Topics in Representation Theory, 2025

Assignment 1.

The ground field is $\mathbb{C}$, unless stated otherwise.

Problem 1 (Cyclic group)**.** Let $G = C_n = \langle r | r^n = e \rangle \simeq \mathbb{Z}/n\mathbb{Z}$ be a cyclic group of order n. Consider the regular representation of C_n. In matrix terms it is given by

$$r \mapsto \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

a) Find all invariant subspaces.

b) Find character of the representation.

c) Find isomorphism with the sum of irreducible representations of C_n.

Problem 2 (Quaternion group)**.** Let $Q = \{\pm 1, \pm x, \pm y, \pm z\}$ be the group defined by relations

$$x^2 = y^2 = z^2 = -1, \quad xy = -yx = z, \quad xz = -xz = -y, \quad yz = -zy = x.$$

a) Determine the conjugacy classes of Q.

b) Find dimensions of irreducible representations of Q.

c) Find one dimensional representation of Q.

Hint: The commutator, i.e. subgroup generated by elements of the form $xyx^{-1}y^{-1}$ belongs to kernel of any one dimensional representation.

d) Find all irreducible representations of Q and write its character table.

Problem 3 (Dihedral group)**.** Let D_6 be a group of symmetries of the regular hexagon. Equivalently, D_6 can be described by generators s, r with relations $r^6 = s^2 = (rs)^2 = e$

a) Find conjugacy classes in D_6.

b) Find dimensions of all irreducible representations of D_6.

c) Find character table of D_6.

d) Explicitly realize two-dimensional irreducible representations of D_6 as matrices.

e) Decompose tensor products of irreducible representations into a sum of irreducibles.

Problem 4 (Ideals in matrix algebra)**.** Let $I \subset \mathrm{Mat}_n$ be a two-sided ideal. Show that either $I = 0$ or $I = \mathrm{Mat}_n$.

Problem 5 (q-Weyl algebra)**.** The q-Weyl algebra A is an algebra generated by x, x^{-1}, y, y^{-1} with defining relations $yx = qxy$ and $xx^{-1} = x^{-1}x = y^{-1}y = yy^{-1} = 1$.

a) Show that elements $x^i y^j$, $i, j \in \mathbb{Z}$ form a basis in A.

b) For which q does this algebra have finite dimensional representations?

Hint: Use determinants.

c) For such q, find all finite dimensional irreducible representations of A, up to isomorphism.

Hint: Let V be an irreducible finite dimensional representation of A, and let v be an eigenvector of y in V. Show that the collection of vectors $\{v, xv, x^2v, \ldots, x^{p-1}v\}$ is a basis of V

d) If q is not a root of unity, what are the two-sided ideals in A?

Problem 6 (Second orthogonality relation)**.** Show that

$$\sum_{i \in \mathrm{Irr}(G)} \chi_i(h)\overline{\chi_i(h')} = \delta_{C,C'} \frac{|G|}{|C|},$$

where h, h' are elements of conjugacy classes C, C'.

Hint: use the fact that character table is a transformation between two orthogonal bases, hence after simple rescaling it becomes unitary

Problem 7 (Frobenius determinant)**.** Let $G = \{g_1, \ldots, g_n\}$. Let x_{g_i} be a variable associated with $g_i \in G$. Let X be a matrix consisting of elements $x_{g_i g_j}$. Then

$$\det X = \prod_{i=1}^{r} P_j(x_{g_1}, \ldots, x_{g_n})^{\deg P_i},$$

where r is number of conjugacy classes in G and P_j are pairwise non-proportional polynomials.

a) Prove this for $G = C_n$.

b)** Prove for arbitrary finite group.

Problem 8 (Quaternion algebra)**.** Here the ground field $k = \mathbb{R}$. Let $\mathbb{H}$ be an algebra of quaternions with basis $1, x, y, z$ over k and multiplications

$$x^2 = y^2 = z^2 = -1, \quad xy = -yx = z, \quad xz = -xz = -y, \quad yz = -zy = x.$$

a) Show that $\mathbb{H}$ is irreducible as left module over itself. In particular, it means that $\mathbb{H}$ is semisimple.

b) Find the algebra of endomorphisms $\mathrm{End}_{\mathbb{H}-\mathrm{mod}}(\mathbb{H})$ of $\mathbb{H}$ as left module over itself.

c) Show $\mathbb{H}$ is not isomorphic to matrix algebra over $\mathbb{R}$ or $\mathbb{C}$.

d) Show that $\mathbb{H} \otimes_{\mathbb{C}}$ (complexification of $\mathbb{H}$) is isomorphic to $\mathrm{Mat}_2(\mathbb{C})$.