- "New" Vershik-Okounkov approach to reps S_n

Idea ~ inductive process

$S_1 \subset S_2 \subset \dots \quad \dots \subset S_{n-1} \subset S_n \subset S_{n+1} \subset \dots$

$S_m \subset S_n$ permutes $\{1 \dots m\} \subset \{1 \dots n\}$

- $V \in \operatorname{Irr} S_n$, $\quad U \in \operatorname{Irr} S_m$

$$\operatorname{Res}^{S_n}_{S_m} V = \bigoplus_{U \in \operatorname{Irr} S_m} U \otimes M_{V,U}$$

multiplicity space

irred. reps of $Z_m(n) = Z_{kS_m}(kS_n)$

Th $Z_m(n)$ is generated by

(a) $Z(kS_m)$ $\quad$ (b) $kS_{[m+1,\dots,n]}$

(c) $J_{m+1}, \dots, J_n$ $\quad$ where $J_\ell = \sum_{i<\ell} (i,\ell)$

Corollary (i) Z^{n}_{n-1} is commutative

(ii) $\mathrm{Res}^{S_n}_{S_{n-1}} V = \bigoplus_{\mathcal{U}_i} \mathcal{U}_i$ - pairwise different

$$J_n\big|_{\mathcal{U}_i} = w_{n,i}\,\mathrm{Id}\big|_{\mathcal{U}_i}$$

- Branching graph

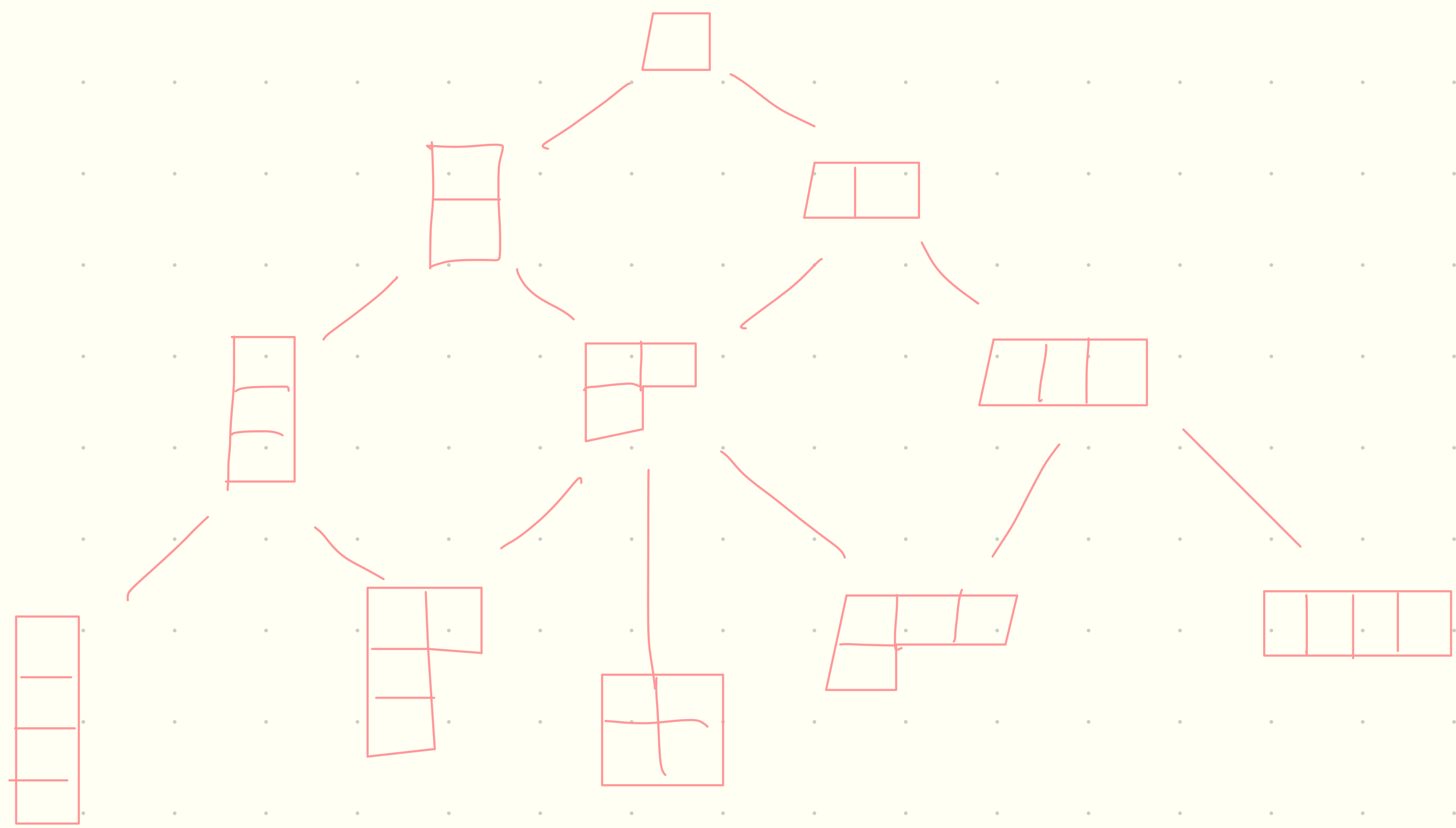
Vertices $\bigcup_{n\geq 1} \mathrm{Irr}(S_n)$

Edges $V^{(n-1)} \longrightarrow V^{(n)}$ iff $\dim M_{V^{(n)}, V^{(n-1)}} = 1$

Here and below $V^{(m)} \in \mathrm{Irr}\, S_m$

$$\begin{tikzcd}[column sep=small]
 & & \mathrm{triv} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \mathrm{triv} \arrow[dl, dash] \arrow[dr, dash] & & \mathrm{sgn} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{triv} & & \mathrm{refl}_3 & & \mathrm{sgn}
\end{tikzcd}$$

Hope



$\forall$ arrow $V^{(n-1)} \to V^{(n)}$ defines a map (up to factor)

Hence $\forall$ path $p: (V^{(m)}, V^{(m+1)}, \dots, V^{(n)})$ defines a map $\varphi_p: V^{(m)} \to V^{(n)}$ (up to resaling)

In particular $\forall$ $p: (V^{(1)} = \mathrm{triv}, V^{(2)}, \dots, V^{(n)})$ defines $\varphi_p: V^{(1)} = \mathrm{triv} \to V^{(n)}$ $\Rightarrow$ defines a vector $v_p = \varphi_p(1)$ up to rescaling

Lemma (a) Vectors $\{v_P \mid P : V^{(1)} \to V^{(n)}\}$ form a basis in $V^{(n)}$

(b) v_P – eigen vectors for $\langle J_1, \dots, J_n \rangle$

Pf Induction $V^{(n)} = V_1^{(n-1)} \oplus \dots \oplus V_k^{(n-1)}$

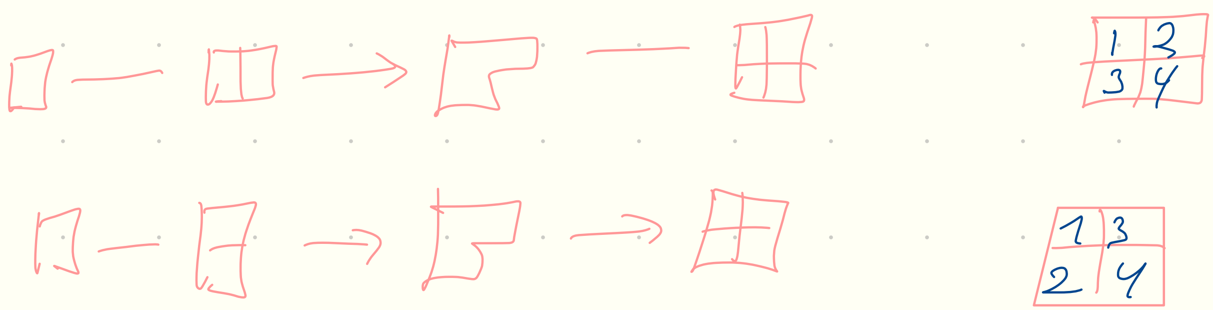
$$\begin{tikzcd} V_1^{(n-1)} \arrow[dr, dash] & \cdots \arrow[d, dash] & V_k^{(n-1)} \arrow[dl, dash] \\ & V^{(n)} & \end{tikzcd}$$

$\Rightarrow$ (a) & (b) $\square$

- Algebra generated by $J_1, \dots, J_n$ – commutative
  Indeed, if $l < m$ $J_l \in \mathbb{k} S_l \subset \mathbb{k} S_{m-1}$ – commute w/ J_m.

Gelfand – Zeitlin subalgebra (Quantum Integrable System)

- Hope



paths P to V_λ $\overset{1:1}{\longleftrightarrow}$ standart tablew of form λ

In particular $\dim V_\lambda = \#\{SYT \text{ of form } \lambda\}$ — Standard Young tableau

Ex $n=4$ triv $\begin{array}{|c|c|c|c|}\hline & & & \\ \hline \end{array}$ $\qquad \frac{4!}{4!}=1 \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array}$

refl_4 $\begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1} \end{array}$ $\qquad \frac{4!}{4\cdot 2}=3 \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}$

$\mathbb{C}^2$ $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$ $\qquad \frac{4!}{3\cdot 2\cdot 2}=2 \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$

- **Notation** $v_p \sim w_p = (w_1, \dots, w_n)$ eigen values of $J_1, \dots, J_n$. Since $J_1 = 0 \Rightarrow w_1 = 0$

Lemma $w_p = w_{p'} \Rightarrow p = p'$

Pf Induction on n. Let $P = (V^{(1)} - V^{(2)} - \dots - V^{(n)})$

$P' = (V^{(1)'} - V^{(2)'} - \dots V^{(n)'})$ $\quad \underline{P} = (V^{(1)} - \dots - V^{(n-1)})$ - truncation of P, similarly for P'.

By induction hypothesis $v_{\underline{P}} = v_{\underline{P}'} \in V^{(n-1)}$

$$\begin{array}{ll} P: & V^{(n-1)} \to V^{(n)} \\ P': & \phantom{V^{(n-1)}} \to V^{(n)'} \end{array}$$

Note $Z(kS_n) \subset Z_{kS_{n-1}}(kS_n)$ – generated by $Z(kS_{n-1})$, J_n

$$Z(kS_{n-1})\big|_{V_p} = Z(kS_{n-1})\big|_{V^{(n-1)}} = Z(kS_{n-1})\big|_{V_{p'}}$$

$$J_n\big|_{V_p} = J_n\big|_{V_{p'}} \Rightarrow Z(kS_{n-1})\big|_{V^{(n)}} = Z(kS_n)\big|_{V_p} = Z(kS_n)\big|_{V_{p'}} = ZkS_n\big|_{V^{(n)'}}$$

Hence $V^{(n)} \cong V^{(n)'}$ $\square$

- Next task: try to modify path preserving target

Simple transformation – in only one place

$Path(P,i) = \{ p' \mid P, p'$ can differ only in i-th place $\}$

$$\begin{tikzcd} & V^{(i)} \arrow[dr, dash] & \\ V^{(i-1)} \arrow[ur, dash] \arrow[dr, dash] & & V^{(i+1)} \\ & V^{(i)'} \arrow[ur, dash] & \end{tikzcd}$$

Let $V_{P,i} = \langle v_{p'} \mid p' \in Path(P,i) \rangle$

<u>Lemma</u> $V_{P,i}$ – irred rep of $Z_{i-1}(i+1)$

$V_{P,i} \simeq M_{V^{(i+1)}, V^{(i-1)}}$

Lemma The following identities hold

$J_i J_{i+1} = J_{i+1} J_i \qquad (i,i+1)^2 = 1 \qquad (i,i+1) J_i = J_{i+1}(i,i+1) - 1$

Pf First two we know. For the third one

$(i,i+1) J_i (i,i+1) = (i,i+1) \sum_{\ell<i} (\ell,i)(i,i+1) = \sum_{\ell<i} (\ell,i+1) = J_{i+1} - (i,i+1)$ $\square$

Def Degenerate affine Hecke algebra $H(2)$ is an algebra w/ generators X_1, X_2, T and relations

$X_1 X_2 = X_2 X_1 \qquad T^2 = 1 \qquad T X_1 = X_2 T - 1$

Rem (i) Easy to deduce $X_1 T = T X_2 - 1$

(ii) We have a map $H(2) \to Z_{i-1}(i+1)$

not injective $\dim H(2) = \infty \qquad H(2) = \langle X_1^{d_1} X_2^{d_2} T^{\varepsilon} \mid d_1, d_2 \geq 0, \varepsilon = 0,1 \}$

Hence $\forall i$ there are more relations. But these are universal!

(iii) $V_{P,i}$ - irred. rep of $H(2)$ (since $Z_{i-1}(i+1)$ generated by $\operatorname{Im} H(2)$ & $Z(S_{i-1})$ but $Z(S_{i-1})$ acts by scalar)

(iv) More generally $H(n-m) \to \mathbb{Z}_m(n)$

- Next goal – classify f.d. irreds of $H(2)$

Since $X_1X_2 = X_2X_1 \Rightarrow$ they have joint eigenvector

Assume $X_1 m = am$, $X_2 m = bm$. There are several cases

<u>Case 1</u> $Tm = m \Rightarrow$ 1 dim rep $b = a+1$

<u>Case 2</u> $Tm = -m \Rightarrow$ 1 dim rep $b = a-1$

<u>Case 3</u> m, Tm – linearly independent. Then

$X_1(Tm) = TX_2m - m = bTm - m$ 2-dim rep.

$X_2(Tm) = TX_1m + m = aTm + m$ denote it by $M(a,b)$

$$T \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad X_1 \mapsto \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix} \quad X_2 \mapsto \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}$$

<u>Lemma</u> $M(a,b)$ is irred iff $a \neq b \pm 1$

$M(a,b) \simeq M(a',b') \iff a=a', b=b'$ or $a=b', b=a'$

<u>Theorem</u> F.d. irred reps of $H(2)$ are classified by (a,b)

(1) $b=a+1$ ~ 1 dim $T=1$

(2) $b=a-1$ ~ 1 dim $T=-1$

(3) $b \neq a \pm 1$, $b \neq a$ $M_{a,b}$ 2dim $M_{a,b} \simeq M_{b,a}$

(4) $b=a$, $M_{a,a}$ 2dim, X_1, X_2 are <u>not</u> diagalizable

<u>Corollary</u> Let $w_p = (w_1, w_2, \dots, w_n)$ Then

① $w_{i+1} = w_i$ $\forall i$

② If $w_{i+1} = w_i \pm 1 \Rightarrow Path(P,i) = \{P\}$

③ If $w_{i+1} \neq w_i \pm 1 \Rightarrow Path(P,i) = \{P, P'\}$ where P' differs from P by permutation of w_i, w_{i+1}

admissible transpositions

④ If $i \leq n-2$, $w_{i+1} = w_i \pm 1 \Rightarrow w_{i+2} \neq w_i$

Pf ①, ②, ③ ~ easy

④ Assume opposit: $w_{i+1} = w_i \pm 1$, $w_{i+2} = w_{i+1} \mp 1$

Then $(i, i+1) v_p = \pm 1 v_p$ but $(i+1, i+2) v_2 = \mp v_p$

Contradicts braid relation

$$(i, i+1)(i+1, i+2)(i, i+1) = (i+1, i+2)(i, i+1)(i+1, i+2)$$ $\square$

- Promised answer

Def $s = (i, j) \in \lambda$ Content $c(s) := i - j$

$\forall$ SYT T gives numbering of boxes $s_1, s_2, \ldots, s_n$

Define $w_T = (c(s_1), c(s_2), \ldots, c(s_n))$

Example

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$(0, 1, 2, -1)$ $\qquad$ $(0, 1, -1, 2)$ $\qquad$ $(0, -1, 1, 2)$

Satisfy properties of Corollary

Def Two weights $(w_1, w_2, \dots, w_n) \sim (w_1', \dots, w_n')$ equiv if one is obtained from another by sequence of admissible transpositions

Lemma Any class $\{w_p\}/\sim$ contains one element of the form $(0, 1, \dots, \lambda_1 - 1, -1, \dots, \lambda_2 - 2, \dots, 1-r, \dots, \lambda_r - r)$ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$ and $\sum \lambda_i = n$

Sketch of proof **Step 1** $\forall w_i \in \mathbb{Z}$. (otherwise, take first not integer value and permute to the first place. But $w_1 = 0$)

Step 2 Let $\lambda_1 = 1 + \max(w_i)$. Move $\lambda_1 - 1$ to the left while we can $\rightsquigarrow$ get begining $0, 1, \dots, \lambda_1 - 1$

Step 3 The next w_{λ_1+1} could not be larger than $\lambda_1 - 1$

It can't be $0, \dots, \lambda_1 - 1$ due to Corollary ④

It can't be ≤ -2. Hence $w_{\lambda_1+1} = -1$.

Then repeat step 2 ....

$\square$

<u>Theorem</u> Basis $V_\lambda \sim \{v_T \mid T$ - SYT of form $\lambda\}$
with action $J_m v_T = c(S_m) v_T$