Introduction to Quantum Groups

Lecture 14

Lusztig's Braid group

- $U_q(\mathfrak{g})$ — QUE $\qquad E_i, K_i^{\pm 1}, F_i$

Construction is not symmetric w.r.t adjoint action G on $\mathfrak{g}$

Cartan subalgebra is fixed.

- W — Weyl group

$W = N(H)/H \iff$ S_i generators $S_i = S_{\alpha_i}$ $\quad \Pi = \{\alpha_1, \dots, \alpha_r\}$

relations $s_i^2 = e$, Braid group relations

<u>Caveat</u> No natural embeding of W into G.

Hence no natural action W on $\mathfrak{g}$

<u>Ex</u> $G = SL_2 \qquad W = \{e, s \mid s^2 = e\} \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \notin SL_2$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ — order 4

<u>Solution</u> (Tits) $\exists\, \widetilde{W}$ central extension of W $\qquad$ Braid group

$e \to (\mathbb{Z}/2\mathbb{Z})^r \longrightarrow \widetilde{W} \to W \to e \qquad \widetilde{W} \subset G \qquad \exists\, B \twoheadrightarrow \widetilde{W}$

- **Def** Braid group B generated by $T_i = T_{\alpha_i}$ $\Pi = \{\alpha_1, \dots, \alpha_r\}$ with relations $\underbrace{T_i T_j T_i \dots}_{m_{ij}} = \underbrace{T_j T_i T_j \dots}_{m_{ij}}$

$m_{ij} = 2$ $a_{ij} = 0$ $m_{ij} = 3$ $a_{ij} a_{ji} = 1$

$m_{ij} = 4$ $a_{ij} a_{ji} = 2$ $m_{ij} = 6$ $a_{ij} a_{ji} = 3$

- **Rk** $\exists$ homomorphism $B \to W$ $T_i \mapsto s_i$

- **Fact** (Matsumoto) Let $w \in W$ $w = s_{i_1} \dots s_{i_\ell} = s_{i'_1} \dots s_{i'_\ell}$ – two reduced (= shortest) expressions.
  Then $\bar{i} \to \bar{i}'$ only using braid relations

Corollary $\forall w \in W$ $\exists!$ element $T_w = T_{i_1} \dots T_{i_\ell} \in B$

- Ex $G = SL_3$ $w_0 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = s_1 s_2 s_1 = s_2 s_1 s_2$

- $E_i^{(r)} = E_i^r / [r]_q!$ $\quad F_i^{(r)} = F_i^r / [r]_q!$ $\quad [r]_q! = \prod_{j=1}^{r} [j]_q$ $\quad [j]_q = \dfrac{q^j - q^{-j}}{q - q^{-1}}$

<u>Def</u> $T_i(E_i) = -F_i K_i$ $\qquad T_i(F_i) = -K_i^{-1} E_i$ $\qquad T_i : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g})$

$$T_i(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{-r} E_i^{(-a_{ij}-r)} E_j E_i^{(r)}$$

$$T_i(K_j) = K_j K_i^{-a_{ij}}$$

$$T_i(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{r} F_i^{(r)} F_j F_i^{(-a_{ij}-r)}$$

- Inverse elements are defined by similar formulas

$T_i^{-1}(E_i) = -K_i^{-1} F_i$ $\qquad T_i^{-1}(F_i) = -E_i K_i$

$$T_i^{-1}(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{-r} E_i^{(r)} E_j E_i^{(-a_{ij}-r)}$$

$$T_i(K_j) = K_j K_i^{-a_{ij}}$$

$$T_i^{-1}(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{r} F_i^{(-a_{ij}-r)} F_j F_i^{(r)}$$

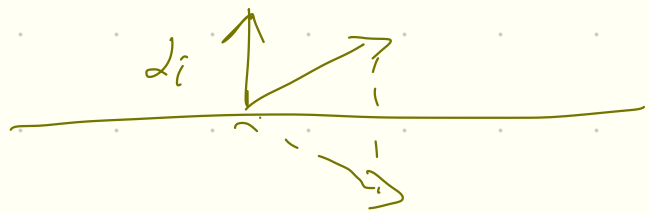
$-\alpha_j$ $\qquad -\alpha_j + (-a_{ij})(-\alpha_i) = -(\alpha_j - a_{ij}\alpha_i)$

• **Remark** $U_q(\mathfrak{g}) = \bigoplus_{\lambda \in Q} U_q(\mathfrak{g})_{(\lambda)}$ $\quad Q$ - root lattice

T_i reflects weights i.e $T_i \ U_q(\mathfrak{g})_{(\lambda)} \longmapsto U_q(\mathfrak{g})_{s_i(\lambda)}$

$(\alpha_j \longmapsto \alpha_i - a_{ij}\alpha_i = \alpha_j - (\alpha_i, \alpha_j)\alpha_i = s_i(\alpha_j))$

(simply laced)



T_i acts on Cartan as reflection $\quad T_i(K_\lambda) = K_{s_i(\lambda)}$

• **Example** $a_{ij} = -1$

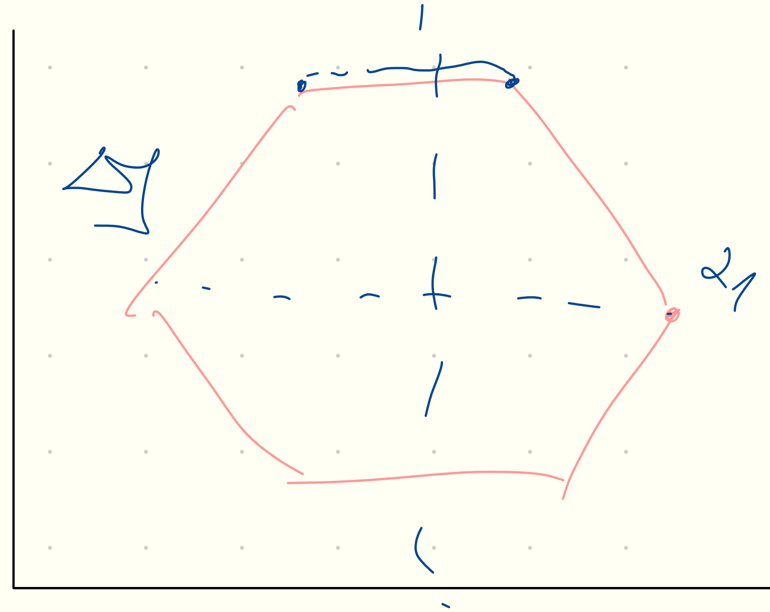
$$T_i(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{-r} E_i^{(-a_{ij}-r)} E_j E_i^{(r)}$$

$$T_i(E_j) = -E_i E_j + q^{-1} E_j E_i = -[E_i, E_j]_q$$

$\big([X,Y] = XY - q^{(\deg X, \deg Y)} YX$ — recall$\big)$

$u_{ij}^+ \rightsquigarrow ad_{q,E_i}^{1-a_{ij}}(E_j) \qquad T_i(E_j) \sim ad_{q,E_i}^{-a_{ij}}(E_j)$

Similarly T_i can be written in terms of adjoint action



- **Th** T_i is automorphism of algebra $U_q(\mathfrak{sl})$

Th $\{T_i, T_i^{-1}, i \in I\}$ satisfy braid group relations

- Some checks

@ $[E_i, F_j] = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q - q^{-1}}$

$$T_i(E_i) = -F_i K_i \qquad T_i(F_i) = -K_i^{-1} E_i$$

$$T_i(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{-r} E_i^{(-a_{ij}-r)} E_j E_i^{(r)}$$

$$T_i(K_j) = K_j K_i^{-a_{ij}}$$

$$T_i(F_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{r} F_i^{(r)} F_j F_i^{(-a_{ij}-r)}$$

$$[T_i(E_i), T_i(F_i)] = [-F_i K_i, -K_i^{-1} E_i] = F_i E_i - K_i^{-1} E_i F_i K_i = [F_i, E_i] = \frac{K_i^{-1} - K_i}{q - q^{-1}}$$

$$= \frac{T(K_i) - T(K_i)^{-1}}{q - q^{-1}}$$

$$[T_i(E_i), T_i(F_j)] = [-F_i K_i, -F_j F_i + q F_i F_j] =$$

$$= [-F_i K_i, -q[F_i, F_j]_q] = q^2 [F_i, [F_i, F_j]_q]_q K_i = q^2 u_{ij} K_i = 0$$

Problem Check $[T_i(E_j), T_i(F_j)] = T_i([E_j, F_j])$ if $a_{ij} = -1$

- Cumbersome check. Let $a_{21}a_{12} = a_{23}a_{32} = 1$

$$E_{ijkl} := E_i E_j E_k E_l$$

$$[T_2(E_1), T_2(E_3)] = [[E_2, E_1]_q, [E_2, E_3]_q]$$

$$= E_{2123} - E_{2321} - q^{-1}E_{1223} + q^{-1}E_{2312} - q^{-1}E_{2132} + q^{-1}E_{3221} + q^{-2}E_{1232} - q^{-2}E_{3212}$$

$$= q^{-1}E_1(q^{-1}E_{232} - E_{223}) + q^{-1}E_3(E_{221} - q^{-1}E_{212}) + E_{2123} - E_{2321}$$

$$= -q^{-1}E_1(qE_{232} - E_{322}) - q^{-1}E_3(E_{122} - qE_{212}) + E_{2123} - E_{2321}$$

$$= E_{2123} - E_{2321} - E_{1232} + E_{3212} = \frac{1}{q+q^{-1}}\Big(E_1(E_{223} - (q+q^{-1})E_{232} + E_{322}) +$$
$$(E_{223} - (q+q^{-1})E_{232} + E_{322})E_1 - E_3(E_{122} - (q+q^{-1})E_{212} + E_{221}) - (E_{122} - (q+q^{-1})E_{212} + E_{221})E_3\Big) = 0$$

- Let $w_0 \in W$ the longest element. For sl_n $w_0 = \begin{pmatrix} 1 & 2 & & n \\ n & n-1 & & 1 \end{pmatrix}$

$w_0 = s_{i_1} \cdot \ldots \cdot s_{i_N}$ – reduced expression $\quad N = \ell(w_0)$

Introduce: $\beta_N = \alpha_{i_N}, \quad \beta_{N-1} = s_{i_N}(\alpha_{i_{N-1}}), \quad \beta_{N-2} = s_{i_N} s_{i_{N-1}} (\alpha_{i_{N-2}}), \ldots,$

$\beta_1 = s_{i_N} \cdot s_{i_{N-1}} \cdot \ldots \cdot s_{i_2}(\alpha_{i_1})$

- Prop $\{\beta_1, \ldots, \beta_N\}$ – all roots of Δ_+, without repetitions

Proof Use property $\forall w \in W$ $\ell(w) = \#\{\alpha \in \Delta_+ \mid w(\alpha) \in \Delta_-\}$

$s_{i_{N-1}} s_{i_N}$ $\quad \ell = 2 \quad \{\beta_{N-1}, \beta_N\} \longmapsto \Delta_-$

$s_{i_{N-2}} s_{i_{N-1}} s_{i_N}$ $\quad \ell = 3 \quad \{\beta_{N-2}, \beta_{N-1}, \beta_N\} \longmapsto \Delta_-$

– – – –

$\square$

- Def Convex order on positive roots $\beta_1, \ldots, \beta_N$
s.t. $\alpha, \alpha', \alpha'' \in \Delta_+$ $\quad \alpha = \alpha' + \alpha''$, $\quad \alpha = \beta_i$, $\alpha' = \beta_{i'}$, $\alpha'' = \beta_{i''}$
then $\quad i' < i < i''$ or $i'' < i < i'$

Prop The order $\beta_1, \dots, \beta_N$ defined above is convex.

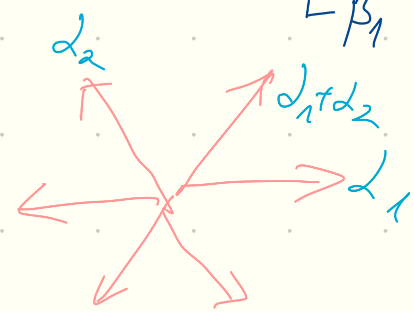
Pf Follows from previous proof $\square$

Cartan-Weyl elements

$$E_{\beta_N} = E_{i_N}, \quad E_{\beta_{N-1}} = T_{i_N}^{-1}(E_{i_{N-1}}), \quad E_{\beta_{N-2}} = T_{i_N}^{-1} T_{i_{N-1}}^{-1}(E_{i_{N-2}})$$

$$E_{\beta_1} = T_{i_N}^{-1} \cdot T_{i_{N-1}}^{-1} \cdot \dots \cdot (E_{i_1})$$

- Example $\mathfrak{g} = \mathfrak{sl}_3$



$$T_i^{-1}(E_i) = -K_i^{-1}F_i$$

$$T_i^{-1}(E_j) = \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} q^{-r} E_i^{(r)} E_j E_i^{(-a_{ij}-r)}$$

$$w_0 = s_1 s_2 s_1 \qquad \beta_3 = \alpha_1, \quad \beta_2 = s_1(\alpha_2) = \alpha_1 + \alpha_2 \qquad \beta_1 = s_1 s_2(\alpha_1) = \alpha_2$$

$$E_{\beta_3} = E_1 \qquad E_{\beta_2} = T_1^{-1}(E_2) = -E_2E_1 + q^{-1}E_1E_2 = -[E_2, E_1]_q$$

$$E_{\beta_1} = T_1^{-1} T_2^{-1}(E_1) = T_1^{-1}(-[E_1, E_2]_q) = -[-K_1^{-1}F_1, -[E_2, E_1]_q]_q =$$

$$= -K_1^{-1}F_1E_2E_1 + q^{-1}K_1^{-1}F_1E_1E_2 + q^{-1}E_2E_1K_1^{-1}F_1 - q^{-2}E_1E_2K_1^{-1}F_1$$

$$= K_1^{-1}E_2\,[E_1, F_1] - [E_1, F_1]\,E_2K_1^{-1}$$

$$= \frac{1}{q-q^{-1}}\left(K_1^{-1}E_2(K_1 - K_1^{-1}) - (K_1 - K_1^{-1})E_2K_1^{-1}\right) = \frac{1}{q-q^{-1}}\left(K_1^{-1}E_2K_1 - K_1E_2K_1^{-1}\right) = E_2$$

Another reduced expr

$w_0 = s_2 s_1 s_2$ $\quad E_{\beta_3} = E_2 \quad E_{\beta_2} = -[E_1, E_2]_q \quad E_{\beta_1} = E_1$

<u>Lessons</u>

(1) E_β has weight β

(2) $\{E_\beta\}$ depends on choice of reduced decomposition

• Problem ⓐ If $a_{i_k i_{k+1}}=0$ then transposing i_k and i_{k+1} we get reduced expr $\vec{i}'$, and Cartan-Weyl elements are unchanged (but reordered $E'_{\beta_k}=E_{\beta_{k+1}}$, $E'_{\beta_{k+1}}=E_{\beta_k}$)

ⓑ If $i_k=i_{k+2}$ and $a_{i_k i_{k+1}}=a_{i_{k+1} i_k}=-1$. Then

$\beta_{k+1}=\beta_k+\beta_{k+2}$ $\qquad E_{\beta_{k+1}}=-[E_{\beta_k},E_{\beta_{k+2}}]_q$

$\vec{i}'$ - replacing $i_k i_{k+1} i_k \longmapsto i_{k+1} i_k i_{k+1}$.

$\{E'_\beta\}$ differs from $\{E_\beta\}$ only $E'_{\beta_{k+1}}$ and E_{β_k} (and order $E'_{\beta_{k+2}}=E_{\beta_k}$, $E'_{\beta_k}=E_{\beta_{k+2}}$)

ⓒ If $\beta_k=\alpha_j$ - simple root then $E_{\beta_k}=E_{\alpha_j}$

Hint ⓐ, $\quad E_{\beta_k}=T_{i_N}^{-1}T_{i_{N-1}}^{-1}\cdots T_{i_{k+1}}^{-1}E_{i_k}$ $\quad E_{\beta_{k+1}}=T_{i_N}^{-1}\cdots T_{i_{k+2}}^{-1}E_{i_{k+1}}$ apply $T_{i_{k+2}}\cdots T_{i_N}$

get $T_{i_{k+1}}^{-1}E_{i_k}=E_{i_k}$ (since $a_{i_k,i_{k+1}}=0$) and $E_{i_{k+1}}$

for $\vec{i}'$ we get $E_{i_{k+1}}=T_{i_k}E_{i_{k+1}}$ and E_{i_k}

(b) Apply $T_{i_{k+3}} \cdots T_{i_N}$ we get $E_{\beta_k} \mapsto T_{i_{k+2}}^{-1} T_{i_{k+1}}^{-1} E_{i_k}$ $E_{\beta_{k+1}} \mapsto T_{i_{k+2}}^{-1} E_{i_{k+1}}$ $E_{\beta_{k+2}} \mapsto E_{i_{k+2}}$
reduces to sl_3 case above

(c) Using (a), (b) and Matsumoto thm $\bar{i} \rightsquigarrow \bar{i}'$ s.t $i_N' = j$
Hence $\beta_N' = \alpha_j$, $E'_{\alpha_j} = E'_{\beta} = E_j \Rightarrow E_{\alpha_j} = E_j$

Problem For $\mathfrak{g} = sl_n$ relate E_β with $e_{ji} \in L^-$

Lemma $\forall \beta \in \Delta_+$ $E_\beta \in \mathcal{U}^+ = U_q(n_+)$
Pf Induction by $ht(\beta)$
Base $ht(\beta) = 1 \Rightarrow \beta = \alpha_j$ for some $j \Rightarrow E_\beta = E_{\alpha_j}$.

Step $\beta = \alpha + \gamma$, $\alpha, \gamma \in \Delta_+$, α - simple

$\exists\, \bar{i}, \bar{i}'$ s.t $\alpha_{i_N} = \alpha_{i_1'} = \alpha$. By Matsumoto thm $\bar{i}$ to $\bar{i}'$
using braid relations. Hence α goes through β.
Hence by (b) $E_\beta = -[E_\gamma, E_\alpha]_q \in \mathcal{U}^+$ $\square$

Lemma For $k \le m \le N$, $T_{i_m}^{-1} T_{i_{m-1}}^{-1} \cdots T_{i_{k+1}}^{-1} E_{i_k} \in \mathcal{U}^+$

Pf If $w_0 = s_{i_1} \cdots s_{i_k} \cdots s_{i_m} \cdots s_{i_N}$ reduced expression of w_0

Then $w_0 = s_{\sigma(i_{m+1})} s_{\sigma(i_{m+2})} \cdots s_{\sigma(i_N)} s_{i_1} \cdots s_{i_k} \cdots s_{i_m}$ is reduced expression

Here $\sigma : \{\text{simple roots}\} \to \{\text{simple roots}\}$ s.t

$w_0(\alpha_j) = \alpha_{\sigma(j)}$, equivalently $w_0 s_j = s_{\sigma(j)} w_0$

Applying previous lemma to new expression $\Rightarrow$ q.e.d $\square$

- **Th** (PBW) (a) Elements $E_{\beta_1}^{a_1} \cdots E_{\beta_N}^{a_N}$ form a basis in $\mathcal{U}^+$

(b) Elements $F_{\beta_1}^{b_1} \cdots F_{\beta_N}^{b_N} K_\lambda E_{\beta_1}^{a_1} \cdots E_{\beta_N}^{a_N}$ form a basis in $\mathcal{U}_q(\mathfrak{g})$

Pf sufficient to prove (a)

Prov that monomials $E_{\beta_1}^{a_1}\dots E_{\beta_k}^{a_k}$ are linearly indep.

Induction on k.

Assume $\sum c_{\bar{a}} E_{\beta_1}^{a_1}\dots E_{\beta_k}^{a_k}=0$.

Apply $T=T_{i_k}\dots T_{i_N}$ We get

$T E_{\beta_j}\in \mathcal{U}^+$, $1\le j\le k-1$ by Lemma $T E_{\beta_k}=T_{i_k}E_{i_k}=-F_{i_k}K_{i_k}\in \mathcal{U}^-\mathcal{U}^0$

Then $\sum c_a \underbrace{T(E_{\beta_1})^{a_1}\dots T(E_{\beta_{k-1}})^{a_{k-1}}}_{\in \mathcal{U}^+\text{, linearly indep by step}}\cdot \underbrace{(-F_{i_k}K_{i_k})^{a_k}}_{\in \mathcal{U}^0\mathcal{U}^-\text{, linearly indep. by } sl_2} \Rightarrow$ q.e.d $\square$

References

- Chary, Pressley A guide to quantum groups
  Sec 8.1
- Tingley Elementary construction of Lusztig's canonical basis