Schur-Weyl duality, Representations of GL_N

- Before: reps of S_n

Today $S_n \curvearrowright \underbrace{V\otimes V\otimes\ldots\otimes V}_{n \text{ copies}}$ $\qquad V = K^N$

<u>Example</u> $V\otimes V = S^2V + \Lambda^2 V$

Recall S^2V — basis $e_i\wedge e_j = e_i\otimes e_j - e_j\otimes e_i$ $\quad$ dim $n(n-1)/2$

$\Lambda^2 V$ — basis $e_i\cdot e_j = e_i\otimes e_j + e_j\otimes e_i$ $\quad$ dim $n(n+1)/2$

Reps of $GL(N)$ $\qquad\qquad + \quad n^2.$

- Th (Double Centralizer) Let $A, B \subset End(U)$, for f.d. U

A s/s and $B = End_A(U)$ Then

(a) $A = End_B(U)$

(b) B is s/s

(c) As rep. of $A\otimes B$ $\qquad U = \bigoplus_{i\in I} V_i \otimes W_i$ $\quad$ where V_i - all irreps A

W_i - all irreps B

<u>Rk</u> a) $B = End_A(\mathcal{U}) \longleftrightarrow B = \{ b \in End(\mathcal{U}) \mid ab = ba,\ \forall a \in A \} = Z_A(End\,\mathcal{U})$

-centralizer of A in $End\,\mathcal{U}$

If A is s/s then $Z_{Z_A(End\,\mathcal{U})}(End\,\mathcal{U}) = A$.

b) $\quad$ Irreps $A \xleftrightarrow{1:1}$ Irreps B

<u>Pf</u> A is s/s $\Rightarrow A = \oplus End(V_i)$

$\mathcal{U} = \oplus V_i^{\oplus k_i} = \oplus V_i \otimes W_i$ where $W_i = Hom_A(V_i, A)$

$B = End_A\,\mathcal{U} = \oplus End(W_i)$ by Schur lemma $\square$

• Return to $S_n \curvearrowright V^{\otimes n}$. Let $A = \mathrm{Im}\,\mathbb{C}[S_n]$

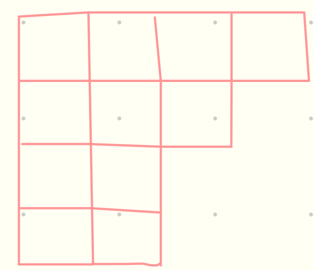
$$V^{\otimes n} = \bigoplus_{|\lambda|=n} V_\lambda \otimes W_\lambda$$

<u>Lemma</u> $W_\lambda \neq 0$ iff $\ell(\lambda) \leq N$

<u>Pf</u> $\dim_{KS_n} \mathrm{Hom}(V_\lambda, V^{\otimes n}) = \dim \mathrm{Hom}_{KS_n}(\mathbb{C}[S_n]a_\lambda b_\lambda, V^{\otimes n}) = \dim \mathrm{Hom}_k(a_\lambda b_\lambda V^{\otimes n})$

Hence $W_\lambda = a_\lambda b_\lambda V^{\otimes n}$. If $\ell(\lambda) > N$

then $b_\lambda V^{\otimes n} = 0$



otherwise fix table

1	2	3	4
5	6	7	
8	9		
10	11		

$$a_\lambda b_\lambda \underbrace{e_1\otimes e_1\otimes\dots\otimes e_1}_{\lambda_1} \underbrace{e_2\otimes e_2\otimes\dots\otimes e_2}_{\lambda_2} = \frac{1}{|Q_\lambda|} \underbrace{e_1\otimes\dots\otimes e_1}_{\lambda_1} \otimes \underbrace{e_2\otimes\dots\otimes e_2}_{\lambda_2} \otimes \dots + \text{other terms} \neq 0$$

$\square$

Let A - image $\mathbb{C}[S_n]$ in $\operatorname{End} V^{\otimes n}$

B - algebra generated by image of GL_N in $\operatorname{End}(V^{\otimes n})$

<u>Th</u> (a) A, B centralizers of each other in $\operatorname{End}(V^{\otimes n})$

(b) A, B are s/s

(c) $V^{\otimes n} = \bigoplus_{\lambda,\ |\lambda|=n,\ \ell(\lambda)\le N} V_\lambda \otimes L_\lambda$

V_λ - irreps for S_n $\qquad$ L_λ - irreps for GL_N

<u>Pf</u> (a) A and B commute

$$A = \operatorname{Im} kS_n = \bigoplus_{|\lambda|=n,\ \ell(\lambda)\le N} \operatorname{End}(V_\lambda) \text{ — s/s}$$

Want $B = \operatorname{End}_A(V^{\otimes n})$

<u>Step 1</u> $\operatorname{End}(V^{\otimes n}) = \operatorname{End}(V)^{\otimes n} \Rightarrow \operatorname{End}_A(V^{\otimes n}) = S^n(\operatorname{End} V)$

Step 2 B is generated by $g\otimes g\cdots\otimes g$ $g\in GL_N$

B contains all elements $b\otimes\cdots\otimes b$ $b\in \operatorname{End} V$
since $\forall b$ can be limit of $g(t)\in GL(V)$
and B is closed.

Step 3 For $\forall W$, S^nW is generated by $w\otimes\cdots\otimes w$

(**Example** S^2W $(v+w)\otimes(v+w)-(v\otimes v)-(w\otimes w)=v\otimes w-w\otimes v)$)

Problem S^nW is irrep of $GL(W)$
Then $\langle v\otimes\cdots\otimes v\rangle\subset S^nW$ is subrep $\square$

Characters

<u>Def</u> A f.d. rep U of $GL(V)$ is <u>algebraic</u> if its matrix elements are polynomial functions on $g_{ij}, \det g^{-1}$

Diagonalizable matrices dense in $GL(V)$ $\Rightarrow$
Character is determined on $\left\{\begin{pmatrix} x_1 & 0 & 0 \\ 0 & \ddots & \\ 0 & & x_N \end{pmatrix}\right\} \subset GL(V)$

<u>Lemma</u> $GL(V)$ - symmetric (Laurent) polynomial on $x_1, \dots, x_n$

<u>Pf</u> $S_N \in GL_N$ $\quad g = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_N \end{pmatrix} \sim wgw^{-1} = \begin{pmatrix} x_{w(1)} & & 0 \\ & \ddots & \\ 0 & & x_{w(N)} \end{pmatrix}$ $\square$

Examples of sym polyn $\quad p_m(x) = \sum_{i=1}^{N} x_i^m$

another $\quad m_\mu(x) = x_1^{\mu_1} \cdots x_N^{\mu_N} +$ (sym permutations)

Def Schur polynomials

$$S_\lambda(x_1,\dots,x_n)=\frac{\det(x_i^{\lambda_j+N-j})}{\det(x_i^{N-j})}=\frac{\sum_{w\in S_N}(-1)^w\prod x_{w(j)}^{\lambda_j+N-j}}{\prod_{i<j}(x_i-x_j)}$$

Properties

(a) S_λ – polynomial of hom. degree $|\lambda|=\lambda_1+\dots+\lambda_r$

(b) $N=2$ $\quad S_\lambda=\dfrac{x_1^{\lambda_1+1}x_2^{\lambda_2}-x_1^{\lambda_2}x_2^{\lambda_1+1}}{x_1-x_2}=x_1^{\lambda_1}x_2^{\lambda_2}+x_1^{\lambda_1-1}x_2^{\lambda_2+1}+\dots+x_1^{\lambda_2}x_2^{\lambda_1}$

(c) $S_\lambda=x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_N^{\lambda_N}+$ low terms $\Rightarrow S_\lambda$ linear indep

$S_\lambda\cdot(x_1^{N-1}x_2^{N-2}\cdots x_{N-1}+\text{low terms})=x_1^{\lambda_1+N-1}x_2^{\lambda_2+N-2}\cdots x_N^{\lambda_N}+\dots$

(lexicographic order $x_1^{a_1}\dots x_N^{a_N}>x_1^{b_1}\dots x_N^{b_N}$ if $\sum a_i=\sum b_i$
$\exists r$ s.t. $a_1=b_1, a_2=b_2,\dots,a_{r-1}=b_{r-1}, a_r>b_r$)

(d) $\forall\mu$ $\quad\prod p_{\mu_i}=\sum_{\lambda,\ \ell(\lambda)\le N}\chi_\lambda(C_\mu)\,S_\lambda(x)$

Frobenius character formula

- Apply to $V^{\otimes n}$

<u>Prop</u> Let $w \in C_\mu$, $g = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_N \end{pmatrix}$. Then $\mathrm{Tr}(w \otimes g)\Big|_{V^{\otimes n}} = \prod p_{\mu_i}$

<u>Pf</u> For $w = (1, \ldots, m_1)(m_1+1, \ldots, m_1+m_2)(m_1+m_2+1, \ldots, m_1+m_2+m_3)(\ldots)$

w preserves basis elements of the form

$e_{\bar{J}}^{\bar{m}} = e_{j_1}^{\otimes m_1} \otimes e_{j_2}^{\otimes m_2} \otimes e_{j_3}^{\otimes m_3} \otimes \ldots$

$g\, e_{\bar{J}} = \prod x_{j_e}^{m_e}\, e_{\bar{J}}^{\bar{m}}$ hence $\mathrm{Tr}(w \otimes g) = \sum_{\bar{J}} \prod x_{j_e}^{m_e} = \prod_e \sum x_{j_e}^{m_e} = \prod p_\mu$ $\square$

On the other hand $\mathrm{Tr}(w \otimes g)\Big|_{V^{\otimes n}} = \sum \chi_\lambda(C_i)\, \chi_{L_\lambda}(g)$

<u>Thm</u> (Weyl character formula) $\chi_{L_\lambda}(g) = S_\lambda(g)$

<u>Corollary</u> $\dim L_\lambda = \prod_{i<j} \dfrac{(\lambda_i - i - \lambda_j + j)}{j-i} = \prod \dfrac{\ell_i - \ell_j}{j-i}$

$\ell_i = \lambda_i + N - i$

Pf $s_\lambda(1)$ — not determined $\sim 0/0$

$$s_\lambda(1,q,\dots,q^{N-1}) = \frac{\det\left(q^{i(\lambda_j+N-j)}\right)}{\det\left(q^{i(N-j)}\right)} = \frac{\prod\left(q^{\lambda_i+N-i}-q^{\lambda_j+N-j}\right)}{\prod_{i<j}\left(q^{N-i}-q^{N-j}\right)}$$

Taking the limit $q \to 1$ $\displaystyle\prod_{i<j} \frac{(\lambda_i - i - \lambda_j + j)}{(j-i)}$ $\square$