Schur functions & reps of S_n

- λ, L_λ – irred. rep GL_N $\qquad s_\lambda(x_1 \dots x_N) = \dfrac{\det\left(x_i^{\lambda_j+N-j}\right)}{\prod_{i<j}(x_i - x_j)}$

<u>Ex</u> $\quad s_{k\left\{\begin{smallmatrix}\square\\ \vdots\\ \square\end{smallmatrix}\right.} = e_k = \sum\limits_{i_1<\dots<i_k} x_{i_1}\cdots x_{i_k} \qquad s_{\underbrace{\square\square\square\square}_{k}} = h_k = \sum\limits_{i_1\le\dots\le i_k} x_{i_1}\cdots x_{i_k}$

- <u>Th</u> (Pieri rule) $\quad s_\lambda e_k = \sum\limits_{\substack{\mu,\ \mu\setminus\lambda\ k \text{ boxes} \\ \text{no boxes in same row}}} s_\mu$

<u>Pf</u> $\quad s_\lambda e_k = \dfrac{1}{\prod(x_i - x_j)}\left(x_1^{l_1} x_2^{l_2} \cdots x_N^{l_N} + \text{asym}\right)\left(\sum\limits_{j_1<\dots<j_k} x_{j_1}\cdots x_{j_k}\right) =$

$$= \frac{1}{\prod(x_i - x_j)}\left(x^{l+\sum 1_{j_i}} + \text{asym}\right) = \sum S_{\lambda + \sum_{i=1}^{k} 1_{j_i}} = \sum_{\substack{\mu,\ \mu\setminus\lambda\ \text{no box in} \\ \text{one row}}} s_\mu$$

note $\quad s_\lambda = 0 \quad$ if $\quad \lambda_{j+1} = \lambda_j + 1$

$$\begin{vmatrix} \cdots & & \cdots \\ x_1^{\lambda_j+N-j} & \cdots & x_N^{\lambda_j+N-j} \\ x_1^{\lambda_{j+1}+N-j-1} & & x_N^{\lambda_{j+1}+N-j-1} \end{vmatrix}$$

- **Rem** $e_1, \dots, e_N$ generate $\mathbb{C}[x_1, \dots, x_N]^{S_N} = \mathbb{C}[e_1, \dots, e_N]$

Hence mult on $e_1 \dots e_N$ gives (in principle) full information

- **Th** (Pieri rule) $s_\lambda \cdot h_k = \sum\limits_{\substack{\mu,\ \mu \setminus \lambda - k \text{ boxes} \\ \text{no boxes in the same row}}} s_\mu$

Plan of Pf $1 + \sum\limits_{k=1}^{\infty} h_k = \prod\limits_{i=1}^{N} (1 + x_i + x_i^2 + \dots) = \prod\limits_{i=1}^{N} \frac{1}{1-x_i}$

$$\prod \frac{1}{1-x_i} \begin{vmatrix} x_1^{l_1} & & x_1^{l_N} \\ \vdots & & \\ x_N^{l_1} & & x_N^{l_N} \end{vmatrix} = \left(\text{Use } \frac{x^l}{1-x} = \sum_{m \geq l} x^m \right) = \begin{vmatrix} \sum\limits_{m_1 \geq l_1} x_1^{m_1} & \dots & \sum\limits_{m_N \geq l_N} x_1^{m_N} \\ \vdots & & \vdots \\ \sum\limits_{m_1 \geq l_1} x_N^{m_1} & \dots & \sum\limits_{m_N \geq l_N} x_N^{m_N} \end{vmatrix}$$

$$= \sum_{m_1 \geq l_1,\ m_2 \geq l_2, \dots, m_N \geq l_N} \left(x_1^{m_1} \cdots x_N^{m_N} + \text{asym} \right) = \sum_{m_1 \geq l_1 > m_2 \geq l_2 > \dots > m_N \geq l_N} \left(x_1^{m_1} \cdots x_N^{m_N} + \text{asym} \right) \qquad \square$$

(If $m_{r+1} \geq l_r$ then we have pairs $x_1^{m_1} \dots x_r^{m_r} x_{r+1}^{m_{r+1}} \dots x_N^{m_N} + \text{asym} + x_1^{m_1} \dots x_r^{m_{r+1}} x_{r+1}^{m_r} \dots x_N^{m_N} + \text{asym} = 0$)

- Return to reps of S_n

λ, $|\lambda| = n$ $\rightsquigarrow V_\lambda$ Specht module S_n

$p_\mu = \prod p_{\mu_i} = \prod p_m^{i_m}$ $\quad p_\mu = \sum \chi_{V_\lambda}(c_\mu) s_\lambda$ Frobenius character formula

Issue p_μ are not linearly independent $\mathbb{C}[x_1,..,x_N]^{S_N} = \mathbb{C}[p_1,..,p_N]$

p_{N+1}, p_{N+2} - are expressed through $p_1,..,p_N$.

Cure 1 Take $N > n$, on degree n $\mathbb{C}[x_1,..,x_N]^{S_N}_n$ $\{p_\mu \,|\, |\mu| = n\}$ - basis

Cure 2 Take inverse limit $N \to \infty$

$$\mathbb{C}[x_1] \leftarrow \mathbb{C}[x_1,x_2]^{S_2} \leftarrow \quad .. \quad \leftarrow \mathbb{C}[x_1,..,x_N]^{S_N} \leftarrow$$

Here maps $\mathbb{C}[x_1,..,x_N]^{S_N} \to \mathbb{C}[x_1,..,x_{N-1}]^{S_{N-1}}$

$$f \mapsto f|_{x_N=0}$$

$$\Lambda := \underleftarrow{\lim}\, \mathbb{C}[x_1 \ldots x_N]^{S_N} = \{ f_M \in \mathbb{C}[x_1, \ldots, x_M]^{S_M} \mid f_M \big|_{x_M = 0} = f_{M-1} \}$$

For given degree n $\quad \left(\mathbb{C}[x_1, \ldots x_N]^{S_N}\right)_n = \langle m_\lambda \mid \lambda \vdash n,\ \ell(\lambda) \leq N \rangle$

For $N \geq n$ $\quad \dim\left(\mathbb{C}[x_1 \ldots x_N]^{S_N}\right)_n = p(n) = \#\{\lambda \mid |\lambda| = n\}$
does not depend on N $\Rightarrow$ restriction map is isomorphism

<u>Examples</u> $\quad e_k \big|_{x_N = 0} = \begin{cases} e_k & k \leq N \\ 0 & \text{otherwise} \end{cases} \Longrightarrow e_k \in \Lambda$

$p_k \big|_{x_N = 0} = p_k \qquad h_k \big|_{x_N = 0} = h_k \qquad \Longrightarrow p_k, h_k \in \Lambda$

As an algebra

$$\Lambda = \mathbb{C}[e_1, e_2, \ldots,\ ] = \mathbb{C}[p_1, p_2, \ldots,\ ] = \mathbb{C}[h_1, h_2, \ldots\ ]$$

For Schur functions $s_\lambda\big|_{x_N=0} = \begin{cases} s_\lambda & \ell(\lambda)<N \\ 0 & \text{otherwise} \end{cases}$

Follows from $s_\lambda = \sum_{T\in SSYT(\lambda)} x^{wt(T)}$ another pf:

$$\frac{1}{\prod_{i<j} x_i-x_j}\begin{vmatrix} x_1^{\ell_1} & x_1^{\ell_2} & x_1^{\ell_3} \\ x_2^{\ell_1} & x_2^{\ell_2} & x_2^{\ell_3} \\ x_3^{\ell_1} & x_3^{\ell_2} & x_3^{\ell_3}\end{vmatrix}\Bigg|_{x_3=0} = \begin{cases} \frac{1}{(x_1-x_2)x_1x_2}\begin{vmatrix} x_1^{\ell_1} & x_1^{\ell_2} \\ x_2^{\ell_1} & x_2^{\ell_3}\end{vmatrix} \\ 0 \text{ if } \ell_3>0\end{cases} = \begin{cases} \frac{1}{x_1-x_2}\begin{vmatrix} x_1^{\lambda_1+1} & x_1^{\lambda_2} \\ x_2^{\lambda_1+1} & x_2^{\lambda_2}\end{vmatrix} \\ 0 \text{ if } \ell_3>0\end{cases}$$

Hence s_λ-basis in Λ

- Scalar product χ_{V_λ} orthonormal basis

Def Introduce scalar product of $\mathbb{C}[x_1,\dots,x_N]^{S_N}$ s.t. s_λ -orthonormal basis

$$\langle \chi_{V_\lambda}, \sum_{g\in C_\mu} g\rangle = \frac{1}{|G|}\sum_{g\in C_\mu}\chi_{V_\lambda}(g) = \frac{|C_\mu|}{|G|}\chi_{V_\lambda}(C_\mu) \qquad \langle s_\lambda, p_\mu\rangle = \chi_{V_\lambda}(C_\mu)$$

Hence $\chi_{V_\lambda} \longleftrightarrow S_\lambda \qquad \frac{|G|}{|C_\mu|}\sum_{g\in C_\mu} g \longleftrightarrow p_\lambda$

<u>Corollary</u> $\langle p_\mu, p_\lambda\rangle = \delta_{\lambda,\mu}\frac{|G|}{|C_\mu|} = z_\lambda$ - cize of centalizer

$\prod m^{i_m} i_m!$

take two sets of vars $x_1,\dots,x_n, y_1,\dots,y_n$

$$\sum S_\lambda(x)\,S_\lambda(y) = \sum \frac{1}{z_\lambda} p_\lambda(x)\,p_\lambda(y)$$

pairing of dual bases $\qquad \sum e_i\otimes e^i = \sum f_i\otimes f^i \in V\otimes V^*$

$$= \sum_i \prod \frac{\left(\sum x_j^m \sum y_k^m\right)^{i_m}}{m^{i_m}\, i_m!} = \prod_m \sum_{i_m} \frac{\left(\sum \frac{x_j^m y_k^m}{m}\right)^{i_m}}{i_m!} = \prod_m \exp\left(\sum_{j,k} \frac{x_j^m y_k^m}{m}\right)$$

$$= \prod_{j,k} \exp\left(\sum_m \frac{(x_j y_k)^m}{m}\right) = \prod_{j,k} \exp(-\log(1-x_j y_k)) = \prod \frac{1}{1-x_j y_k}$$

<u>Th</u> (Cauchy identity) $\sum s_\lambda(x)s_\lambda(y) = \prod \frac{1}{1-x_j y_k}$

Representation theory interpretation

$S_n \times S_n$ $\quad \bigoplus V_\lambda \otimes V_\lambda = \bigoplus V_\lambda \otimes V_\lambda^* = kS_n$ — <u>Regular rep</u>

since $w \sim w^{-1} \Rightarrow \chi_{V_\lambda} = \overline{\chi_{V_\lambda}} \Rightarrow V_\lambda \simeq V_\lambda^*$

$GL_N \times S_n$ $\quad \bigoplus V_\lambda \otimes L_\lambda = (\mathbb{C}^N)^{\otimes n}$ — Schur-Weyl duality

$GL_N \times GL_M$ $\quad \bigoplus V_\lambda \otimes V_\lambda = \mathbb{C}[Mat_{M\times N}]$ ~ c.f. Peter-Weyl thm

Howe duality

$\mathbb{C}[\{(t_{ij})\}] = \mathbb{C}[t_{11},\dots,t_{NM}]$

$$\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N \end{pmatrix}^{-1} \begin{pmatrix} t_{ij} \end{pmatrix} \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_m \end{pmatrix} = \left(x_i^{-1} t_{ij} y_j\right) \Rightarrow \chi_{\mathbb{C}[Mat]} = \prod \frac{1}{1-x_i y_j^{-1}} = \sum S_\lambda(x) S_\lambda(y^{-1})$$

$$\bigoplus V_\lambda \otimes V_\lambda^*$$

- $\langle s_\lambda, p_\mu \rangle = \chi_{V_\lambda}(C_\mu) \Rightarrow s_\lambda = \sum \frac{1}{z_\lambda} \chi_{V_\lambda}(C_\mu) p_\mu$

$s_\lambda \cdot s_\mu = \sum c_{\lambda\mu}^{\nu} s_\nu$ If $|\lambda| = n$, $|\mu| = m \Rightarrow |\nu| = n+m$
(S_n, S_m, S_{n+m})

$$\left(\sum_{\bar{i}'} \frac{1}{|z_{\bar{i}'}|} \chi_\lambda(C_{\bar{i}'}) p_{\bar{i}'}\right)\left(\sum_{\bar{i}''} \frac{1}{|z_{\bar{i}''}|} \chi_\mu(C_{\bar{i}''}) p_{\bar{i}''}\right) = \sum_{\bar{i}} \frac{1}{|z_{\bar{i}}|} p_{\bar{i}} \sum_{\bar{i}'+\bar{i}''=\bar{i}} \frac{|z_{\bar{i}}|}{|z_{\bar{i}'}|\,|z_{\bar{i}''}|} \chi_\lambda(C_{\bar{i}'}) \chi_\mu(C_{\bar{i}''})$$

Frobenius formula $= \sum_{\bar{i}} \frac{1}{|z_{\bar{i}}|} p_{\bar{i}} \chi_{\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} V_\lambda \boxtimes V_\mu}(C_{\bar{i}})$

Hence $\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} V_\lambda \boxtimes V_\mu = \sum c_{\lambda\mu}^{\nu} V_\nu$

Digression G, G' groups. $\rho: G \to GL(V)$, $\rho': G' \to GL(V')$
Then $V \otimes V'$ is rep. of $G \times G'$.
$(g, g') \mapsto \rho(g) \otimes \rho(g')$ Notation $V \boxtimes V'$

Th a) If V, V' are irrep. $\Rightarrow V \boxtimes V'$ is irrep
b) All irreps of $G \times G'$ have this form.

Example

$$L_\square \otimes L_\lambda = \bigoplus_{\nu,\, \nu=\lambda+\square} L_\nu$$

$$s_1 s_\lambda = e_1 \cdot s_\lambda = h_1 \cdot s_\lambda = \sum_{\nu,\, \nu=\lambda+\square} s_\nu \iff \mathrm{Ind}_{S_n}^{S_{n+1}} V_\lambda = \bigoplus_{\nu,\, \nu=\lambda+\square} V_\nu$$

- Involution $p_k^* = (-1)^{k-1} p_k \quad p_\mu^* = (-1)^{c_\mu} p_\mu$

In terms of S_n – tensor product with sgn

$s_\lambda^* = s_{\lambda'}$. Claim $h_n^* = e_n$. Indeed

$$1+\sum_{n>0} h_n z^n = \prod \frac{1}{1-x_i z} = \exp\Big(\sum_{i,k} \frac{x_i^k}{k} z^k\Big) = \exp\Big(\sum \frac{p_k}{k} z^k\Big)$$

$$1+\sum_{n>0} e_n z^n = \prod (1+x_i z) = \exp\Big(\sum_{i,k} (-1)^{k-1}\frac{x_i^k}{k} z^k\Big) = \exp\Big(\sum (-1)^k \frac{p_k}{k} z^k\Big)$$

Involution permutes e_n & h_n

(In terms of GL_N: $V \mapsto V[1]$, change of parity

If V super vector space $V = V_{even} \oplus V_{odd}$

Then $V[1]_{even} = V_{odd} \quad V[1]_{odd} = V_{even}$)

- Main lesson

Symmetic polynomials $\longleftrightarrow$ Reps of $GL(N)$

Symmetic polynomials $\longleftrightarrow$ Reps of S_n