Introduction to Quantum Groups

Lecture 12

Functions on quantum SL_2

- $\mathbb{C}[Mat_n] = \mathbb{C}[t_{ij}]$ $\quad 1 \le i,j \le n$

  $\mathbb{C}[Mat_n] \to \mathbb{C}[Mat_n] \otimes \mathbb{C}[Mat_n]$ $\quad$ dual to $M_1 M_2$

  $\Delta t_{ij} = \sum t_{ik} \otimes t_{kj}$

  $\Delta T = T \otimes T \in \mathbb{C}[Mat]^{2} \otimes Mat$ $\quad T = \begin{pmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nn} \end{pmatrix} = \sum t_{ij} E_{ij}$

$\varepsilon(t_{ij}) = \delta_{ij}$ $\quad$ dual to evaluation at E

- $S: \mathbb{C}[GL_n] \to \mathbb{C}[GL_n]$ $\quad \mathbb{C}[t_{ij}][det^{-1}]$

  $T \mapsto T^{-1}$

- $\mathbb{C}[SL_n] = \mathbb{C}[t_{ij}] / det - 1$

- Comodules

$GL_2 \times V \to V \qquad V = \mathbb{C}^2$

$\Delta : \mathbb{C}[V] \to \mathbb{C}[GL_2] \otimes \mathbb{C}[V]$ homomorphism of algebras

$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ coassosiativity, cf. $(g_1 g_2)v = g_1(g_2 v)$

$\mathbb{C}[V] \to \mathbb{C}[GL_2] \otimes \mathbb{C}[V] \rightrightarrows \mathbb{C}[GL_2] \otimes \mathbb{C}[GL_2] \otimes \mathbb{C}[V]$

- In coordinates $\quad \mathbb{C}[V] = \mathbb{C}[x_1, x_2]$

$\Delta : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \Delta x_1 = t_{11} \otimes x_1 + t_{12} \otimes x_2$

- Dual module $\quad V^* \times GL_2 \to V^* \qquad \mathbb{C}[V^*] \to \mathbb{C}[V^*] \otimes \mathbb{C}[GL_2]$

$(x_1, x_2) \mapsto (x_1, x_2) \otimes \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$

- $\mathbb{C}_q[V] = \mathbb{C}\langle x_1, x_2\rangle / (x_1x_2 - q^{-1}x_2x_1)$

$$\Delta : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad X \mapsto T \otimes X$$

coassos.

$$(\Delta \otimes id)\Delta X = (\Delta \otimes id)(T \otimes X) = T \otimes T \otimes X = (id \otimes \Delta)(T \otimes X) = (id \otimes \Delta)\Delta X$$

- homomorphism $x_1x_2 = q^{-1}x_2x_1 \mapsto$

$$(t_{11} \otimes x_1 + t_{12} \otimes x_2)(t_{21} \otimes x_1 + t_{22} \otimes x_2) = q^{-1}(t_{21} \otimes x_1 + t_{22} \otimes x_2)(t_{11} \otimes x_1 + t_{12} \otimes x_2)$$

$t_{11}t_{21} = q^{-1}t_{21}t_{11}$ $\qquad t_{12}t_{22} = q^{-1}t_{22}t_{12}$

$t_{11}t_{22} + q\, t_{12}t_{21} = q^{-1}t_{21}t_{12} + t_{22}t_{11}$

- $\mathbb{C}_q[V^*] \to \mathbb{C}_q[V^*] \otimes \mathbb{C}_q[G]$ transposed relations

$$t_{11} t_{12} = q^{-1} t_{12} t_{11} \qquad t_{21} t_{22} = q^{-1} t_{22} t_{21}$$

$$t_{11} t_{22} + q\, t_{21} t_{12} = q^{-1} t_{12} t_{21} + t_{22} t_{11}$$

- $t_{12} t_{21} = t_{21} t_{12}$

Overall $\mathbb{C}[Mat_2]_q = \mathbb{C}\langle t_{11}, t_{12}, t_{21}, t_{22} \rangle /$ all relations

$$t_{11} t_{21} = q^{-1} t_{21} t_{11}, \quad t_{12} t_{22} = q^{-1} t_{22} t_{12} \qquad t_{12} t_{21} = t_{21} t_{12}$$

$$t_{11} t_{12} = q^{-1} t_{12} t_{11}, \quad t_{21} t_{22} = q^{-1} t_{22} t_{21}$$

$$t_{11} t_{22} - t_{22} t_{11} + (q - q^{-1}) t_{12} t_{21} = 0$$

- Matrix form

$$R\, T_1 T_2 = T_2 T_1 R \qquad R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

$$R_{ii'}^{jj'} t_{jk} t_{j'k'} = t_{i'k'} t_{ik} R_{jj'}^{kk'}$$

Equivalently

$$\hat{R} T_1 T_2 = T_1 T_2 \hat{R} \qquad \hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & (q-q^{-1}) & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

Quantization of $\{T_1, T_2\} = [r, T_1 \otimes T_2]$

- In order to define $\mathbb{C}[GL_2]_q$, $\mathbb{C}[SL_2]_q$ we need $q\det(T) = t_{11} t_{22} - q^{-1} t_{12} t_{21}$

<u>Problem</u> Show directly that ⓐ $\Delta q\det = q\det \otimes q\det$
ⓑ $q\det$ is central

<u>Def</u> $\mathbb{C}[GL_2]_q = \mathbb{C}[Mat_2]_q [q\det^{-1}]$

$$\mathbb{C}[SL_2]_q = \mathbb{C}[Mat_2]_q / q\det - 1$$

- A - f.d. Hopf algebra $\rightsquigarrow$ A^* Hopf algebra

<u>Prop</u> Let $A^\circ \subset A^*$ be a subalgebra generated by matrix elements of f.d. reps of A.
Then A° is Hopf algebra

<u>Rk</u> Peter-Weyl $\mathbb{C}[G] = \bigoplus L_\lambda \otimes L_\lambda^*$ for G s/s
Here by definition $U_q(\mathfrak{sl}_2)^\circ = \bigoplus L_{\lambda q} \otimes L_{\lambda q}^*$

<u>Proof</u> V_1, V_2 - f.d reps $\quad t_{ij} \in \mathrm{End}(V_1)$, $t_{ke} \in \mathrm{End}(V_2)$

Product $\quad t_{ij} \cdot t_{ke}(x) = t_{ij} \otimes t_{ke}(\Delta(x)) \Rightarrow t_{ij} t_{ke} \in \mathrm{End}(V_1 \otimes V_2)$

Coproduct $\quad \Delta t_{ij}(x_1 \otimes x_2) = t_{ij}(x_1 \cdot x_2) \qquad \Delta t_{ij} = \sum t_{ik} \otimes t_{kj}$ $\square$

Th $\mathbb{C}[SL_2]_q \simeq U_q(sl_2)^{0,I}$ (Matrix elements of type I reps)

Pf • $t_{11}, t_{12}, t_{21}, t_{22} \mapsto$ matrix elements of $\mathbb{C}^2$

$\varphi: \mathbb{C}[SL_2]_q \to U_q(sl_2)^{0,I}$ $\quad x \mapsto \begin{pmatrix} t_{11}(x) & t_{12}(x) \\ t_{21}(x) & t_{22}(x) \end{pmatrix}$

• $\mathbb{C}^2 \otimes_\Delta \mathbb{C}^2 \xrightarrow{\tilde{R}} \mathbb{C}^2 \otimes_\Delta \mathbb{C}^2$ intertwining operator

$\tilde{R}\,\Delta(x) = \Delta(x)\,\tilde{R}$

Hence $\forall x \in U_q(sl_2)$

$$R^{kk'}_{ii'}(t_{kj}\, t_{k'j'})(x) = \tilde{R}^{kk'}_{ii'}(t_{kj} \otimes t_{k'j'})\Delta(x) = (t_{ik} \otimes t_{i'k'})\Delta(x) R^{kk'}_{jj'} =$$

$$= t_{ik}\, t_{i'k'}(x)\, R^{kk'}_{jj'}$$

Therefore $\tilde{R}\, T_1 T_2 = T_1 T_2 \tilde{R}$.

• $\xi_1 \otimes \xi_2 - q\,\xi_2 \otimes \xi_1 \subset \mathbb{C}^2 \otimes \mathbb{C}^2$ generates trivial $U_q\, sl_2$ module

I.e. $\forall x \in U_q(sl_2)$ $\quad \Delta(x)(\xi_1\otimes\xi_2 - q\,\xi_2\otimes\xi_1) = \varepsilon(x)$

$\qquad = (t_{11}t_{12} - q\,t_{12}t_{11})(x)\,\xi_1\otimes\xi_1 + (t_{11}t_{22} - q\,t_{21}t_{12})\,\xi_1\otimes\xi_2 \Rightarrow q\det(x) = \varepsilon(x)$

q' comm $\qquad\qquad$ q det !

We have homomorphism of Hopf algebras

- ψ is surjective since $\forall$ f.d. rep $U_q(sl_2)$ is direct summand in a product of $V^{\otimes n}$ for some n.

- Introduce filtrations

$B = \mathbb{C}[SL_2]_q \quad B_0 \subset B_1 \subset B_2 \subset \dots \quad B_k$ generated by $t_{i_1 j_1} \dots t_{i_\ell j_\ell}$ $\ell \le k$

$A = U_q(sl_2)^{0,I} \quad A_0 \subset A_1 \subset A_2 \subset \dots \quad A_k$ generated by $End(L_\ell)$ $\ell \le k$

<u>Problem</u> Show that $End(L_\ell)$ are linearly independent for different $\ell \in \mathbb{Z}_{\ge 0}$

<u>Hint</u> Define action of $U_q(sl_2)$ on $U_q(sl_2)^0$.
Use casimir C_q

Hence $\dim A_k = 1+2^2+\ldots+(k+1)^2$ (We use $End(L_e) \hookrightarrow U_q(sl_2)^0$ since L_e is irrep)

- B is linearly generated by

$t_{11}t_{21}=q^{-1}t_{21}t_{11}$, $t_{12}t_{22}=q^{-1}t_{22}t_{12}$
$t_{11}t_{12}=q^{-1}t_{12}t_{11}$, $t_{21}t_{22}=q^{-1}t_{22}t_{21}$
$t_{11}t_{22}-t_{22}t_{11}+(q-q^{-1})t_{12}t_{21}=0$
$t_{11}t_{22}-qt_{12}t_{21}=1$ $t_{12}t_{21}=t_{21}t_{12}$

If we have both t_{11}, t_{22} we can simplify

$$t_{12}^{k_1}t_{21}^{k_2} \qquad t_{11}^{k_0}t_{12}^{k_1}t_{21}^{k_2}, \qquad t_{22}^{k_0}t_{12}^{k_1}t_{21}^{k_2} \qquad k_0>0,\ k_1, k_2 \geq 0$$

Hence $\dim B_k/B_{k-1} \leq (k+1) + \frac{k(k+1)}{2} + \frac{k(k+1)}{2} = (k+1)^2$

$\psi(B_k) \subset A_k$ ψ surj $\Rightarrow$ ψ is isomorphism $\square$

Problem* Let $\mathcal{U}_\hbar(\mathfrak{g})$ – quantum universal enveloping alg.

Let $\Delta_n = (\Delta \otimes id \otimes \ldots \otimes id) \circ \ldots (\Delta \otimes id) \circ \Delta : \mathcal{U}_\hbar(\mathfrak{g}) \to \mathcal{U}_\hbar(\mathfrak{g})^{\otimes n}$

Let $A = \{ x \in \mathcal{U}_\hbar(\mathfrak{g}) \mid (id - \varepsilon)^{\otimes n} \Delta_n x \in \hbar^n \, \mathcal{U}_\hbar(\mathfrak{g})^{\otimes n};\ \forall n \}$

Show that A is Hopf subalg. cocomm up to first order in $\hbar$

Remark We found $\mathbb{C}[G^*]$ in $\mathcal{U}_\hbar(\mathfrak{g})$

References

- Chary, Pressley A guide to quantum groups Sec 7.1
- Korogodski Soibelman Algebras of functions on quantum group Sec 3.1, 3.3
- Etingof, Schiffmann Lectures on quantum groups Ch 10