Where we are

Reps (finite group) G $\forall g \mapsto \rho(g) \in Mat(n, \mathbb{R})$
Reps of algebras $G \to \mathbb{R}G$
s/s reps $\oplus n_i V_i$ - irreds
subreps $\oplus r_i V_i$, if $n_i = 1$ $A \twoheadrightarrow \oplus End(V_i)$ surj
density thm

<u>Def</u> A - s/s algebra if A is s/s as A mod

<u>Th</u> $k = \bar{k}$ $(k = \mathbb{C})$ A s/s $\iff$ $A = \oplus End_k(V_i) = \oplus Mat_{d_i}(k)$

<u>Pf</u> $A = \oplus d_i V_i$ as A-mod

Schur lemma $\Rightarrow$ $End_{A\text{-mod}} A = \oplus Mat_{d_i}(k)$

On the other hand $\forall \varphi \in End_{A\text{-mod}}(A)$, let $\varphi(1)=a \in A$

$\forall x \in A \quad \varphi(x)=\varphi(x\cdot 1) = x\,\varphi(1) = xa \Rightarrow$

$\varphi = \rho_a$ - right mult by $\underline{a \in A}$

$\rho_a \circ \rho_b(x) = xba = \rho_{ba}(x) = \rho_{a \times_{op} b}\, x$ Hence $End_{A\text{-mod}} A = A^{op}$

(Opposite product $a \times_{op} b = ba$)

$A^{op} \simeq \bigoplus Mat_{d_i}(k) \Rightarrow \quad A \simeq \bigoplus Mat_{d_i}(k)$ $\square$

$A = \bigoplus d_i\, k^{d_i} \Rightarrow d_i = \dim V_i$

- **Th** Let $A = \bigoplus_{i=1}^{\ell} Mat(d_i, k)$ Then irreps A are $V_1, \dots, V_\ell$, $V_i = k^{d_i}$ Any f.d. rep of A - sum of V_i

Example $kC_2 = Mat_1 \oplus Mat_1$

$k[x]/_{x^2-1} \simeq k[x]/_{x-1} \oplus k[x]/_{x+1}$

Any rep C_2 - sup of 2 irreps

Pf V - rep $(e_1, \dots, e_n)$ - basis V.

$\phi: nA \to V \quad (a_1, \dots, a_n) \mapsto \sum a_i e_i$ surj map of A-mod

$W = \ker \phi, \quad V = nA/W \qquad nA = \oplus n d_i V_i \qquad W = \oplus l_i V_i$

$A^{\oplus n}/W \simeq \oplus (nd_i - r_i) V_i$ change $\overbrace{\text{basis}}$ — first r_i summands in $nd_i V_i$ $\square$

Corollary G, $|G| < \infty$ d_i - dimensions of irreps

$\sum d_i^2 = |G| = \dim kG$

Prop Let A be f.d. algebra Let V be irred rep

Then $\dim V < \infty$

Pf $\forall e \in V$ $\rho(A)e$ - invariant subspace

$0 < \dim \rho(A)e \leq \dim A \Rightarrow \rho(A)e = V$ $\square$

Prop V – f.d. rep of A. Then $\exists$ filtration of subreps $0 = V_0 \subset V_1 \subset \dots \subset V_n = V$ s.t. $\forall i$ V_i / V_{i-1} – irred rep.

Pf Induction by $\dim V$. If V is irred. – ok
if not – $\exists$ $V_1 \subset V$ irred. Let $U = V/V_1$,
$\dim U < \dim V$
$\exists$ filtration $0 = U_0 \subset U_1 \subset \dots \subset U_{n-1} = U$ s.t. U_i / U_{i-1} – irred
Define V_i to be preimage of U_{i-1} $\square$

In coordinates:

① $\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$

② $\begin{array}{c} \begin{array}{ccc} V_1 & & \end{array} \\ \begin{pmatrix} \rho_{V_1} & \times & \times \\ 0 & & \\ \vdots & \rho_{V/V_1} & \\ 0 & & \end{pmatrix} \end{array}$

③ $\begin{pmatrix} \rho_{V_1} & & & \times \\ 0 & \rho_{V_2/V_1} & & \\ & 0 & \ddots & \\ 0 & & 0 & \rho_{V_n/V_{n-1}} \end{pmatrix}$

Rk Here A arbitrary

Example $A = k[x]$ $\quad V = \bigoplus J_{n_i}(\lambda_i)$ – Jordan Blocks

$$x \mapsto \underbrace{\begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}}_{n_i}$$

Filtration $\quad V_i = \langle e_1, \dots, e_i \rangle$

$V_i / V_{i-1} \quad x \mapsto (\lambda) \qquad V_1/V_0 \simeq V_2/V_1 \simeq \dots \simeq V_{n_i}/V_{n_i-1} = J_1(\lambda_i)$

$J_n(\lambda_i) \neq n J_1(\lambda_i)$

Def $\mathrm{rad}(A) = \{ a \in A \mid \rho(a) = 0 \quad \forall$ f.d. irred. rep $\rho \}$

Ex $A = \bigoplus \mathrm{End}(V_i) \quad \mathrm{Ker}\, \rho_i = (x \dots x \underset{i\text{-th}}{0} x \cdot x) \quad \mathrm{Rad}\, A = \bigcap \mathrm{Ker}\, \rho_i = 0$

Prop $\mathrm{rad}(A)$ is 2-sided ideal

Pf Need $\forall a \in A \quad a\, \mathrm{rad}\, A \subset \mathrm{rad}\, A \qquad \mathrm{rad}\, A\, a \subset \mathrm{rad}\, A$ $\square$

Prop Let A be f.d. algebra.

(i) Let I nilp 2-sided ideal, i.e $\exists n$ s.t. $x_1 \cdot \ldots \cdot x_n = 0$ $\forall x_1, \ldots, x_n \in I$ $(I^n = 0)$

Then $I \subset \operatorname{rad} A$

(ii) $\operatorname{Rad} A$ is nilpotent ideal.

Thus $\operatorname{Rad} A$ – maximal nilpotent 2-sided ideal

Pf (i) If $I^n = 0$, V-irred rep. Assume $\exists e$ s.t $Ie \neq 0$

Then $Ie = V \Rightarrow \exists x \in I$ s.t. $xe = e \Rightarrow x^n e = e$ – contradiction

(ii) A as a left A module has filtration

$0 = A_0 \subset A_1 \subset \ldots \subset A_n = A$ $\qquad V_i / V_{i-1}$ irred

In this basis $\operatorname{rad} A \sim \begin{pmatrix} 0 & * & & * \\ & \ddots & \ddots & \vdots \\ & & & * \\ & 0 & & 0 \end{pmatrix}$ i.e.

$(\operatorname{rad} A) A_i \subset A_{i-1}$ $\forall i$ $\Rightarrow$ $(\operatorname{rad} A)^n = 0$ $\square$

Th $k=\bar{k}$ (i) A f.d algebra. Then it has f.m. irred reps $V_1, \dots, V_r$ and $A/_{\mathrm{rad} A} \simeq \bigoplus \mathrm{End}(V_i)$

(ii) A s/s $\iff A = \bigoplus \mathrm{Mat}_{d_i}(k) \iff \mathrm{rad}\, A = 0$

Example $A = k[x]/x^n$ $\quad \mathrm{rad} A = (x) = \langle x, x^2, \dots, x^{n-1} \rangle$

Reps $X \sim \bigoplus J_{n_i}(0)$ $\quad n_i \leq n$ $\quad \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix} n_i$

irrep $J_1(0)$ $\quad \mathrm{Ker} = \mathrm{rad} A$ $\quad A/_{\mathrm{rad} A} = k$

Example $A = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$ algebra of upper triangular matrices

$\mathrm{Rad}\, A = \left\{ \begin{pmatrix} 0 & \ddots & * \\ 0 & & 0 \end{pmatrix} \right\}$ strictly upper triangular matrices

$A/_{\mathrm{Rad} A} = \left\{ \begin{pmatrix} * & \ddots & 0 \\ 0 & & * \end{pmatrix} \right\} = \bigoplus_{i=1}^{n} k$ $\quad \mathrm{rad} A$ - nilpotent

Pf (i) Map $\Phi : A \to \bigoplus \mathrm{End}(V_i)$ surj by density thm

(ii) $\mathrm{Rad}\, A = 0 \Rightarrow A = \bigoplus \mathrm{End}(V_i)$

$A = \bigoplus \mathrm{End}(V_i)$, $\mathrm{Rad}(\mathrm{End}(V_i)) = 0 \Rightarrow \mathrm{Rad}(\bigoplus \mathrm{End}(V_i)) = 0$ $\square$

Rem True for $k \neq \bar{k}$ but have to add $\mathrm{Mat}(n, H)$
f.d division algebra over k
For $k = \mathbb{R}$ ~ $H = \mathbb{R}, \mathbb{C}, \mathbb{H}$

s/s algebras	f.d. algebras
$A = \bigoplus \mathrm{Mat}(d_i, k)$	$\mathrm{rad}\, A$
$\dim A = \sum d_i^2$	$A/\mathrm{rad}\, A = \bigoplus \mathrm{Mat}(d_i, k)$
$\lvert G \rvert = \sum d_i^2$	