Introduction to Quantum Groups

Lecture 15

Factorization of universal $\mathcal{R}$ matrix

- For $U_q(sl_2)$ define $t \in (U_q(sl_2)^\circ)^* = \mathbb{C}[SL_2]_q^*$ s.t

$\forall \ell \geq 0$ L_ℓ - $\ell+1$ dim rep $L_\ell = \bigoplus_{\substack{-\ell \leq m \leq \ell \\ 2 \mid m+\ell}} (L_\ell)_{(m)}$

$$t\Big|_{L_\ell(m)} = \sum_{a-b+c=m} (-1)^b q^{ac-b} F^{(a)} E^{(b)} F^{(c)} \qquad E^{(a)} = \frac{E^a}{[a]_q!}$$

<u>Rk</u> t well-defined $\qquad t : (L_\ell)_{(m)} \longmapsto (L_\ell)_{(-m)}$

<u>Rk</u> For $q=1$ $\quad \sum (-1)^b \frac{F^a}{a!} \frac{E^b}{b!} \frac{F^c}{c!} = e^F e^{-E} e^F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

i.e. classically $t \in SL_2$ moreover $t \in N(H)$

t reflection

G acts on f.d reps $\circlearrowleft$ $\rightsquigarrow$ q-deform acts on f.d reps $U_q(\mathfrak{g})$

- Problem (a) For $\xi \in (L_\ell)_{(m)}$ $\quad E^a F^b \xi = \sum_{t\geq 0} F^{(b-t)} E^{(a-t)} \begin{bmatrix} m-b+a \\ t \end{bmatrix} \xi$

(b) Let $\xi_\ell \in (L_\ell)_{(\ell)}$ be h.w. vector. Let $\tilde{\xi}_m = F^{\left(\frac{\ell-m}{2}\right)} \xi_\ell \in (L_\ell)_{(m)}$

Then $t\tilde{\xi}_m = (-1)^{\frac{\ell-m}{2}} q^{-\frac{\ell+m+2}{2}\frac{\ell-m}{2}} \tilde{\xi}_{-m}$

(c) $tFv = -EKtv \quad tKv = K^{-1}tv \quad tEv = -K^{-1}Ftv \quad \forall v \in L_\ell$

Hint (b) $\sum_{a-b+c=m} (-1)^b q^{ac-b} F^{(a)} E^{(b)} F^{(c)} F^{\left(\frac{\ell-m}{2}\right)} v_\ell = \left|\ \text{using } F^{(i)}F^{(j)} = \begin{bmatrix} i+j \\ i \end{bmatrix} F^{(i+j)} \text{ and (a)} \right|$

$= \sum_{a-b+c=m} (-1)^b q^{ac-b} \begin{bmatrix} c+\frac{\ell-m}{2} \\ c \end{bmatrix}_q \begin{bmatrix} \frac{\ell+m}{2}+b-c \\ b \end{bmatrix}_q \begin{bmatrix} \frac{\ell+m}{2} \\ a \end{bmatrix}_q F^{\left(\frac{\ell+m}{2}\right)} v_\ell = \left|\ \text{Use } \begin{bmatrix} n \\ k \end{bmatrix} = (-1)^k \begin{bmatrix} -n+k-1 \\ k \end{bmatrix}\right.$

$\sum_j \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} m \\ k-j \end{bmatrix} q^{j(n+m)-nk} = \begin{bmatrix} n+m \\ k \end{bmatrix}$

follows from $(x+y)^{n+m} = (x+y)^n (x+y)^m$, $xy = q^2 yx$

$= \sum_a (-1)^{m+a} q^{-a-\frac{\ell+m+2}{2}\frac{\ell-m}{2}} \begin{bmatrix} \frac{\ell+m}{2} \\ a \end{bmatrix} \begin{bmatrix} a-1 \\ \frac{\ell+m}{2} \end{bmatrix} F^{\left(\frac{\ell+m}{2}\right)} v_\ell$

$=$ | using $0 \leq a \leq \ell$ only $a=0$ remains | $= (-1)^m q^{-\frac{\ell+m+2}{2}\frac{\ell-m}{2}} F^{\left(\frac{\ell+m}{2}\right)} v_\ell$

- In other words, $\mathcal{U}_q(sl_2)^{\circ}$ has basis $C^{\ell}_{m'm}$ - matrix elements $\tilde{\xi}_m \mapsto \tilde{\xi}_{m'} \in L_\ell$

  $t \in (\mathcal{U}_q(sl_2)^{\circ})^*$ $\quad t(C^{\ell}_{m'm}) = (-1)^{\frac{\ell-m}{2}} q^{-\frac{\ell+m+2}{2}\frac{\ell-m}{2}} \delta_{m+m',0}$

- In $t^{-1}xt = T(x)$ (in any f.d rep) where T is a generator of Lusztig's braid group.

- In other words $t^{-1}Et = -FK$ $\quad t^{-1}Kt = K^{-1}$ $\quad t^{-1}Ft = -K^{-1}E$
- Remark One can consider $\mathbb{C}[t^{\pm 1}] \ltimes \mathcal{U}_q(sl_2)$

  Any L_ℓ is rep of this algebra

- Let $\bar{R} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(q-q^{-1})^n}{[n]_q!} E^n \otimes F^n$

Problem Prove $\bar{R}^{-1} = \sum \frac{(-1)^n q^{-\binom{n}{2}}(q-q^{-1})^n}{[n]_q!} E^n \otimes F^n$

- Problem $\Delta(t) = \bar{R}^{-1}\, t \otimes t$

Equivalently $\quad \forall\, \xi_1 \in L_{e_1} \quad \xi_2 \in L_{e_2} \quad t(\xi_1 \otimes \xi_2) = \bar{R}^{-1}(t\xi_1 \otimes t\xi_2)$

Hint Possible plan of computation

(a) If $\Delta(t)\, v_1 \otimes v_2 = \bar{R}^{-1}(t \otimes t)\, v_1 \otimes v_2 \Rightarrow \Delta(t)\Delta(E)\, v_1 \otimes v_2 = \bar{R}^{-1}(t \otimes t)\Delta(E)\, v_1 \otimes v_2$

(b) Hence sufficient to check for ξ_1 s.t $F\xi_1 = 0$

(c) If $F\xi_1 = 0 \Rightarrow E\, t\xi_1 = 0 \Rightarrow \bar{R}^{-1}\, t\xi_1 \otimes t\xi_2 = t\xi_1 \otimes t\xi_2$

$\Delta t(\xi_1 \otimes \xi_2) = \qquad\qquad = t\xi_1 \otimes t\xi_2$

- <u>Lemma</u> $\bar{R}\Delta(x)\bar{R}^{-1} = (T^{-1}\otimes T^{-1})\Delta(T(x))$

$\Delta(t) = \bar{R}^{-1} t\otimes t$

$t^{-1}xt = T(x)$

<u>Pf</u> $\Delta(T^{-1}(x)) = \Delta(txt^{-1}) = \bar{R}^{-1} t\otimes t\, \Delta(x)\, t^{-1}\otimes t^{-1}\bar{R} = \bar{R}^{-1}(T^{-1}\otimes T^{-1})(\Delta(x))\bar{R}$ $\square$

<u>Remark</u> $\bar{R}$ intertwines two coproducts Δ and $(T^{-1}\otimes T^{-1})\circ\Delta\circ T$

- <u>Th</u> Let $R_H = e^{\hbar H\otimes H/2}$ $\quad R = R_H\bar{R}$ Then

$R\Delta(x)R^{-1} = \Delta^{op}(x)$

universal R matrix

<u>Pf</u> $R\Delta(E)R^{-1} = R_H(T^{-1}\otimes T^{-1})\Delta(T(E))R_H^{-1} = R_H(T^{-1}\otimes T^{-1})\Delta(-FK)R_H^{-1}$

$= R_H(T^{-1}\otimes T^{-1})(-FK\otimes K + 1\otimes FK)R_H^{-1} = q^{H\otimes H/2}(E\otimes K^{-1} + 1\otimes E)q^{-H\otimes H/2}$

$= E\otimes 1 + K\otimes E = \Delta^{op}(E)$

$q^{H\otimes H/2} E\otimes 1 = E\otimes K\, q^{H\otimes H/2}.$ $\square$

- General case

$U_q(\mathfrak{g})$ $E_i, K_i^{\pm 1}, F_i$ $i \in I$ $\rightsquigarrow t_i \in (U_q(\mathfrak{g})^{0,I})^*$

More explicitly $\xi \in L_\lambda$, if $K_i \xi = q^m \xi$ then

$$t_i \xi = \sum_{a-b+c=m} (-1)^b q^{ac-b} F_i^{(a)} E_i^{(b)} F_i^{(c)} \xi$$

Remark (a) Morally $SL_2 \xrightarrow{\phi_i} G$, $t \mapsto t_i$

(b) Morally $t_i \in G$, moreover $t_i \in N(H)$ quantum Weyl group

- Th (a) $T_i(x) = t_i^{-1} x t_i$, where T_i - from Lusztig group

(b) $\underbrace{t_i t_j \dots}_{m_{ij}} = \underbrace{t_j t_i \dots}_{m_{ij}}$ Braid relations

For proof $\operatorname{rk}\mathfrak{g} = 1, 2.$

Corollary T_i automorphisms, T_i satisfy braid relations

- **Lemma** $\Delta(t_i) = \tilde{R}_i^{-1}\, t_i \otimes t_i$ where $\bar{R} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(q-q^{-1})^n}{[n]_q!} E_i^n \otimes F_i^n$

  Follows from $\operatorname{rk}\mathfrak{g} = 1$ above

Corollary $\bar{R}_i \Delta(x) \bar{R}_i^{-1} = T_i^{-1} \otimes T_i^{-1} \Delta(T(x))$

Remark t_i - quantum Weyl group, but not group-like w.r.t. Δ. Classically $r|_H = 0$ but $r|_{N(H)} \neq 0$ $r|_{S_i H} \sim E_i \wedge F_i$

- Fix reduced decomp $w_0 = s_{i_1} \cdots s_{i_N}$

$E_{\beta_N} = E_{i_N}$, $E_{\beta_{N-1}} = T_{i_N}^{-1}(E_{i_{N-1}})$ $\ldots$ $E_{\beta_1} = T_{i_N}^{-1} T_{i_{N-1}}^{-1} \cdots T_{i_2}^{-1}(E_{i_1})$

$F_{\beta_N} = F_{i_N}$, $F_{\beta_{N-1}} = T_{i_N}^{-1}(F_{i_{N-1}})$ $\ldots$ $F_{\beta_1} = T_{i_N}^{-1} T_{i_{N-1}}^{-1} \cdots T_{i_2}^{-1}(F_{i_1})$

- <u>Th</u> $R\Delta(x)R^{-1} = \Delta^{op}(x)$ where $R = R_H \cdot R_{\beta_1} \cdot \ldots \cdot R_{\beta_N}$

$R_H = q^{\sum H_j \otimes H^j}$ $\{H_j\}, \{H^j\}$ - dual bases in $\mathfrak{h}$

$\overline{R}_{\beta_k} = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(q-q^{-1})^n}{[n]_q!} E_{\beta_k}^n \otimes F_{\beta_k}^n = T_{i_N}^{-1} \cdot \ldots \cdot T_{i_{k+1}}^{-1}(\overline{R}_{i_k})$

- <u>Prop</u> $\Delta E_{\beta_k} = \prod_{l>k} \overline{R}_{\beta_l}^{-1} \left(E_{\beta_k} \otimes K_{\beta_k} + 1 \otimes E_{\beta_k}\right) \prod_{l>k} \overline{R}_{\beta_l}$

- <u>Prop</u> $\langle E_{\beta_1}^{m_1} \cdot \ldots \cdot E_{\beta_N}^{m_N} F_{\beta_1}^{n_1} \cdot \ldots \cdot F_{\beta_N}^{n_N} \rangle = \prod_{k=1}^{N} \delta_{n_k, m_k} \frac{[n_k]_q!}{(q-q^{-1})^{n_k}} q^{-\binom{n_k}{2}}$

References

- Chary, Pressley A guide to quantum groups Sec 8.1
- Lusztig Introduction to quantum groups Ch 5, 37, 39
- Korogodski Soibelman Algebras of functions on quantum group Sec 2.2, 4.3, 4.4