Schur functions and representation theory

- Recall λ, $\ell(\lambda) \le N \rightsquigarrow L_\lambda$ rep GL_N,

$$(k^N)^{\otimes n} = \sum_{\lambda,\ |\lambda|=n,\ \ell(\lambda)\le N} V_\lambda \otimes L_\lambda$$ Schur-Weyl duality

$$\chi_{L_\lambda}\left(\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_N \end{pmatrix}\right) = S_\lambda(x_1 \ldots x_N)$$ Weyl formula

$$S_\lambda(x_1, \ldots, x_N) = \frac{\det(x_i^{\lambda_i + N - j})}{\det(x_i^{N-j})}$$ – Schur polynomial

$$\dim L_\lambda = \frac{\prod_{i<j}(\ell_i - \ell_j)}{\prod_{i<j}(j-i)}$$ $\ell_i = \lambda_i + N - i$ – shifted parts

<u>Example</u> (a) $\lambda = \square$ $\ell_1 = N$, $\ell_2 = N-2, \ldots, \ell_N = 0$

$$\dim L_\square = \frac{N \cdot N-1 \cdots 2}{N-1 \cdots 1} = N$$ $V^{\otimes 1} = V$

(b) $\lambda = \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\Big\}k$ $\quad \ell_1 = N, \ \ell_2 = N-1, \dots, \ell_k = N-k+1, \ \ell_{k+1} = N-k-1, \dots, \ell_N = 0$

$\qquad\qquad N-1 \qquad N-2 \qquad\qquad N-k \qquad\qquad N-k-1 \qquad\qquad 0$

$$\dim L_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = \frac{N!/k}{(k-1)!\,(N-k)!} = \frac{N!}{k!\,(N-k)!} = \binom{N}{k}$$ – agrees with

$L_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$ corresp. to $V_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = \mathrm{sgn}$ rep S_n in $V^{\otimes n} \Rightarrow L_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = \Lambda^k V$

(c) $\lambda = \underbrace{\square\square\square}_{k}$ $\quad \ell_1 = N+k-1, \ \ell_2 = N-2, \ \dots, \ \ell_N = 0$

$\qquad\qquad N-1 \qquad\qquad N-2 \qquad\qquad\qquad 0$

$$\dim L_{\square\square\square} = \frac{(N+k-1)^{\downarrow N-1}}{(N-1)!} = \frac{(N+k-1)!}{(N-1)!\,k!} = \binom{N+k-1}{k}$$ – agrees with

$L_{\square\square\square}$ corresp. to $V_{\square\square\square} = \mathrm{triv}$ rep S_n in $V^{\otimes n} \Rightarrow L_{\square\square\square} = S^k V$

(d) $\lambda = \underbrace{\begin{smallmatrix}\square\square\square\square\square\\\square\square\square\square\square\end{smallmatrix}}_{k}\}2$ $\quad \dim L_\lambda = C_k$ – Catalan number

<u>Prop</u> $L_{\lambda + 1^N}$ is isomorphic to $L_\lambda \otimes \det V = L_\lambda \otimes \Lambda^N V$

<u>Pf</u> $L_\lambda \otimes \Lambda^N V \subset V^{\otimes |\lambda| + N}$, and $S_{\lambda + 1^N} = S_\lambda \cdot \prod x_i$ $\square$

<u>Def</u> $L_{\lambda - r\cdot 1^N} = L_\lambda \otimes (\det V^*)^r$ – algebraic rep of $GL(V)$

Now L_λ defined for $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \in \mathbb{Z}$

<u>Th</u> (a) Any f.d. algebraic rep of $GL(V)$ is completely reducible and decomp into sum of L_λ (which are pairwise non-isom)

(b) (Peter-Weyl theorem for $GL(V)$) Let R – algebra of polyn. funct. on $GL(V)$. This is $GL(V)\times GL(V)$ rep

$(\rho(g,h)\phi(x) = \phi(g^{-1}xh),\ \ g,h,x \in GL(V),\ \phi \in R)$

<u>Then</u> $R = \bigoplus_\lambda L_\lambda^* \otimes L_\lambda$

c.f. $kG = \bigoplus_{V \in Irr(G)} End(V)$

<u>Idea of pf</u> (a) Any $U \subset R^{\oplus \dim U}$. Any f.d subspace in R can be embedded into $V^{\oplus m} \otimes (\det V)^r$ for some m, r

We know that $V^{\otimes m} \otimes (\det V^*)^r$ are S/S

(b) For any irrep U, $Hom_{GL(V)}(U, R) \simeq \{ f: GL(V) \to U^* \text{ s.t } f(xg) = g^{-1} f(x)\} = U^*$

Remark For SL_N $\det V$ - trivial rep, hence $L_{\lambda+1^N} \simeq L_\lambda$

The same thm holds, irreps are labelled by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N-1} \geq 0$

(One can extend rep of SL_N to rep of GL_N
since center of SL_N acts as $\begin{pmatrix} \omega & & \\ & \ddots & \\ & & \omega \end{pmatrix} \mapsto \omega^i \rightsquigarrow \begin{pmatrix} x & \\ & x \end{pmatrix} \mapsto x^i$)

- Want basis in L_λ. For this consider sequence

$GL_1 \subset GL_2 \subset \ldots \subset GL_{N-1} \subset GL_N \subset \ldots$

embedding $\left(\begin{array}{ccc|c} & GL_{N-1} & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 & \ldots & 0 & 1 \end{array}\right) \subset GL_N$

Def $\lambda = (\lambda_1, \ldots, \lambda_N)$, $\mu = (\mu_1, \ldots, \mu_{N-1})$ interlace if

$$\begin{array}{ccccccc} \lambda_1 & & \lambda_2 & & \lambda_3 & \ldots & \lambda_N \\ & \geq \mu_1 \geq & & \geq \mu_2 \geq & & & \mu_{N-1} \geq \end{array}$$

Notation $\lambda \succ \mu$

Th $L_\lambda \big|_{GL(N-1)} = \bigoplus_{\lambda \succ \mu} L_\mu$

Pf Want $S_\lambda(x_1, \ldots, x_{N-1}, 1) = \sum_{\lambda \succ \mu} S_\mu(x_1, \ldots, x_{N-1})$

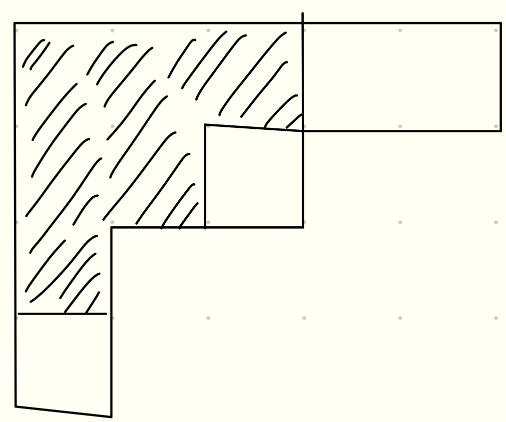
By example

$$s_\lambda(x_1,x_2,1) = \frac{1}{(x_1-x_2)(x_1-1)(x_2-1)} \begin{vmatrix} x_1^{\lambda_1+2} & x_2^{\lambda_1+2} & 1 \\ x_1^{\lambda_2+1} & x_2^{\lambda_2+1} & 1 \\ x_1^{\lambda_3} & x_2^{\lambda_3} & 1 \end{vmatrix} =$$

$$\frac{1}{(x_1-x_2)(x_1-1)(x_2-1)} \begin{vmatrix} x_1^{\lambda_1+2}-x_1^{\lambda_2+1} & x_2^{\lambda_1+2}-x_2^{\lambda_2+1} & 0 \\ x_1^{\lambda_2+1}-x_1^{\lambda_3} & x_2^{\lambda_2+1}-x_2^{\lambda_3} & 0 \\ x_1^{\lambda_3} & x_2^{\lambda_3} & 1 \end{vmatrix} = \sum_{\mu_1,\mu_2} \frac{1}{x_1-x_2} \begin{vmatrix} x_1^{\mu_1+1} & x_2^{\mu_1+1} \\ x_1^{\mu_2} & x_2^{\mu_2} \end{vmatrix}$$

Since $\displaystyle \frac{x_i^{\lambda_1+2}-x_i^{\lambda_2+1}}{x_i-1} = \sum_{\lambda_2\le\mu_1\le\lambda_1} x_i^{\mu_1+1}$ $\qquad$ $\displaystyle \frac{x_i^{\lambda_2+1}-x_i^{\lambda_3}}{x_i-1} = \sum_{\lambda_3\le\mu_2\le\lambda_2} x_i^{\mu_2}$ $\square$

- In terms of partitions $\mu\subset\lambda$ s.t $\lambda\setminus\mu$ – no elements with in one column



- Sequence $GL(1) \subset GL(2) \subset \dots \subset GL(N)$ $\qquad \lambda^{(N)} = \lambda$

$$L_\lambda = \bigoplus_{\lambda^{(N)} \succ \lambda^{(N-1)}} L_{\lambda^{(N-1)}} = \bigoplus_{\lambda^{(N)} \succ \lambda^{(N-1)} \succ \lambda^{(N-2)}} L_{\lambda^{(N-2)}} = \dots = \bigoplus_{\lambda^{(N)} \succ \lambda^{(N-1)} \succ \dots \succ \lambda^{(1)}} L_{\lambda^{(1)}} \quad \text{— 1 dim space}$$

hence we get (projective) basis labelled by

Gelfand-Tsetlin pattern

$$\begin{array}{ccccccccc}
\lambda_1^{(N)} & & \lambda_2^{(N)} & & \lambda_3^{(N-1)} & & & & \lambda_N^{(N)} \\
 & \geq & & \geq \quad \geq & & & & & \\
 & & \lambda_1^{(N-1)} & & \lambda_2^{(N-1)} \geq & & & & \\
 & & & \geq \quad \geq & & & & & \\
 & & & & \lambda_1^{(N-2)} \geq & & & & \\
 & & & & & \ddots & & & \\
 & & & & & & \lambda_1^{(1)} & &
\end{array}$$

Semistandard Young tableu

weekly increase on row

strongly increase column

$$\begin{array}{ccccc}
1 & 1 & 1 & 2 & 4 \\
2 & 3 & 3 & 4 & 6 \\
4 & 4 & 5 & & \\
5 & 5 & & & \\
6 & 6 & & &
\end{array}$$

$\left(\text{Gelfand-Tsetlin patterns} \longleftrightarrow \text{paths in branching graph} \qquad \begin{array}{c} \lambda^{(1)} \\ | \\ \lambda^{(2)} \\ | \\ \vdots \\ | \\ \lambda^{(N)} \end{array} \right)$

- Character $L_\lambda = \bigoplus_{\lambda > \mu} L_\mu$

$$\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N \end{pmatrix} \qquad \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_{N-1} \end{pmatrix}$$

$\deg = |\lambda|$ $\qquad$ $\deg x_\mu$

Hence $\begin{pmatrix} 1 & & \\ & \ddots & \\ & & x_N \end{pmatrix}\Bigg|_{L_\mu} = x_N^{|\lambda| - |\mu|}$

<u>Th</u> $\displaystyle s_\lambda(x_1, .., x_N) = \sum_{T\text{ - S/S form } \lambda} x_1^{\#1 \text{ in } T} x_2^{\#2 \text{ in } T} \cdots x_N^{\#N \text{ in } T}$

<u>Remark</u> Symmetricity is nontrivial.

Example ① $S_{(k)} = S_{\begin{array}{|c|c|c|c|}\hline \, & \, & \, & \, \\ \hline\end{array}} = \sum\limits_{1\le i_1\le\dots\le i_k\le N} x_{i_1}\cdots x_{i_k} = h_k$ $\quad \begin{array}{|cccc|}\hline i_1 & i_2 & \quad & i_k \\ \hline\end{array}$

<u>complete</u> homogenious symmetric polynomial

$L_{(k)} \simeq S^k\mathbb{C}^N$ basis $\{e_{i_1}\cdots e_{i_k} \mid \mathrm{Sym}(e_{i_1}\otimes\dots\otimes e_{i_k})\}$

② $S_{(1^k)} = S_{\begin{array}{|c|}\hline \, \\ \hline \, \\ \hline \, \\ \hline\end{array}} = \sum\limits_{1\le i_1<\dots<i_k\le N} x_{i_1}\cdots x_{i_k} = e_k$ $\quad \begin{array}{|c|}\hline i_1 \\ i_2 \\ \vdots \\ i_k \\ \hline\end{array}$

elementary symmetric polynomial

$L_{(1)^k} = \Lambda^k\mathbb{C}^n$

Tensor products

$$L_\lambda \otimes L_\mu = \bigoplus c_{\lambda\mu}^{\nu} L_\nu$$

Clebsh-Gordan Littlewood Richardson coefficients

In terms of characters

$$s_\lambda \cdot s_\mu = \sum c_{\lambda\mu}^{\nu} s_\nu \qquad \text{(note } |\nu| = |\lambda| + |\mu|\text{)}$$

For simplicity we cover only particular cases

<u>Th</u> (Pieri rule) $\displaystyle s_\lambda e_k = \sum_{\mu,\ \mu\setminus\lambda \text{ no boxes in same row}} s_\mu$

<u>Ex</u>