Reference: Etingof et al Introduction to rep. th

- **Def** Associative algebra A is vector spac w/ bilinear map $A \times A \to A$ $(a,b) \mapsto ab$ s.t. $(ab)c = a(bc)$

A **unit** in A is $1 \in A$ s.t. $1 \cdot a = a \cdot 1 = a$

Below algebra — ass. algebra with 1. Assume $A \neq 0$

- **Examples** ⓐ $Mat(n, k)$ - algebra of matrices
  More invariantly V - vector space
  $End(A)$ - algebra of endomorphisms

ⓑ $k[x]$ - polynomials — commutative
$k[x_1, \dots, x_m]$ - Basis $x_1^{a_1} \cdots x_m^{a_m}$

ⓒ $k\langle x_1, \dots, x_n \rangle$ - free algebra Basis $x_{i_1} \cdots x_{i_k}$

ⓓ kG - group algebra. Basis $\langle e_g \mid g \in G \rangle$
multiplication $e_g e_h = e_{gh}$

- **Def** Homomorphism of algebras: linear map $\varphi: A \to B$ s.t $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$ & $f(1) = 1$

Def Representation of algebra is a homomorphism $\rho: A \to End(V)$

V - rep. space

$\dim V$ - dim of rep

- **Example** (a) zero rep $V = 0$

(b) Reps of G $\longleftrightarrow$ Reps of $k[G]$

(c) $V = A$ $\quad a \cdot x = ax$

If $A = kG$ - regular representation

(d) $\mathbb{C}[x]$ $(k=\bar{k})$ reps - operator $\rho(x) \curvearrowright V$

Jordan normal form

$$\rho(x) \sim \operatorname{diad}(J_{k_1}(\lambda_1), \dots, J_{k_n}(\lambda_n)) = \begin{pmatrix} \lambda_1 & 1 & 0 & & & & & & \\ & \lambda_1 & 1 & & & & & & \\ & & \lambda_1 & & & & & & \\ & & & \lambda_2 & 1 & & & & \\ & & & & \lambda_2 & & & & \\ & & & & & \lambda_3 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \lambda_n & 1 \\ & & & & & & & & \lambda_n \end{pmatrix}$$

Def Subrep, irrep, direct sum as for groups

Def Subrepresentation $U \subset V$ s.t $\forall a \in A$ $\rho(a) U \subset U$

Def Rep V is irreducible if for any subrep $U \subset V$ we have $U = 0$ or $U = V$

Def For two reps $\rho_{1,2} : A \to \operatorname{End} V_i$ Direct sum $V = V_1 \oplus V_2$
$$\rho(g) = \begin{pmatrix} \rho(g_1) & 0 \\ 0 & \rho(g_2) \end{pmatrix}$$

Def $U \subset V$ subrep hence $V/U = \{v + U \mid v \in V\}$ - quotient rep

In matrix terms $\rho \to {}_{V/U}^{U}\begin{pmatrix} \rho_1 & * \\ 0 & \rho_2 \end{pmatrix} - \rho_2$ quotient rep.

<u>Def</u> V is indecomposable if it is not isomorphic to direct sum of two nonzero reps

Any irrep is indecomp. Converse not true in general.

<u>Ex</u> (continued) ⓐ Reps $\mathbb{C}[x]$ — $\{J_{\lambda_1,k_1}, \dots, J_{\lambda_n,k_n}\}$

irreps ——— $J_1(\lambda)$ $\rho(x) = (\lambda)$

indecomp reps ——— $J_{\lambda,k}$ – one Jordan block

ⓑ Reps of kG Mashke thm. $\Rightarrow$

indecomp. rep. $\Longleftrightarrow$ irrep

- <u>Def</u> $I \subset A$ left ideal if $\forall a \in A$, $aI \subset I$

right ——— // ——— $Ia \subset I$

two-sided ideal ——— // ——— $Ia, aI \subset I$

<u>Examp</u> (a) $0 \subset A$ $\quad A \subset A$ - two-sided ideals

(b) $A = Mat(n, k)$ Left ideals w/ generator: AX

If $X = 0$, $AX = 0$ | If $X = E$ (or X invertable), $AX = A$ | If $X = \begin{pmatrix} 1 & \\ 0 & \\ \vdots & O \\ 0 & \end{pmatrix}$, $AX = \begin{pmatrix} * & \\ * & O \\ * & \end{pmatrix}$

(c) $\phi: A \to B$ homomorphism, then $\operatorname{Ker}\phi$ - two-sided ideal

<u>Pf</u> $a \in A$, $i \in I$ $\phi(i) = 0 \Rightarrow \phi(ai) = \phi(ia) = 0$

<u>Remark</u> Ideal stronger then subalgebra

c.f Normal subgroup stronger then subgroup

(d) $S \subset A$ any subset $I = \langle a_1 s a_2 \mid a_1, a_2 \in A, s \in S \rangle$ - two-sided ideal generated by S.

Def Let $I \subset A$ two-sided ideal. $A/I = \{a + I \mid a \in A\}$ quotient has algebra structure $(a+I)(b+I) = ab + I$

C.f. $H \subset G$ normal subgroup $\Rightarrow G/H$ has group str

Def If $f_1, \dots, f_m$ -elements of the free algebra $k\langle x_1, \dots, x_n\rangle$ then $k\langle x_1, \dots, x_n\rangle / (f_1, \dots, f_m)$ algebra with generators $x_1, \dots, x_n$ and relations $f_1, \dots, f_m$

Example $C_2 = \{e, r\}$, $r^2 = e$.

kC_2 - group algebra $1 = e_e$ $x = e_r$, $x^2 = 1$

$$kC_2 = k[x]/(x^2-1) \simeq k[x]/(x-1) \oplus k[x]/(x+1) \simeq k \oplus k$$

Chinese remainder theorem

$(x^2-1) = (x-1)(x+1)$

Directly $1_1 = \frac{1}{2}(1+x)$, $1_2 = \frac{1}{2}(1-x) \in \mathbb{C}[x]/x^2-1$

Then $1_i 1_i = 1_i$ $1_1 1_2 = 0$

$(a1_1 + b1_2)(a'1_1 + b'1_2) = (aa'1_1 + bb1_2)$ $\mathbb{C}[x]/(x^2-1) \simeq \mathbb{C} \oplus \mathbb{C}$

- Example Weyl algebra $k\langle x, y\rangle/(xy - yx + 1)$

- No f.d. reps $Tr(xy - yx) = 0$ $Tr(1) = \dim V$
- $\exists$ infinite d. reps e.g. $V = x^\lambda \mathbb{C}[x, x^{-1}] = \langle x^{\lambda+n} \mid n \in \mathbb{Z}\rangle$

  or $V = C^\infty(\mathbb{R})$ $x \cdot f = xf$ $y \cdot f = \frac{\partial}{\partial x} f$.

- Recall

  Th (Schur) $\varphi \in \mathrm{Hom}_a(V_1, V_2)$ If V_1, V_2 are irred then φ is isomorphism or $\varphi = 0$

Corollary $k = \bar{k}$ V irred $\varphi : V \to V \Rightarrow \varphi = \lambda\, \mathrm{Id}$

Def A semisimple (or completely reducible) rep $V = \oplus n_i V_i$, V_i (different) irreps

Th Let $V = \oplus n_i V_i$, V_i different irreps, $W \subset V$ subrep.

a) $W \simeq \oplus r_i V_i$ $0 \le r_i \le n_i$ $k = \bar{k}$

b) Inclusion $\phi : W \hookrightarrow V$ is $\oplus \phi_i$

$\phi_i : r_i V_i \to n_i V_i$ $X_i \in Mat(r \times n)$

$\phi_i(v_1, \dots, v_{r_i}) = (v_1 \dots v_{r_i}) X_i$

Pf Induction on $\sum n_i$

Step $P \subset W$ irrep P isom to some V_i, i is unique

By Schur lemma $\pi_{ij}(v) = x_{ij} v$. Here $\pi_{ij} : V \to V_i$ jth summ

Using GL_{n_i}, $\exists X$ $X(x_{i1}, \dots, x_{in_i}) = (1 \dots 0)$. Then

P isom first V_i in $n_i V_i$ $W = P \oplus \operatorname{Ker} \pi_{i,1}$

- **Theorem (density)** $\rho: A \to \operatorname{End}(V)$ $\quad k=\bar{k}$

a) V — f.d. irrep. Then $\operatorname{Im}\rho = \operatorname{End}_k V$

b) $V = V_1 \oplus \dots \oplus V_n$ $\quad V_i$ nonisomorphic irreps
Then $\operatorname{Im}\rho = \bigoplus' \operatorname{End}_k V_i$

Corollary a) V irrep G. Then $(\dim V)^2 \le |G|$
b) $V_1, \dots, V_n$ irreps G. Then $\sum (\dim V_i)^2 \le |G|$
In particular finitely many irreps

Rem V rep. A, $\dim V = d$. Then $\operatorname{End}(V) = dV$ as A mod
$\operatorname{End} V = \begin{pmatrix} \vdots & \vdots & & \vdots \end{pmatrix}$ — columns corresp V.

Pf a) Let $e_1, \dots, e_d$ - basis V. Want to prove $\forall f_1, \dots, f_d \in V$ $\exists a \in A$ s.t. $\rho(a) v_i = w_i$.

Assume contrary $W = (\rho(a)e_1, \rho(a)e_2, \dots, \rho(a)e_d) \subset nV$ proper submodule
By assumption $W \simeq rV$ $\quad r < d$

Hence $\exists$ matrix $X \in Mat(r\times d, k)$ s.t $W = \{(u_1, \dots, u_r)X \mid u_i \in V\}$

For instance $(e_1, \dots, e_d) = (u_1, \dots, u_r)X$ for some $u_1, \dots, u_r$

But $e_1, \dots, e_d$ lin. independent
$rk X \le r$ $\exists$ lin relation between columns of X $\Big\} \Rightarrow$ contradiction

b) Let B - image A in $End(V)$.
Note $End\, V_i = d_i V_i$, where $d_i = \dim V_i$

- $B \subset \bigoplus End(V_i) \simeq \bigoplus d_i V_i$ - as A-mod
- Hence $B = \bigoplus r_i V_i$ $\quad r_i \le d_i$
- projetion $A \to End\, V_i$ surjective by a) hence $r_i \ge d_i$
  Hence $B \simeq \bigoplus d_i V_i \simeq \bigoplus End\, V_i$

$\square$