- Ground field k usually $k=\mathbb{C}$

Etingof et al Introduction to Representation theory

For Lecture 1 see also Simon Wadsley Representation theory

- <u>Def</u> Representation of group G is a homomorphism $\rho: G \to GL(V)$

  V – vector space (representation space)

  $\dim V$ – dimension of representation

- Fix a basis $V=\langle e_1, \dots, e_n \rangle$ $\rho: G \to GL(n)$ - group of invertible matrices

- In the begining
  group is finite
  reps ar finite-dimensional.

- **Example** $G = S_3$ $V = k^3$

$e \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ $\quad (12) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \mapsto \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$ $\quad (13) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \mapsto \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$

$(23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$ $\quad (123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \mapsto \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$ $\quad (132) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \mapsto \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$

More generally $G = S_n$ $V = k^n$

- **Example** $G \curvearrowright X$ equiv $G \to \mathrm{Aut}(X)$

$V = kX = \langle e_x \mid x \in X \rangle$ $\quad G \to \mathrm{End}(kX)$ $\quad g e_x = e_{gx}$

$X = \{1, \dots, n\}$ $\quad G \to S_n \to GL_n$

- **Example** Regular representation $X = G$

$V = \langle e_x \mid x \in G \rangle$ $\quad g e_x = e_{gx}$

- **Def** Subspace $U \subset V$ is called subrepresentation if for $\forall g \in G$, $\rho(g)U \subset U$

Example (continued) $S_3 \curvearrowright \mathbb{C}^3$

two invariant subspaces $U_1 = \langle e_1 + e_2 + e_3 \rangle$ $U_2 = \langle \sum x_i e_i \mid \sum x_i = 0 \rangle$

- **Def** V_1, V_2 - two reps of G. Linear map $\varphi: V_1 \to V_2$ is intertwining operator (homomorphism of reps) if $\forall v \in V_1$, $g \in G$ $\varphi(\rho_1(g)v) = \rho_2(g)(\varphi(v))$

$$\begin{tikzcd} V_1 \arrow[r, "\varphi"] \arrow[d, "\rho_1(g)"'] \arrow[dr, phantom, "G"] & V_2 \arrow[d, "\rho_2(g)"] \\ V_1 \arrow[r, "\varphi"] & V_2 \end{tikzcd}$$

– commutative diagram

- **Def** Isomorphism of reps $\Longleftrightarrow$ isom of vector spaces & homomorphism of reps

In matrix terms $\exists \varphi$ s.t $\varphi \rho_1(g) \varphi^{-1} = \rho_2(g)$
(all matrices are conjugated)

- **Def** For two reps $\rho_{1,2} : G \to GL(V_i)$ Direct sum $V = V_1 \oplus V_2$
$$\rho(g) = \begin{pmatrix} \rho(g_1) & 0 \\ 0 & \rho(g_2) \end{pmatrix}$$

- **Example** (continued) $S_3 \curvearrowright \mathbb{C}^3$ If we change basis $e_1, e_2, e_3 \mapsto e_1+e_2+e_3,\ e_2-e_1,\ e_3-e_1$ Then
$$\rho(g) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho_2(g) & \\ 0 & & \end{pmatrix} \qquad V \simeq U_1 \oplus U_2$$

- **Def** Rep V is irreducible if for any subrep $U \subset V$ we have $U = 0$ or $U = V$

- **Example** ⓐ Any 1-dim rep is irreducible
  ⓑ $\mathcal{U}_2$ above is also irreducible
  (since no 1-dim subreps)

- **Th** (Schur) $\varphi \in Hom_G(V_1, V_2)$, $\varphi \neq 0$

  - If V_1 irred then φ is injection
  - If V_2 irred then φ is surjection

If V_1, V_2 are both irred then φ is isomorphism

(Note that here field is arbitrary)

Pf a) $Ker\varphi \subset V_1$ is subrep, $Ker\varphi \neq V_1 \Rightarrow Ker\varphi = 0$
b) ... $\square$

Corollary $k = \mathbb{C}$ V irred $\varphi : V \to V \Rightarrow \varphi = \lambda\, \mathrm{Id}$

Pf Let λ be any eigenvalue of φ.

Hence $\ker(\varphi - \lambda\, \mathrm{Id}) \neq 0$ Hence $\varphi - \lambda\, \mathrm{Id} = 0$ $\square$

Corollary G - commutative hence any irrep is 1-dim

Pf $\forall a \in G$, $\rho(a)\rho(g) = \rho(g)\rho(a)$ hence $\rho(a)$ intertwiner

Hence $\forall a$ $\rho(a) = \lambda_a \mathrm{Id}$

Hence $\dim V = 1$ $\square$

- <u>Example</u> $G = C_n = \mathbb{Z}/n\mathbb{Z} = \langle r \mid r^n = e \rangle$

Eigenvalues of $\rho(r) = A$ – n-th roots of unity. Let $\omega = e^{\frac{2\pi i}{n}}$.

$$A \sim \mathrm{diag}(\underbrace{\omega, \dots, \omega}_{a_1}, \underbrace{\omega^2, \dots, \omega^2}_{a_2}, \dots\dots, \underbrace{\omega^{n-1}, \dots, \omega^{n-1}}, \underbrace{\omega^0, \dots, \omega^0}_{a_0})$$

$$\begin{pmatrix} \omega & & & & & & \\ & \ddots & & & & & \\ & & \omega & & & & \\ & & & \omega^2 & & & \\ & & & & \ddots & & \\ & & & & & \omega^2 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{pmatrix} \sim A$$

Jordan Normal form

$$\mathbb{C}^m = R_1^{a_1} \oplus R_2^{a_2} \oplus \dots \oplus R_0^{a_0}$$

Here R_j is 1dim rep of C_n s.t $r \mapsto \omega^j$

$r^k \mapsto \omega^{kj}$

- Th (Mashke) G - finite group V (f.d) rep. Then
  $V \simeq V_1 \oplus \ldots \oplus V_n$ direct sum of irreducibles

Rem Here $k = \mathbb{C}$ or $\operatorname{char} k = 0$ or $(\operatorname{char} k, |G|) = 1$

Example a) $G = \mathbb{Z}$ $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$
only 1 irred subrep
b) $G = C_p = \mathbb{Z}/p\mathbb{Z}$ $\operatorname{char} k = p$ $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

- Lemma (Under the theorem assumptions) Let $U \subset V$ is subrep. Then $\exists U' \subset V$ subrep, s.t. $V \simeq U \oplus U'$

- Lemma $\Rightarrow$ Theorem

Induction by $\dim V$. If V is irrep, then o.k.
Otherwise $\exists U \subset V$ subrep $\Rightarrow \exists U' \subset V$ subrep $V = U \oplus U'$,
apply induction to U, U'

Pf (Of the Lemma) For any decomposition $V = U \oplus U'$ we can assign projector P s.t $P|_U = id$, $P|_{U'} = 0$

Going back, let P be s.t. $P^2 = P$ and $\operatorname{Im} P = U$

Then $V = U \oplus \operatorname{Ker} P$.

Choose any decomposition $V = U \oplus U'$, take corresponding P.

Averaging with respect to G: $P^G = \frac{1}{|G|} \sum_{h \in G} \rho(h) P \rho(h)^{-1}$ Then

ⓐ $\forall g \quad \rho(g) P^G \rho(g)^{-1} = \frac{1}{|G|} \sum \rho(g)\rho(h) P \rho(h)^{-1} \rho(g)^{-1} = \Big|_{h' = gh} \frac{1}{|G|} \sum_{h' \in G} \rho(h') P \rho(h')^{-1} = P^G$

ⓑ $\forall v \in V, \forall h \in G \quad P \rho(h)^{-1} v \in U \Rightarrow \rho(h) P \rho(h)^{-1} v \in U \Rightarrow P^G v \in U$

ⓒ $\forall u \in U \; \forall h \in G \quad \rho(h)^{-1} u \in U \Rightarrow P \rho(h^{-1}) u = \rho(h^{-1}) u \Rightarrow P^G u = u$

Hence $P^G \cdot P^G = P^G$. Due to ⓐ $\operatorname{Ker} P^G$ is subrep.

And we have $V = U \oplus \operatorname{Ker} P^G$ $\square$

Rem The same pf works for $G=K$ compact group.
$P^K = \frac{1}{Vol_K}\int_K \rho(h) P \rho(h^{-1})\, dh$ where dg is left invariant measure on K (Haar measure)

Rem For $k=\mathbb{C}$ there is another proof.
Consider any positive definite hermitian product
$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ $\quad \langle x, y\rangle = \sum K_{ij} \bar{x}_i y_j$
Averaging with respect to G
$\langle x, y\rangle^G = \frac{1}{|G|} \sum_{h\in G} \langle \rho(h)x, \rho(h)y\rangle$

(In Matrix terms $K^G = \frac{1}{|G|}\sum_h \rho(h)^* K \rho(h)$)
ⓐ $\langle \cdot, \cdot \rangle^G$ - hermition, positive definite

ⓑ $\langle \cdot, \cdot \rangle^G$ is G-invariant, namely $\forall x, y$ $\langle \rho(g)x, \rho(g)y\rangle = \langle x, y\rangle$
Then $U^\perp$ is subrep, $V = U \oplus U^\perp$

Problem $G = C_n = \mathbb{Z}/n\mathbb{Z} = \langle r \mid r^n = e \rangle$ Regular rep

$C_n \to$ $\quad r \mapsto \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$

(a) Find invariant subspaces

(b) Find isom with sum of irreduccibles

Problem (a) Repr $U_2 = \langle \sum_{i=1}^{3} x_i e_i \mid \sum x_i = 0 \rangle$ by matrices

(b) S_3 acts on $\mathbb{C}^2$ by symmetries of equilateral triangle. Show that this rep isom to U_2

(c) Generalize constructions (a) & (b) for S_n. Show that resulting $n-1$-dim reps are isomorphic. Show that they are irreducible.