Introduction to quantum groups

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Problems for the Skoltech course, Fall 2023. There are mistakes here. If you find any, please write to mbersht@gmail.com

Definitions and hints can be found in the slides and references.

1 Poisson algebras and quantization

Problem 1.1. *Show that Moyal formula defines associative product.*

Problem 1.2 (*)**.** *Find an example of the Poisson algebra which cannot be quantized.*

Problem 1.3 (*)**.** *Show that* $HH^2(U(\mathfrak{g})) = 0$ *for semisimple Lie algeba* $\mathfrak{g}$.

Problem 1.4. *Show that distribution* T^{Π} *is integrable.*

2 Poisson-Lie groups and Lie bialgebras

Problem 2.1. *a) Let* G *is Poisson-Lie group,* $H \subset G$ *is Poisson-Lie subgroup. Show that* $C^\infty(G)^H$ *is Poisson subalgebra.*

b) Let $H \subset G$ *is subgroup such that* $\Pi|_H \in TH \otimes TG + TG \otimes TH$. *Show that* $C^\infty(G)^H$ *is Poisson subalgebra.*

Problem 2.2. *Let* $G = GL(2)$*, Poisson-Lie structure defined by* r *matrix with* $r = \frac{1}{4} h \otimes h + e \otimes f$, $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Find brackets of* a, b, c, d. *Check skew-commutativity. Check Poisson-Lie property.*

Problem 2.3. *For any finite dimensional Lie algebra* $\mathfrak{g}$ *there exists bijection between bialgebra structures on* $\mathfrak{g}$ *and Manin triples with* $\mathfrak{q}_+ \simeq \mathfrak{g}$.

Problem 2.4. *Find Lie bialgebra structure (i.e.* δ*) for Examples 1 and 3 and* $\mathfrak{g} = \mathfrak{sl}_2$.

3 Classical r-matrices

Let $\delta_r(a) = \mathrm{ad}_a\, r$.

Problem 3.1. *Show that δ_r maps to $\Lambda^2\mathfrak{g}$ if and only if $r_{12} + r_{21} \in (\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$.*

Problem 3.2. *Let $r = r^S + r^A$, where $r^S = \alpha\Omega$, $r^A \in \Lambda^2\mathfrak{g}$. Show that a) $\delta_r = \delta_{r^A}$. b) $[[r, r]] = [[r^A, r^A]] + \alpha^2 c$.*

Problem 3.3. *Show that $\mathfrak{g} \hookrightarrow D(\mathfrak{g})$ and $(\mathfrak{g}^*)^{coop} \hookrightarrow D(\mathfrak{g})$ are embeddings of Lie bialgebras.*

Problem 3.4. *For standard bialgebra structure for simple $\mathfrak{g}$*
a) Find $\delta(h_i)$, $\delta(e_\alpha)$, $\delta(e_{-\alpha})$, where α is simple.
b) Find Lie algebra $\mathfrak{g}^*$.
c) Show that $r = \sum h_i \otimes h^i + 2\sum_{\alpha\in\Delta_+} e_\alpha \otimes e_{-\alpha}$ defines the same δ and satisfies CYBE.

Problem 3.5 (*)**.** *Let $r \in \Lambda^2\mathfrak{g}$ satisfy MCYBE. Show that $\Pi = (\lambda_g)_* r - (\rho_g)_* r$ is Poisson bracket.*

4 Dual Poisson-Lie groups, symplectic leaves

Problem 4.1 (*)**.** *Let G be Poisson-Lie group. For $\alpha \in \mathfrak{g}^*$ let $\alpha_l \in \Omega^1(G)$ be corresponding left invariant 1-form. Define vector field by the formula $V_(\alpha) = \Pi(\alpha_l) \in \mathrm{Vect}G$. Show that map $\mathfrak{g}^* \to \mathrm{Vect}(G)$ is Lie algebra homomorphism.*

Problem 4.2. *a) Find explicitly double Bruhat cells for $SL(2)$ with standard Poisson bracket (see Problem 2.2).*
b) Find symplectic leaves on $SL(2)$. Find Casimir functions (i.e. generators of the Poisson center) on each double Bruhat cell.

5 Quantum group and algebras. Example of $\mathfrak{sl}_2$

Problem 5.1. *Let $B \in U(\mathfrak{g})$ such that $\Delta(B) = B\otimes 1 + 1\otimes B$. Show that $B \in \mathfrak{g}$.*

Problem 5.2 (*)**.** *Let $U_\hbar(\mathfrak{g})$ is quantization of universal enveloping $U(\mathfrak{g})$. Let $\delta(a) = \frac{\Delta(a)-\Delta^{op}(a)}{\hbar} \bmod \hbar$. Show that δ satisfies cocycle and coJacobi conditions.*

Problem 5.3. *Let $g(\hbar) = 1 + O(\hbar) \in U(\mathfrak{h})[[\hbar]]$ is group like element (i.e. $\Delta(g) = g\times g$), then $g(\hbar) = \exp(\alpha H\hbar)$, for $\alpha \in \mathbb{C}[[\hbar]]$.*

Problem 5.4. *Show that relation $[E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}$ agrees with coproduct Δ.*

Problem 5.5. *Show that exists homomorphism $U_\hbar(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2)[[\hbar]]$ such that $E \mapsto e$, $H \mapsto h$, and $F \mapsto \Phi(c, h)f$, where $c \in U(\mathfrak{sl}_2)$ is Casimir element.*

6 Hopf algebras

Problem 6.1. *Show that S is antihomomorphism of algebra and coalgebra.*

Problem 6.2. *a) Find formulas for action of E, H, F in basis v_m.*
b) Define basis $\tilde{v}_m$.

Problem 6.3. *a) Show existence of natural morphisms $V^* \otimes V \to \mathbb{C}$ and $\mathbb{C} \to V \otimes V^*$.*
b) Show that $(V \otimes W)^* = W^* \otimes V^*$.

Problem 6.4 (*). *a) Show directly that $L_1 \otimes L_l \simeq L_{l+1} \oplus L_{l-1}$, for $l \geq 1$.*
b) Show that $L_{l_1} \otimes L_{l_2} = \oplus L_l$, where summation region $|l_1 - l_2| \leq l \leq l_1 + l_2$ and $l + l_1 + l_2$ is even.

7 Quantum R-matrices

Problem 7.1. *For $U_\hbar(\mathfrak{sl}_2)$ show that $\Delta^{op}(E)R = R\Delta(E)$.*

Problem 7.2. *a) Show that $C_\hbar = FE + \frac{e^{\hbar(H+1)} + e^{-\hbar(H+1)}}{(e^\hbar - e^{-\hbar})^2}$ is central.*
b) Find action of $C_\hbar$ and $e^{-\hbar H}u$ on L_m.
c) Let $\Phi_\hbar^{-1}\colon U(\mathfrak{sl}_2)[[\hbar]] \to U_\hbar(\mathfrak{sl}_2)$ isomorphism. Let $c = fe + h(h+2)/4$. Find $\Phi_\hbar^{-1}(c)$. $\Phi_\hbar^{-1}(e^{\hbar c})$, relate to central elements above.

8 Drinfeld-Jimbo quantum groups

Problem 8.1. *Show that Serre relations are equivalent to*

$$(\mathrm{Ad}_{E_i}^{\mathrm{coop}})^{1-a_{ij}} E_j = 0, \quad (\mathrm{Ad}_{F_i})^{1-a_{ij}} F_j = 0.$$

Problem 8.2. *Show that $\forall u \in \tilde{U}$ we have $S^2(u) = K_{2\rho} u K_{-2\rho}$*

Problem 8.3. *Show that $[F_k, u_{i,j}^+] = 0$.*

9 Finite dimensional Representation of $U_q(\mathfrak{g})$

Problem 9.1. *Show the $V \otimes \mathbb{C}_\sigma \otimes \mathbb{C}_\sigma \simeq V$ for any finite dimensional representation V.*

Problem 9.2. *a) Describe $\mathbb{C}^n = L\varpi_1$ as representation of $U_q(\mathfrak{sl}_n)$.*
b) Check formula for intertwining operator $\tilde{R}\colon \mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^n$

$$\tilde{R} = \sum_i E_{ii} \otimes E_{ii} + \sum_{i<j} (E_{ij} \otimes E_{ji} + E_{ji} \otimes E_{ij} + (q - q^{-1}) E_{jj} \otimes E_{ii}).$$

c)(*) Hecke algebra H_N for $\mathfrak{sl}_N$ is an algebra with generators $T_1, \ldots, T_{N-1}$ and relations

$$(T_i - q)(T_1 + q^{-1}) = 0, \;\; T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \;\; T_i T_j = T_j T_i, \; |i - j| > 1.$$

Construct an action of H_N on $(\mathbb{C}^n)^{\otimes N}$ commuting with $U_q(\mathfrak{sl}_n)$. This is q-analog of Schur-Weyl duality.

Problem 9.3 (*). *a) For* $\mathfrak{g} = \mathfrak{sl}_n$ *construct representation* L_{ϖ_k}*, where* ϖ_k *is* k*-th fundamental weight. This is* q*-analog of* $\Lambda^k\mathbb{C}^n$*.*

b) For $\mathfrak{g} = \mathfrak{sl}_n$ *construct representation* $L_{k\varpi_1}$*. This is* q*-analog of* $S^k\mathbb{C}^n$*.*

10 Drinfeld double. Drinfeld theorem.

Problem 10.1 (*). *Show that relations* $\langle a_{(1)}, b_{(1)}\rangle a_{(2)} * b_{(2)} = b_{(1)} * a_{(1)}\langle a_{(2)}, b_{(2)}\rangle$ *is equivalent to* $b * a = \langle a_{(1)}, b_{(1)}\rangle a_{(2)} * b_{(2)}\langle a_{(3)}, S^{-1}(b_{(3)})\rangle$

Problem 10.2. *Show nondegeneracy of the pairing* $U_\hbar(\mathfrak{b}^+) \otimes U_\hbar(\mathfrak{b}^-) \to \mathbb{C}$*.*

Problem 10.3. *Show that* $(F^n, E^m) = \frac{[n]_q!}{(q-q^{-1})^n)} q^{-\binom{n}{2}}\delta_{m,n}$*.*

Problem 10.4 (*). *For* $\mathfrak{g} = \mathfrak{sl}_2$ *and* $V = \mathbb{C}^2$ *compute* C_V*.*

11 RTT realization

Problem 11.1. *Show that* $\langle RL_1^+L_2^+ - L_2^+L_1^+R, -\rangle = 0$*.*

Problem 11.2. *a) Deduce quadratic relations on* E, F, H *from RTT relations.*
b) Deduce Serre relations from RTT relations.*

Problem 11.3. *Prove that* $U(R)$ *is generated by* $l_{ii}^+, l_{i,i+1}^+, l_{ii}^-, l_{i+1,i}^-$*.*

Problem 11.4. *Find formulas for* $L^+ = (\rho \otimes \mathrm{id})\mathcal{R}$ *and* $L^- = (\mathrm{id} \otimes \rho)\mathcal{R}^{-1}$ *for* $U_q(\mathfrak{sl}_2)$*.*

12 Functions on quantum group SL_2

Problem 12.1. *Show that qdet is central and group-like.*

Problem 12.2. *Show that* $L_l \otimes L_l^*$ *are linearly independent in* $U_q(\mathfrak{sl}_2)^\circ$ *for* $l \geq 0$*.*

Problem 12.3 (*). *Let* $U_\hbar(\mathfrak{g})$ *be quantum universal enveloping algebra. Let* $A = \{x \in U_\hbar(\mathfrak{g}) | (id - \epsilon)\Delta_n(x) \in U_\hbar(\mathfrak{g})^{\otimes n}, \forall n\}$*. Show that* A *is a Hopf algebra, cocommutative up to first order in* $\hbar$*.*

13 Functions on quantum group SL_n

Problem 13.1 (*). *a)* $t^\Lambda_{-w_0(\Lambda),\Lambda} t^{\Lambda'}_{-\mu,\lambda} = q^{(\Lambda,\lambda)-(w_0(\Lambda,\mu))} t^{\Lambda'}_{-\mu,\lambda} t^\Lambda_{-w_0(\Lambda),\Lambda}$*.*

b) $t^\Lambda_{-\Lambda,w_0(\Lambda)} t^{\Lambda'}_{-\mu,\lambda} = q^{(\Lambda,\mu)-(w_0(\Lambda),\lambda)} t^{\Lambda'}_{-\mu,\lambda} t^\Lambda_{-\Lambda,w_0(\Lambda)}$*.*

c) Elements $t^\Lambda_{-w_0(\Lambda),\Lambda}$*,* $t^{\Lambda'}_{-\Lambda',w_0(\Lambda')}$ *form commutative subalgebra.*

Problem 13.2 (*). *a) Subalgrebra* A_+ *is generated by* $t^{1,\dots,k}_{i_1,\dots,i_k}$*.*

b)Subalgebra A_- *is generated by* $t^{n-k+1,\dots,n}_{i_1,\dots,i_k}$*.*

c) Commutative subalgebra from c) above is generated by $t^{1\dots k}_{n-k+1\dots n}$*,* $t^{n-k+1\dots n}_{1\dots k}$*.*

Problem 13.3 (*). *For given* $J = \{j_1 < \dots J_{r-1}\}$, $I = \{i_1, \dots, i_r\}$, $K = \{k_0, \dots k_r\}$ *show relation*

$$\sum_{s=0}^{r} \mathrm{sgn}(J, k_s)(-q)^{-s} t^{i_1 \dots i_r}_{j_1 \dots k_s, \dots, j_{r-1}} t^{i_1 \dots i_r}_{k_0, \dots, \hat{k_s} \dots, k_r} = 0$$

Problem 13.4 (*). *Show that center of* $\mathbb{C}[Mat_n]_q$ *is generated by qdet.*

14 Lusztig's braid group

Problem 14.1 (*). *Check that* $[T_i(E_j), T_i(F_j)] = T_i([E_j, F_j])$ *for* $a_{ij} = -1$.

Problem 14.2 (*). *Fix reduced expression of* $w_0 = s_{i_1} \cdots w_{i_N}$.

a) If $a_{i_k, i_{k+1}} = 0$ *then reversing* $i_k, i_k + 1$ *we get reduced expression* $\vec{i'}$ *with the same (but reordered) set of Cartan-Weyl elements.*

b) If $i_k = i_{k+2}$, $a_{i_k, i_{k+1}} = a_{i_{k+1}, i_{k+2}} = -1$ *then* $\beta_{k+1} = \beta_k + \beta_{k+2}$, $E_{\beta_{k+1}} = -[E_{\beta_k}, E_{\beta_{k+2}}]_q$. *Replacing* $i_k, i_{k+1}, i_k \to i_{k+1} i_k i_{k+1}$ *we get reduced expression* $\vec{i'}$ *and the set of Cartan-Weyl elements* $\{E'_\beta\}$ *differs from* $\{E'_\beta\}$ *only by* $E'_{\beta_{k+1}}$ *and* $E_{\beta_{k+1}}$.

c) If $\beta_k = \alpha_j$ *then* $E_{\beta_k} = E_j$.

Problem 14.3 (*). *Relate* l^-_{ij} *generators in RTT realization and Cartan-Weyl elements.*

15 Factorization of the universal R matrix

Problem 15.1 (*). *a) For* $v \in L_l[m]$ *show that* $E^{(a)} F^{(b)} v = \sum_{t \geq 0} F^{(b-t)} E^{(a-t)} \begin{bmatrix} m-b+a \\ t \end{bmatrix}_q$.

b) Let $v_l \in L_l$ *be highest weight vector. Let* $\tilde{v}_m = F^{(\frac{l-m}{2})} v_l \in L_l[m]$. *Show that*

$$t \tilde{v}_m = (-1)^{\frac{l-m}{2}} q^{-\frac{l-m}{2} \frac{l+m+2}{2}} \tilde{v}_{-m}.$$

c) Show that $tFv = -EKtv$, $tKv = K^{-1} tv$, $tEv = -k^{-1} Ftv$.

Problem 15.2 (*). *Let* $\bar{\mathcal{R}} = \sum_{n \geq 0} q^{\binom{n}{2}} \frac{(q-1/q)^n}{[n]_q!} E^n \otimes F^n$. *Show that*
$\bar{\mathcal{R}}^{-1} = \sum_{n \geq 0} (-1)^n q^{-\binom{n}{2}} \frac{(q-1/q)^n}{[n]_q!} E^n \otimes F^n$

Problem 15.3 (*). *Show that* $\Delta(t) = \bar{\mathcal{R}}^{-1} t \otimes t$