Topics in Representation Theory, 2025

Assignment 2.
The ground field is $\mathbb{C}$, unless stated otherwise.

Problem 1. Using Frobenius character formula show that $\chi_{(n-1,1)}(C_{\vec{i}}) = i_1 - 1$. Here i_1 denotes number od cycles of length 1 in conjugacy class $\vec{i}$.

Hint. It is sufficient to take $N = 2$, i.e. to find coefficient of $x_1^{n-1}x_2$ in $(1 - x_1^{-1}x_2)\prod_m (x_1^m + x_2^m)^{i_m}$.

Problem 2. Show that $s_\lambda(x_1, \ldots, x_{N-1}, 0) = s_\lambda(x_1, \ldots, x_{N-1})$ if $l(\lambda) \leq N - 1$ and 0 otherwise.

Problem 3. Let V be a finite dimensional complex vector space, and let $GL(V)$ be the group of invertible linear transformations of V. Then S^nV and Λ^mV ($m \leq \dim(V)$) are representations of $GL(V)$ in a natural way. Show that they are irreducible representations.

Hint: Choose a basis $\{e_i\}$ in V. Find a diagonal element H of $GL(V)$ such that $\rho(H)$ has distinct eigenvalues (where ρ is one of the above representations). This shows that if W is a subrepresentation, then it is spanned by a subset S of a basis of eigenvectors of $\rho(H)$. Use the invariance of W under the operators $\rho(1 + E_{ij})$ for all $i \neq j$ to show that if the subset S is nonempty, it is necessarily the entire basis.

Problem 4 (Piery formula)**.** Show that $s_\lambda h_k = \sum_\lambda s_\mu$ where h_k is a complete homogeneous symmetric polynomial and the sum is over all partitions μ obtained from λ by adding k elements, no two in the same column.

Problem 5 (Transposition)**.** Show that $V_\lambda \otimes \mathrm{sgn} = V_{\lambda^t}$, where sgn is one dimensional sign representation and λ' is the conjugate partition to λ.

Hint: Use Vershik-Okounkov approach. Another possible plan: use that $\mathbb{C}[S_n]b_\lambda a_\lambda \simeq V_\lambda$ and authomorphism $\phi\colon \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ sending w to $(-1)^{l(w)}w$ for any permutation w.

Problem 6 (Gelfand-Zeitlin subalgebra)**.** Show that subalgebra of group algebra kS_n generated by Jucys-Murphy elements $J_1, \ldots, J_n$ coincides with subalgebra generated by centers of $Z(S_1), Z(S_2), \ldots, Z(S_n)$.

Hint: Prove tow inclusions by induction

Problem 7 (Action in Gelfand-Zeitlin basis)**.** Let $\{v_T | T \text{ SYT of form } \lambda\}$ be a Gelfans-Zeitlin basis in V_λ. Show that

(a) If i and $i+1$ are in the same row of T then $(i, i+1)v_T = v_T$.

(b) If i and $i+1$ are in the same column of T then $(i, i+1)v_T = -v_T$.

(c) Let i, $i+1$ are not in the same row neither in same column of T. Let T' be standard Young tablew obtained for T by permuting i and $i+1$. Then $(i, i+1)v_T = av_T + bv_{T'}$, where $a = (c(s_{i+1}) - c(s_i))^{-1}$ where s_i, s_{i+1} are boxes numbered by $i.i+1$ in T and $c(s)$ is a content of box in λ.

Hint: Use description of representations of degenerate Hecke algebra $H(2)$.

Problem 8 (Murnaghan-Nakayama rule)**.** (a) The border strip is a connected skew Young diagram without 2×2 boxes. The height $\mathrm{ht}(\rho)$ of border strip ρ is a number which is one less than the number of rows it touches. Show the Murnaghan?Nakayama formula

$$s_\lambda p_k = \sum\nolimits_\mu (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu$$

where summation goes over all μ such that μ/λ is border strip of k boxes.

(b) Let $C_{(n)}$ be conjugacy class in S_n corresponding to the cycle of length n. Show that $\chi_{V_\lambda}(C_{(n)}) = (-1)^i$ if λ is hook partition $(n-i, 1^i)$ and zero if λ is not a hook.

(c) (*) Show that

$$\chi_{V_\lambda}(C_\mu) = \sum_T (-1)^{\mathrm{ht}(T)}.$$

The summation runs over tableaux of shape λ such that each integer i appears μ_i times, the integers in every row and column are weakly increasing and the set of squares filled with the integer i form a border strip. The height, $\mathrm{ht}(T)$, is the sum of the heights of the border strips in T.

Hint: (a) Use definition of s_λ as ratio of determinants. (b) Use (a) and Frobenius character formula. (c) Use scalar product $\langle s_\lambda, p_\mu \rangle = \chi_{V_\lambda}(C_\mu)$, induction on $l(\mu)$, and (a).