Introduction to Quantum Groups

Lecture 13

Functions on quantum SL_n

Def $\mathbb{C}[Mat_n]_q$ – is an algebra gen. by t_{ij} $\quad 1 \le i,j \le n$

$\tilde{R}\, T_1 T_2 = T_1 T_2 \tilde{R}$ where

$$\tilde{R} = \sum_i q\, E_{ii} \otimes E_{ii} + \sum_{i<j} \left(E_{ji} \otimes E_{ij} + E_{ij} \otimes E_{ji} + (q-q^{-1}) E_{jj} \otimes E_{ii} \right)$$

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

$$\Delta t_{ij} = \sum_k t_{ik} \otimes t_{kj}$$

- Without matrix notations $\displaystyle\sum_{k,k'} \tilde{R}^{kk'}_{ii'}\, t_{kj}\, t_{k'j'} = \sum_{k,k'} t_{ik}\, t_{i'k'}\, \tilde{R}^{jj'}_{kk'}$

- More explicitly

(i) $t_{ij} t_{i'j} = q\, t_{i'j} t_{ij} \quad i<i' \qquad t_{ij} t_{ij'} = q\, t_{ij'} t_{ij} \quad j<j'$

(ii) $t_{ij} t_{i'j'} = t_{i'j'} t_{ij} \quad i<i',\ j>j'$

(iii) $t_{ij} t_{i'j'} = t_{i'j'} t_{ij} + (q-q^{-1})\, t_{ij'} t_{i'j} \quad i<i',\ j<j'$

Remark ⓐ $RT_1T_2 = T_2T_1R$ ⓑ $q \to 1$ $\{g \otimes g\} = [r, g \otimes g]$

Sklyanin bracket

ⓒ Transposition $t_{ij} \mapsto t_{ji}$ is authomorphism since $\hat{R}^t = \hat{R}$

- Comodules Def $S_q(V) = \mathbb{C}\langle x_1, \dots, x_n \rangle / x_i x_j - q^{-1} x_j x_i$ $i<j$

q deform SV

as a vector space $S_q(V)$ has basis $x_1^{a_1} \dots x_n^{a_n}$ $a_i \in \mathbb{Z}_{\geq 0}$

$\Delta: S_q(V) \to \mathbb{C}[Mat_n]_q \otimes S_q(V)$ $\Delta(x_i) = \sum t_{ij} \otimes x_j$

Def $\Lambda_q V = \mathbb{C}\langle \xi_1, \dots, \xi_n \rangle / \xi_i \xi_j + q \xi_j \xi_i$ $i \leq j$

q deform ΛV has basis $\xi_1^{a_1} \dots \xi_n^{a_n}$ $a_i = \{0, 1\}$

$\Delta: \Lambda_q(V) \to \mathbb{C}[Mat_n]_q \otimes \Lambda_q(V)$ $\Delta(\xi_i) = \sum t_{ij} \otimes \xi_j$

Prop For both $S_q V$ and $\Lambda_q V$ Δ is homom. of algebras

- Relations are quadratic

Eigenvalues of $\hat{R}$ $\quad q$ of multiplicity $\binom{n+1}{2}$

$-q^{-1}$ of multiplicity $\binom{n}{2}$

Since $\hat{R}$ consist of n blocks (q)

$\frac{n^2-n}{2}$ blocks $\begin{pmatrix} 0 & 1 \\ 1 & q-q^{-1} \end{pmatrix}$

- Bilinears of ξ_i belongs to $\Lambda^2_q \mathbb{C}^n \subset \mathbb{C}^n \otimes \mathbb{C}^n$

$\Lambda^2_q \mathbb{C}^n = \mathrm{Ker}(\hat{R} + q^{-1})$ hence $(\hat{R}+q^{-1})\begin{pmatrix} \xi_1 \otimes \xi_1 \\ \xi_2 \quad \xi_2 \end{pmatrix} = 0$

$$\xi_1^2 = \xi_2^2 = 0 \quad \Longleftarrow \quad \begin{pmatrix} q+q^{-1} & & & \\ & q^{-1} & 1 & \\ & 1 & q & \\ & & & q+q^{-1} \end{pmatrix} \begin{pmatrix} \xi_1 \xi_1 \\ \xi_1 \xi_2 \\ \xi_2 \xi_1 \\ \xi_2 \xi_2 \end{pmatrix} = 0$$

$$\xi_1 \xi_2 + q\, \xi_2 \xi_1 = 0$$

- Bilinears of x_i belongs to $S^2_q \mathbb{C}^n \subset \mathbb{C}^n \otimes \mathbb{C}^n$

$$\Lambda^2_q \mathbb{C}^n = \operatorname{Ker}(\hat{R} - q) \quad \text{hence} \quad (\hat{R} - q)\left(\begin{matrix} \xi_1 \\ \xi_2 \end{matrix} \otimes \begin{matrix} \xi_1 \\ \xi_2 \end{matrix} \right) = 0$$

$$x_1 x_2 - q^{-1} x_2 x_1 = 0 \quad \Longleftarrow \quad \begin{pmatrix} 0 & & & \\ & -q & 1 & \\ & 1 & -q^{-1} & \\ & & & 0 \end{pmatrix} \begin{pmatrix} x_1 x_1 \\ x_1 x_2 \\ x_2 x_1 \\ x_2 x_2 \end{pmatrix} = 0$$

q-Minors

$$S_q^{\cdot} V = \bigoplus_k S_q^k V \qquad \Lambda_q^{\cdot} V = \bigoplus_k \Lambda_q^k V$$

$$\Delta : S_q^k V \to \mathbb{C}[Mat_n]_q \otimes S_q^k V \qquad \Delta : \Lambda_q^k V \to \mathbb{C}[Mat_n]_q \otimes \Lambda_q^k V$$

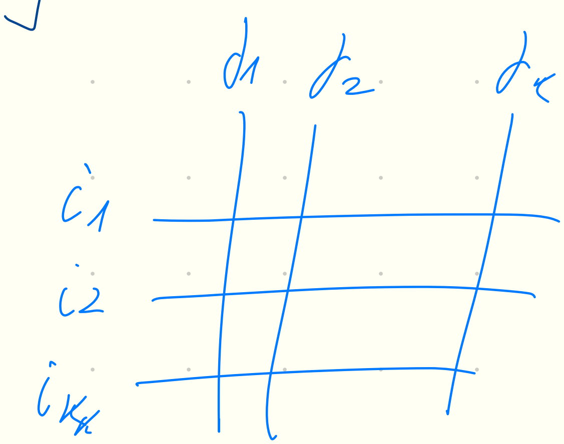
- For $I = \{1 \le i_1 < i_2 < \dots < i_r \le n\}$ $\quad J = \{1 \le j_1 < j_2 < \dots < j_k \le n\}$

define t_I^J by $\Delta \xi_I = \sum t_I^J \otimes \xi_J$, where $\xi_I = \xi_{i_1} \cdots \xi_{i_k}$

- $\Delta(\xi_{i_1} \cdots \xi_{i_k}) = \Delta \xi_{i_1} \Delta \xi_{i_2} \cdots \Delta \xi_{i_k} = \left(\sum_{l_1} t_{i_1 l_1} \otimes \xi_{l_1}\right) \cdots \left(\sum_{l_k} t_{i_k l_k} \otimes \xi_{l_k}\right)$

$$= \sum_{j_1 < \dots < j_k} \left(\sum_{\sigma \in S_k} (-q)^{-|\sigma|} \, t_{i_1 j_{\sigma(1)}} \cdots t_{i_k j_{\sigma(k)}} \right) \otimes \xi_J$$

$|\sigma| = \#\{i, j \mid i < j, \ \sigma(i) > \sigma(j)\}$



- **Prop** $\Delta t_I^J = \sum_K t_I^K \otimes t_K^J$

Proof $\Delta t_I^J \otimes \xi_J = (\Delta \otimes id) \Delta \xi_I = (id \otimes \Delta) \Delta \xi_I = (id \otimes \Delta)(t_I^K \otimes \xi_K)$

$$= t_I^K \, t_K^J \otimes \xi_J$$

Remark For $|I| = |J| = k$ $\{t_I^J\}$ – matrix element $\Lambda^k \mathbb{C}^n$
coproduct $\Delta T = T \otimes T$

- $qdet = t_{1,\dots,n}^{1,\dots,n}$

Corol (of prop) $\Delta t_{1,\dots,n}^{1,\dots,n} = t_{1,\dots,n}^{1,\dots,n} \otimes t_{1,\dots,n}^{1,\dots,n} \iff \Delta qdet = qdet \otimes qdet$

- **Prop** (Laplace expansion) ⓐ For given $J_1 \sqcup J_2 = J$

$$sgn(J_1, J_2) \, t_I^J = \sum_{I_1 \sqcup I_2 = I} t_{I_1}^{J_1} \, t_{I_2}^{J_2} \, sgn_q(I_1, I_2)$$

(b) For given $I_1 \sqcup I_2 = I$

$$\mathrm{sgn}(I_1, I_2)\, t_I^J = \sum_{J_1 \sqcup J_2 = J} t_{I_1}^{J_1} t_{I_2}^{J_2} \mathrm{sgn}_q(J_1, J_2)$$

where $\mathrm{sgn}_q(A,B) = \begin{cases} 0 & \text{if } A \cap B \neq \emptyset \\ (-q)^{-\#\{a\in A,\, b\in B \mid a > b\}} \end{cases}$

(c) If $J_1 \cap J_2 \neq \emptyset$ then r.s $=0$. Similarly for $I_1 \cap I_2 \neq \emptyset$

Remark For $|J_1| = 1$ or $|J_2| = 1$ (a) column expansion
For $|I_1| = 1$ or $|I_2| = 1$ (b) row expansion

Proof (b, c) $\xi_{I_1} \xi_{I_2} = \xi_I \, \mathrm{sgn}_q(I_1, I_2)$

$$\mathrm{sgn}_q(I_1, I_2) \sum t_I^J \otimes \xi_J = \mathrm{sgn}_q(I_1, I_2) \Delta \xi_I = \Delta(\xi_{I_1} \xi_{I_2}) =$$

$$= \sum t_{I_1}^{J_1} \otimes \xi_{J_1} \sum t_{I_2}^{J_2} \otimes \xi_{J_2} = \sum \mathrm{sgn}_q(J_1, J_2)\, t_{I_1}^{J_1} t_{I_2}^{J_2} \otimes \xi_J$$

(B) => (a) use authomorphism $t_{ij} \leftrightarrow t_{ji}$

$t_I^J \leftrightarrow t_J^I$ $\square$

<u>Problem</u> (Plücker relations) For given

$J = \{j_1 < \dots < j_{r-1}\}$, $K = \{k_0 < \dots < k_r\}$, $I = \{i_1 < \dots < i_r\}$

$$\sum_{s=0}^{r} \mathrm{sgn}(J, \{k_s\}) (-q)^{-s} \, t_{j_1, \dots, k_s, \dots, j_{r-1}}^{i_1, \dots, i_r} \, t_{k_0, \dots, \hat{k}_s, \dots, k_r}^{i_1, \dots, i_r} = 0$$

- Hopf algebra structure

$$T^\vee = \sum E_{ij} \, t_{1, \dots, \hat{i}, \dots, n}^{1 \dots \hat{j}, \dots n} (-q)^{i-j}$$

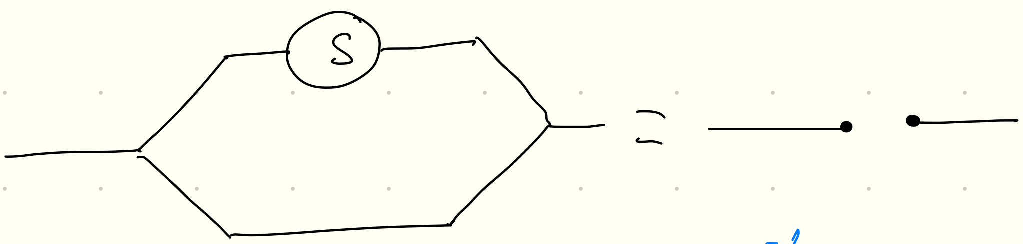
<u>Prop</u> $T^\vee T = q\det \mathrm{Id}_n = T T^\vee$

Corollary $T\,\mathrm{qdet} = T T^\vee T = \mathrm{qdet}\, T \Rightarrow \mathrm{qdet}$ central

Def $\mathbb{C}[GL_n]_q = \mathbb{C}[t_{ij}, \mathrm{qdet}^{-1}]$, $\quad \mathbb{C}[SL_n]_q = \mathbb{C}[t_{ij}] / (\mathrm{qdet} - 1)$

Lemma $S(T) = T^\vee \mathrm{qdet}^{-1}$ satisfies antipode properties

Pf



$S(T) \cdot T \longrightarrow \mathrm{qdet}^{-1} T^\vee T = E_n$

Remark S antiautomorphism

- **Theorem** $\mathbb{C}[Mat_n]_q$ has basis $t_{11}^{a_{11}} \cdot t_{12}^{a_{12}} \cdot \ldots \cdot t_{1n}^{a_{1n}} \cdot t_{21}^{a_{21}} \cdot \ldots \cdot t_{2n}^{a_{2n}} \cdot \ldots \cdot t_{nn}^{a_{nn}}$
$a_{11}, \ldots, a_{nn} \in \mathbb{Z}_{\geq 0}$

Plan of pf Use diamond lemma for lexicogr. order $t_{11} < t_{12} < \ldots < t_{1n} < t_{21} < \ldots < t_{nn}$ $\square$

- **Th** $\underbrace{\mathbb{C}[SL_n]_q \simeq U_q(\mathfrak{sl}_n)^{0,I}}_{\text{Isom. Hopf. algebras}} = \bigoplus_{\lambda \in P_+} L_\lambda \otimes L_\lambda^*$ ← type I reps

Plan of pf Constuct homomorphism $\varphi : \mathbb{C}[SL_n]_q \to U_q(\mathfrak{sl}_n)^0$ using $\mathbb{C}^n \in U_q(\mathfrak{sl}_n)$-mod
$\forall \lambda \in P_+ \quad L_\lambda \subset (\mathbb{C}^n)^{\otimes N}$ for some $N \Rightarrow$ surj. of φ
Compare sizes (using Th above) $\Rightarrow$ inj of φ $\square$

- **Problem** Center of $\mathbb{C}[Mat_n]_q$ is generated by $q\det$

Hint Show that if $y \in S[Mat_n]_q$ is central, then its highest term w.r.t. lexicographical order above is $(t_{11} \cdot t_{22} \cdot \ldots \cdot t_{nn})^d$ for some $d \in \mathbb{Z}_{\geq 0}$

- **Def** $I \subset A$ is primitive if $I = \operatorname{Ann} M$, M - simple A-mod

Th Primitive ideals in $\mathbb{C}[G]_q \longleftrightarrow$

- For $v \in (L_\lambda)(\mu)$, $\ell \in (L_\lambda^*)(\nu)$ $t^\lambda_{\ell,v} \in U_q(\mathfrak{g})^\circ$ - corresp matrix element

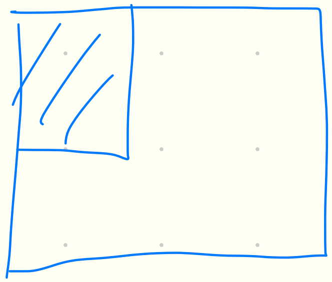
We abbreviate $t^\lambda_{\ell,v}$ to $t^\lambda_{\nu,\mu}$ Note $L_\lambda^* \simeq L_{-w_0(\lambda)}$

- **Example** For $\mathfrak{g}=\mathfrak{sl}_n$, ϖ_k – k th fundamental weight

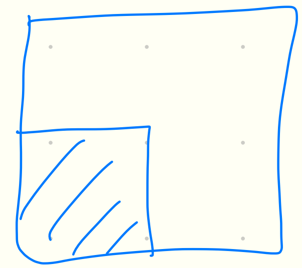
$t^{\varpi_k}_{-w(\varpi_k),\varpi_k} = t^{1,\dots,k}_{i_1,\dots,i_k}$ where $i_s = w(s)$ $\qquad t^{\varpi_k}_{-w(\varpi_k),w_0(\varpi_k)} = t^{n-k+1,\dots,n}_{i_1,\dots,i_k}$

In particular

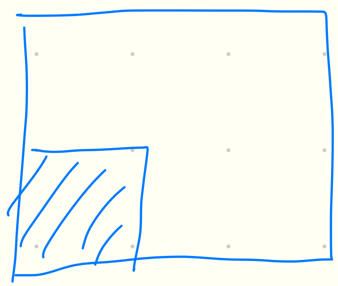
$t^{\varpi_k}_{-\varpi_k,\varpi_k} = t^{1,\dots,k}_{1,\dots,k}$



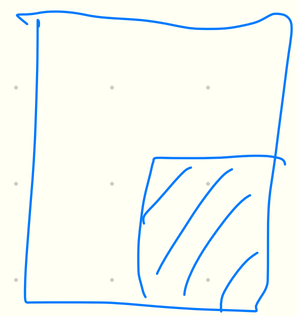
$t^{\varpi_k}_{-w_0(\varpi_k),\varpi_k} = t^{1,\dots,k}_{n-k+1,\dots,n}$



$t^{\varpi_k}_{-\varpi_k,w_0(\varpi_k)} = t^{n-k+1,\dots,n}_{1,\dots,k}$



$t^{\varpi_k}_{-w_0(\varpi_k),w_0(\varpi_k)} = t^{n-k+1,\dots,n}_{n-k+1,\dots,n}$



- **Problem** (a) $t^{\lambda}_{-w_0(\lambda),\lambda}\, t^{\lambda'}_{-\nu,\mu} = q^{(\lambda,\mu)-(w_0(\lambda),\nu)}\, t^{\lambda'}_{-\nu,\mu}\, t^{\lambda}_{-w_0(\lambda),\lambda}$

(b) $t^{\lambda}_{-\lambda,w_0(\lambda)}\, t^{\lambda'}_{-\nu,\mu} = q^{(\lambda,\nu)-(w_0(\lambda),\mu)}\, t^{\lambda'}_{-\nu,\mu}\, t^{\lambda}_{-\lambda,w_0(\lambda)}$

(c) Elements $t^{\lambda}_{-w_0(\lambda),\lambda}$, $t^{\lambda}_{-\lambda,w_0(\lambda)}$ form commutative algebra

- Hint ⓐ Use $R\,T_1T_2 = T_2T_1R$.

Universal R matrix $R = \bar{R}_H \bar{R}$ where

$R = q^{H_j \otimes H^j}$, H_j, H^j - dual bases, $\bar{R} \in U_q(n^+) \otimes U_q(n^-)$

$R\, v_\lambda \otimes v = q^{(\lambda,\mu)} v_\lambda \otimes v$ where $v_\lambda \in (L_\lambda)_{(\lambda)}$ $v \in (L_{\lambda'})_{(\mu)}$

$$(e_{-w_0(\lambda)} \otimes e,\ T_1T_2R\, v_\lambda \otimes v) = q^{(\lambda,\mu)}\, t^{\lambda'}_{-\mu,\nu}\, t^{\lambda}_{-\lambda, w_0(\lambda)}$$

$$\parallel$$

$$(e_{-w_0(\lambda)} \otimes e,\ R\,T_2T_1\, v_\lambda \otimes v) = q^{(w_0(\lambda),\nu)}\, t^{\lambda}_{-\lambda, w_0(\lambda)}\, t^{\lambda'}_{-\nu,\mu}$$

ⓑ Use $R_{21}^{-1}\, TT = TT\, R_{21}^{-1}$

- Def A_+ - subalgebra generated by $t^{\lambda}_{-\mu,\lambda}$

  A_- - subalgebra generated by $t^{\lambda}_{-\mu, w_0(\lambda)}$

Th (Triangular decomposition)

Map $A_+ \otimes A_- \to \mathbb{C}[G]_q$ is surj

Problem (a) A_+ is generated by $t^{1,\dots,k}_{i_1,\dots,i_k}$

(b) A_- is generated by $t^{n-k+1,\dots,n}_{i_1,\dots,i_k}$

(c) Commutative subalgebra (c) above is generated by $t^{n-k+1,\dots,n}_{1,\dots,k}$ $t^{1,\dots,k}_{n-k+1,\dots,n}$

Hint If $\lambda = \sum \ell_i \varpi_i$ $\quad L^q_\lambda \hookrightarrow \otimes L^{\otimes \ell_k}_{\varpi_k}$

References

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- Noumi, Yamada, Mimachi Finite dimensional representations of the quantum group $GL_q(n,\mathbb{C})$ and zonal spherical functions on $U_q(n-1)\backslash U_q(n)$
- Hodges Levasseur Primitive Ideals of $\mathbb{C}[SL_3]_q$