Def $\rho : A \to \operatorname{End} V$ rep. Character
$\chi_V : A \to k$ $\chi_V(a) = \operatorname{Tr} \rho(a)$

Prop $U \subset V$ subrep $\chi_V = \chi_U + \chi_{V/U}$

Pf
$$\begin{array}{cc} U & V/U \\ \left(\begin{array}{cc} \rho_U & * \\ 0 & \rho_{V/U} \end{array}\right) & \end{array}$$
$\square$

Th A f.d algebra, $k = \mathbb{C}$, $V_1, \dots, V_r$ - all irreds $\chi_i = \chi_{V_i}$
Then (i) $\chi_1, \dots, \chi_r$ are linearly independent
(ii) $\chi_1, \dots, \chi_r$ - form a basis on $\left(A^{ss} / [A^{ss}, A^{ss}]\right)^*$
where $A^{ss} = A / \operatorname{rad} A$

Pf $\chi_1, \dots, \chi_r$ vanish on $\operatorname{rad} A$, hence suffices to consider $A^{ss} = A / \operatorname{rad} A = \bigoplus \operatorname{End}(V_i)$

$\chi_i : \bigoplus \operatorname{End} V_i \to k$ linearly indep
$(a_1, \dots, a_r) \mapsto \operatorname{Tr} a_i$

Lemma Let $A = \mathrm{Mat}_n(k)$ $\quad [A,A] := \langle [x,y] \mid x, y \in A \rangle$
where $[x,y] = xy - yx$. Then $[A,A] = \{z \mid tr\, z = 0\} = \mathfrak{sl}_n(k)$

Hint $[E_{ij}, E_{je}] = E_{ie} \quad i \neq e$ $\qquad \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$

$[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$ $\qquad \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $\square$

- **Th** (Jordan-Hölder) V - f.d rep of A $\quad k = \mathbb{C}$

$0 = V_0 \subset V_1 \subset \dots \subset V_n = A$ two filtrations s.t.

$0 = V_0' \subset V_1' \subset \dots \subset V_m' = A$ $\quad W_i = V_i / V_{i-1}$, $W_j' = V_j' / V_{j-1}'$ - irred. reps

Hence $n = m$ and $\exists$ permutation $\sigma \in S_n$ s.t. $W_{\sigma(i)} \cong W_i'$

Pf $\chi_A = \sum \chi_{W_i} = \sum \chi_{W_i'}$ $\square$

Rem Thm holds for $\forall$ ground field k.

Def n in Jordan–Hölder thm is length of rep V

- G – finite group. $A = kG = \bigoplus_{i=1}^{\ell} \mathrm{End}(V_i)$

Can restrict character to G.

$[A,A] = \langle gh - hg \mid g,h \in G \rangle = \langle g_1 - g_2 \mid g_1 \sim g_2 \rangle$

conjugated

Hence χ_V constant on conjugacy classes

Basis in $A/[A,A] = \langle \mathbb{1}_C \mid C\text{-conj classes in } G \rangle$

$\bigoplus$ = Functions on conj. classes $= \langle \mathbb{1}_C \mid \text{c.c.} \rangle = \langle \chi_i \mid V_i \in \text{Irreds} \rangle$

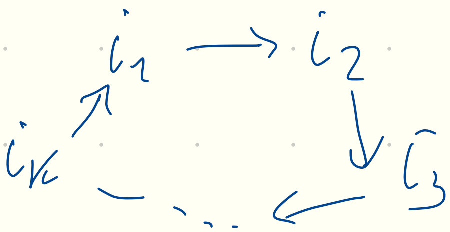
$C_1, \dots, C_\ell$ – set of conj classes

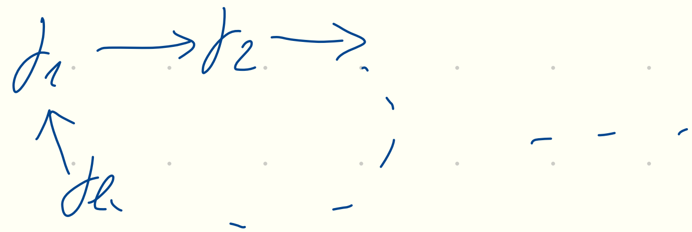
Th $|G| = \sum d_i^2$, $\#$ irreps $= \#$ conj classes

Example $G = S_3$

Digression Conjugacy classes in S_n

Any permutation: product of indep. cycles





$$\sigma\Big((i_1,\dots,i_k)(j_1,\dots,j_l)(\dots)\Big)\sigma^{-1} = (\sigma(i_1),\dots,\sigma(i_k))(\sigma(j_1),\dots,\sigma(j_l))\dots$$

Permutations $\alpha \sim \beta$ are conjugated $\Longleftrightarrow$ have the same lengths of independent cycles

Return to S_3

		e	$(1\,2)$ $(1\,3)$ $(2\,3)$	$(1\,2\,3)$ $(1\,3\,2)$
trivial rep	χ_{triv}	1	1	1
sign rep	χ_{sgn}	1	-1	1
	$\chi_{\mathbb{C}^3}$	3	1	0
Recall $\mathbb{C}^3 = U_2 + triv$	χ_{U_2}	2	0	-1
	χ_{reg}	6	0	0

\# irreps $= 3$

$6 = 1+1+2^2$

Prop $\chi_{reg}(g) = \begin{cases} |G| & g = e \\ 0 & \text{otherwise} \end{cases}$

$\chi_{reg} = \sum d \chi_{V_i}$ $\quad d_i = \dim V_i$

Example: For $G = S_3$ $\quad \chi_{reg} = \chi_{triv} + \chi_{sgn} + 2\chi_{U_2}$

Example $G = C_n = \{e, r, r^2, \dots, r^{n-1} \mid r^n = e\}$

$\omega = e^{2\pi i/n}$ - primitive root of 1

$R_j : r^a \mapsto \omega^{aj}$

	e	r	r^2	r^3	$\dots$	r^{n-1}
R_0	1	1	1	1		1
R_1	1	ω	ω^2	ω^3		
R_2	1	ω^2	ω^4	ω^6	$\vdots$	
R_3	1	ω^3	ω^6	ω^9	$\vdots$	
$\vdots$		$-$	$-$	$-$		
R_{n-1}	1					ω^{n^2}
χ_{reg}	n	0	$\dots$	$\dots$	$\dots$	0

$\chi_{reg} = \sum \chi_{R_j}$

Indeed

$$\chi_{reg}(r^a) = \sum_{j=0}^{n-1} \omega^{aj} = \frac{1-\omega^{an}}{1-\omega^a} = 0$$

- What can we do with characters?
  - $\chi_{V_1} \oplus \chi_{V_2} = \chi_{V_1 \oplus V_2}$
- Dual representation V^*

Action $\forall \varphi \in V^*$ $\quad g\varphi \in V^*$ defined by

$$(g\varphi)(v) = \varphi(g^{-1}v) \quad \forall v$$

This is group homomorphism since

$$(g(h\varphi))v = h\varphi(g^{-1}v) = \varphi(h^{-1}g^{-1}v) = (gh)\varphi(v)$$

In matrix terms $\quad \rho_{V^*}(g) = (\rho_V(g)^t)^{-1}$

$$\rho_V(g) = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_2 \\ & & & & & & \ddots \end{pmatrix} \qquad \rho_{V^*}(g) = \begin{pmatrix} \lambda_1^{-1} & & & & 0 \\ & \ddots & & & \\ & & \lambda_1^{-1} & & \\ & & & \lambda_2^{-1} & \\ & & & & \ddots \\ & & & & & \lambda_2^{-1} \end{pmatrix}$$

$\exists\, n$ s.t $g^n = 1 \Rightarrow \forall i\ \lambda_i \in \sqrt[n]{1}$, $|\lambda_i| = 1$ $\lambda_i^{-1} = \overline{\lambda_i}$

$\rho_{V^*}(g) = \overline{\rho_V(g)} = \rho_V(g^{-1})$

Remark $g^n = 1 \Rightarrow \rho_V(g)$ - diagonalizable

If $\chi_V(g) \in \mathbb{R} \Rightarrow V \simeq V^*$ (e.g for S_3 not for C_n) and for S_n

- Tensor product

Def $V \otimes W$ of vector spaces is a quotient of space with basis $v \otimes w$ $v \in V$, $w \in W$ by subspace

$$a v \otimes w - a(v \otimes w) \qquad a \in k$$
$$(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w \qquad v, v_1, v_2 \in V$$
$$v \otimes (a w) - a(v \otimes w) \qquad w, w_1, w_2 \in W$$
$$v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2$$

- If $\{e_i\}$ - basis in V, $\{f_j\}$ - basis in W $\Rightarrow$ $\{e_i \otimes f_j\}$ - basis in $V \otimes W$

Elements of the form $v \otimes w$ - pure tensors
not any element of $V \otimes W$ is pure, e.g $e_1 \otimes f_1 + e_2 \otimes f_2$ - not pure

In physics / diff geom tensors $V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^*$

- $V^* \otimes W = \mathrm{Hom}(V, W)$

$\xi \otimes w \mapsto (\varphi(v) = \xi(w) v)$

In matrix terms $e^i \otimes f_j \mapsto E_{ji} = \begin{array}{cc} & i \\ j & \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \end{array}$

- Tensor product of operators $A \otimes B(v \otimes w) = Av \otimes Bw$

In matrix terms $A \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $B \sim \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$

$$A\otimes B(e_i\otimes f_j) = \lambda_i e_i \otimes \mu_j f_j = \lambda_i \mu_j\, e_i \otimes f_j$$

$$A\otimes B = \begin{pmatrix} \lambda_1\mu_1 & & & \\ & \lambda_1\mu_2 & & \\ & & \lambda_2\mu_1 & \\ & & & \lambda_2\mu_2 \end{pmatrix}$$

$$\mathrm{Tr}(A\otimes B) = \mathrm{Tr}A \cdot \mathrm{Tr}B$$

- Tensor product of reps

<u>Def</u> V_1, V_2 - reps of G

$V_1 \otimes V_2$ rep of G $\quad \rho_{V_1\otimes V_2}(g) = \rho_{V_1}(g) \otimes \rho_{V_2}(g)$

In terms of characters $\chi_{V_1\otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$

<u>Remark</u> In general - no structer V^*, $V\otimes W$ on A-mod. Isomorphism $A \simeq kG$ gives this structure

<u>Ex</u> C_n $\quad R_j \otimes R_{j'} = R_{j+j' \bmod n}$

<u>Def</u> Virtual reps $V = \sum n_i V_i \quad n_i \in \mathbb{Z}$

$\chi_V = \sum n_i \chi_{V_i}$

Ring of virtual reps $= \Theta =$ Ring of functions on G invariant under conjug. $k[G]^{\mathrm{Ad}\,G}$

$=$

(Functions on $\{C_1, \ldots, C_\ell\}$)

- $\mathrm{Hom}(V,W) = V^* \otimes W$

$\chi_{V^* \otimes W}(g) = \overline{\chi_V(g)}\,\chi_W(g)$

<u>Lemma</u> $\mathrm{Hom}_G(V,W) = (V^* \otimes W)^G$

Pf $\varphi(gv) = g\,\varphi(v)$

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd} \iff \forall g \quad g\,\varphi(g^{-1}v) = \varphi(v)$$

$\square$

Lemma $P = \frac{1}{|G|}\sum g$ Projector $U \to U^G$ $\forall U$

Pf $g P u = \frac{1}{|G|}\sum g h u = P u$

$$P^2 = \frac{1}{|G||G|}\sum h_1 h_2 = \frac{1}{|G||G|}\sum_{h_1, h} h = \frac{1}{|G|}\sum h = P$$

$\square$

Hence $\dim (V^* \otimes W)^G = \operatorname{Tr} P|_{V^*\otimes W} = \frac{1}{|G|}\sum_g \chi_{V^*\otimes W}(g) = \frac{1}{|G|}\sum .$

Def Hermitian scalar product

$$\langle \chi, \chi' \rangle = \frac{1}{|G|}\sum \overline{\chi(g)}\,\chi'(g) = \frac{1}{|G|}\sum_{i=1}^{l} |C_i|\,\overline{\chi(h_i)}\,\chi'(h_i)$$

where $h_i \in C_i$ set of representatives.

Arguments above $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W)$

Schur lemma $\Rightarrow$ $\langle \chi_i, \chi_j \rangle = \delta_{ij}$

Th $\chi_{V_1} \dots \chi_{V_l}$ form an orthonormal basis Θ

Rem $\left\{ \sqrt{\frac{|G|}{|C_i|}} \mathbb{1}_{C_i} \right\}$ – form another orthonormal basis

Character matrix ~ unitary after
rescaling of columns