \ell^-_{i\,i} \otimes \ell_{i+1,i}$$

$$\Delta E_iK_i^{-1} = E_iK_i^{-1} \otimes K_{i+1}^{-1} + K_i^{-1} \otimes E_iK_i^{-1}$$

$$\Delta E_i = E_i \otimes K_iK_{i+1}^{-1} + 1 \otimes E_i$$

$$\Delta \ell^{\pm}_{ij} = \sum \ell^{\pm}_{ik} \otimes \ell^{\pm}_{kj}$$

- <u>Problem</u> a) Check directly quadratic relations on $E_i, F_i, K_i$
  b)* Check Serre relations

- <u>Problem</u> Show that $\mathcal{U}(R)$ is generated by $e^+_{ii}, e^-_{ii}, e^+_{i\,i+1}, e^-_{i+1,i}$ $\Longrightarrow$ <u>surjectivity</u>

- <u>Rk</u> Different "classical" limits

$$\mathcal{U}_q(\mathfrak{g}) \begin{array}{l} \nearrow \ \mathcal{U}(\mathfrak{g}) \\ \searrow \ \mathbb{C}[G^*]^{coop} \end{array}$$

For $\mathfrak{sl}_2$, take generators $\begin{pmatrix} 1 & 0 \\ (q^{-1}-q)E & 1 \end{pmatrix}\begin{pmatrix} K^{-1/2} & 0 \\ 0 & K^{+1/2} \end{pmatrix}$ $\begin{pmatrix} K^{1/2} & 0 \\ 0 & K^{-1/2} \end{pmatrix}\begin{pmatrix} 1 & (q-q^{-1})F \\ 0 & 1 \end{pmatrix}$

$K^{\pm\frac{1}{2}}$, $(q-q^{-1})K^{1/2}F$, $(q-q^{-1})EK^{-1/2}$ $\quad [(q-q^{-1})EK^{-1/2}, (q-q^{-1})K^{1/2}F] = (q-q^{-1})(K-K^{-1}) \xrightarrow[\hbar \to 0]{} 0$

- From universal $R$ matrix

$$R \in U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-) \subset U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n)$$

- $\rho : U_q(\mathfrak{gl}_n) \to Mat_n$ $n$-dim rep $\qquad (\rho\otimes\rho)R = R$

- Let $L^+ = (\rho \otimes id) R \in Mat_n \otimes U_q(\mathfrak{gl}_n) \qquad L^- = (id \otimes \rho) R^{-1}$

  Since $R \in U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-)$ $\qquad$ $L^+$ - upper triangular in $\mathfrak{b}^-$
  $L^-$ - lower triangular in $\mathfrak{b}^+$

- Problem Compute $(\rho \otimes id) R$, $(id \otimes \rho) R^{-}$ for $sl_2$

- $$R = R_H \left(1 + \sum (q-q^{-1}) E_i \otimes F_i + \ldots\right) \mapsto (\rho \otimes id)(R_H) \left( 1 + \begin{pmatrix} 0 & (q-q^{-1})F_1 & & \\ & 0 & & \\ & & \ddots & (q-q^{-1})F_{n-1} \\ & & & 0 \end{pmatrix} + \right)$$

Yang - Baxter

- $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \xmapsto{p\otimes p\otimes id} R_{12}L_1^+L_2^+ = L_2^+L_1^+R_{12}$

$R_{23}R_{12}^{-1}R_{13}^{-1} = R_{13}^{-1}R_{12}^{-1}R_{23} \xrightarrow{id\otimes p\otimes p} R_{12}L_1^-L_2^- = L_2^-L_1^-R_{12}$

$R_{13}R_{12}R_{23}^{-1} = R_{23}^{-1}R_{12}R_{13} \xrightarrow{p\otimes id\otimes p} R_{12}L_1^+L_2^- = L_2^-L_1^+R_{12}$

algebra hom

- $(id\otimes\Delta)R = R_{13}R_{12} \xmapsto{p\otimes id\otimes id} \Delta L^+ = L_2^+\otimes L_1^+$

$(\Delta\otimes id)R = R_{13}R_{23} \longmapsto (\Delta\otimes id)R^{-1} = R_{23}^{-1}R_{13}^{-1} \xrightarrow{id\otimes id\otimes p}$

coalg. homom.

$\Delta L^- = L_2^-\otimes L_1^-$

## Injectivity

Recall for $\forall \lambda \in P_+$ $\exists$ $L_\lambda$ - f.d. rep $U_q(\mathfrak{gl}_n)$

Hence $\forall \lambda$ $L_\lambda^+ = (\rho_{\mathbb{C}^n} \otimes \rho_{L_\lambda})R$, $L_\lambda^- = (\rho_{L_\lambda} \otimes \rho_{\mathbb{C}^n})R^{-1}$ — satisfy RLL relations

$$\begin{tikzcd} U_q(\mathfrak{gl}_n) \arrow[r] \arrow[rr, bend right] & U(R) \arrow[r, dashed] & \mathrm{Mat}_{L_\lambda} \end{tikzcd}$$

If $I = \mathrm{Ker}(U_q(\mathfrak{gl}_n) \to U(R))$, then $I \subset \cap \mathrm{Ker}\, \rho_{L_\lambda} = 0$

see Lecture 09

$\square$

# References


- Ding, Frenkel Isomorphism of two realizations of quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$