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Contents

Chapter 1

Linear differential operators

1.1 Definitions and main examples

Let $\Omega \subset \mathbb{R}^d$ be an open subset. Denote $\mathcal{C}^\infty(\Omega)$ the set of all infinitely differentiable complex valued smooth functions on Ω. The Euclidean coordinates on $\mathbb{R}^d$ will be denoted $x_1, \dots, x_d$. We will use short notations for the derivatives

$$\partial_k = \frac{\partial}{\partial x_k}$$

and we also introduce operators

$$D_k = -i\, \partial_k, \quad k = 1, \dots, d. \tag{1.1.1}$$

For a multiindex

$$\mathbf{p} = (p_1, \dots, p_d)$$

denote

$$\begin{aligned}
&|\mathbf{p}| = p_1 + \cdots + p_d \\
&\mathbf{p}! = p_1! \dots p_d! \\
&\mathbf{x}^\mathbf{p} = x_1^{p_1} \dots x_d^{p_d} \\
&\partial^\mathbf{p} = \partial_1^{p_1} \dots \partial_d^{p_d}, \quad D^\mathbf{p} = D_1^{p_1} \dots D_d^{p_d}.
\end{aligned}$$

The derivatives, as well as the higher order operators $D^\mathbf{p}$ define linear operators

$$D^\mathbf{p} : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega), \quad f \mapsto D^\mathbf{p} f = (-i)^{|\mathbf{p}|} \frac{\partial^{|\mathbf{p}|} f}{\partial x_1^{p_1} \dots \partial x_d^{p_d}}.$$

More generally, we will consider *linear differential operators* of the form

$$\begin{aligned}
&A = \sum_{|\mathbf{p}| \le m} a_\mathbf{p}(x) D^\mathbf{p} \\
&a_\mathbf{p}(x) \in \mathcal{C}^\infty(\Omega) \\
&A : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega).
\end{aligned} \tag{1.1.2}$$

We will define the *order* of the linear differential operator by

$$\operatorname{ord} A = \max |\mathbf{p}| \quad \text{such that} \quad a_\mathbf{p}(x) \neq 0. \tag{1.1.3}$$

Main examples are

1. Laplace operator

$$\Delta = \partial_1^2 + \cdots + \partial_d^2 = -(D_1^2 + \dots D_d^2) \tag{1.1.4}$$

2. Heat operator

$$\frac{\partial}{\partial t} - \Delta \tag{1.1.5}$$

acting on functions on the $(d+1)$-dimensional space with the coordinates $(t, x_1, \dots, x_d)$.

3. Wave operator

$$\frac{\partial^2}{\partial t^2} - \Delta. \tag{1.1.6}$$

4. Schrödinger operator

$$i\,\frac{\partial}{\partial t} + \Delta. \tag{1.1.7}$$

1.2 Principal symbol of a linear differential operator

Symbol of a linear differential operator (1.1.2) is a function

$$a(x,\xi) = \sum_{|\mathbf{p}|\le m} a_{\mathbf{p}}(x)\xi^{\mathbf{p}}, \quad x \in \Omega \subset \mathbb{R}^d, \quad \xi \in \mathbb{R}^d. \tag{1.2.1}$$

If the order of the operator is equal to m then the *principal symbol* is defined by

$$a_m(x,\xi) = \sum_{|\mathbf{p}|=m} a_{\mathbf{p}}(x)\xi^{\mathbf{p}}. \tag{1.2.2}$$

The symbols (1.2.1), (1.2.2) are polynomials in d variables $\xi_1, \dots, \xi_d$ with coefficients being smooth functions on Ω.

For the above examples we have the following symbols

1. For the Laplace operator Δ the symbol and principal symbol coincide

$$a = a_2 = -(\xi_1^2 + \cdots + \xi_d^2) \equiv -\xi^2.$$

2. For the heat equation the full symbol is

$$a = i\,\tau + \xi^2$$

while the principal symbol is ξ^2.

3. For the wave operator again the symbol and principal symbols coincide

$$a = a_2 = -\tau^2 + \xi^2.$$

4. The symbol of the Schrödinger operator is

$$-(\tau + \xi^2)$$

while the principal symbol is ξ^2.

Exercise 1.1 *Prove the following formula for the symbol of a linear differential operator*

$$a(x, i\xi) = e^{-i\,x\cdot\xi} A\left(e^{i\,x\cdot\xi}\right). \tag{1.2.3}$$

Here we use the notation

$$x \cdot \xi = x_1\xi_1 + \cdots + x_d \cdot \xi_d$$

for the natural pairing $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$.

Exercise 1.2 *Given a linear differential operator A with* constant *coefficients denote $a(\xi)$ its symbol (it does not depend on x for linear differential operators with constant coefficients). Prove that the exponential function*

$$u(x) = e^{i\,x\cdot\xi}$$

is a solution to the linear differential equation

$$A\,u = 0$$

iff *the vector ξ satisfies*

$$a(\xi) = 0.$$

Exercise 1.3 *Prove that for a pair of smooth functions $u(x)$, $S(x)$ and a linear differential operator A of order m the expression of the form*

$$e^{-i\,\lambda\,S(x)} A\left(u(x)e^{i\,\lambda\,S(x)}\right)$$

is a polynomial in λ of degree m. Derive the following expression for the leading coefficient of this polynomial

$$e^{-i\,\lambda\,S(x)} A\left(u(x)e^{i\,\lambda\,S(x)}\right) = i^m u(x) a_m(x, S_x(x))\lambda^m + O(\lambda^{m-1}). \tag{1.2.4}$$

Here

$$S_x = \left(\frac{\partial S}{\partial x_1}, \ldots, \frac{\partial S}{\partial x_d}\right)$$

is the gradient of the function $S(x)$.

Exercise 1.4 *Let A and B be two linear differential operators of orders k and l with the principal symbols $a_k(x, \xi)$ and $b_l(x, \xi)$ respectively. Prove that the superposition $C = A \circ B$ is a linear differential operator of order $\leq k + l$. Prove that the principal symbol of C is equal to*

$$c_{k+l}(x, \xi) = a_k(x, \xi)\, b_l(x, \xi) \tag{1.2.5}$$

in the case $\operatorname{ord} C = \operatorname{ord} A + \operatorname{ord} B$. *In the case of strict inequality* $\operatorname{ord} C < \operatorname{ord} A + \operatorname{ord} B$ *prove that the product (1.2.5) of principal symbols is identically equal to zero.*

The formula for computing the full symbol of the product of two linear differential operators is more complicated. We will give here the formula for the particular case of one spatial variable x.

Exercise 1.5 *Let $a(x, \xi)$ and $b(x, \xi)$ be the symbols of two linear differential operators A and B with one spatial variable. Prove that the symbol of the superposition $A \circ B$ is equal to*

$$a \star b = \sum_{k\geq 0} \frac{(-i)^k}{k!} \partial_\xi^k a\, \partial_x^k b. \tag{1.2.6}$$

1.3 Change of independent variables

Let us now analyze the transformation rules of the principal symbol $a(x,\xi)$ of an operator A under smooth invertible changes of variables

$$y_i = y_i(x), \quad i = 1, \dots, n. \tag{1.3.1}$$

Recall that the first derivatives transform according to the chain rule

$$\frac{\partial}{\partial x_i} = \sum_{k=1}^{d} \frac{\partial y_k}{\partial x_i} \frac{\partial}{\partial y_k}. \tag{1.3.2}$$

The transformation law of higher order derivatives is more complicated. For example

$$\frac{\partial^2}{\partial x_i \partial x_j} = \sum_{k,l=1}^{d} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} \frac{\partial^2}{\partial y_k \partial y^l} + \sum_{k=1}^{d} \frac{\partial^2 y_k}{\partial x_i \partial x_j} \frac{\partial}{\partial y_k}$$

etc. However it is clear that after the transformation one obtains again a linear differential operator of the same order m. More precisely define the operator

$$\tilde{A} = \sum (-i)^{|\mathbf{p}|} a_{\mathbf{p}}(y) \frac{\partial^{|\mathbf{p}|}}{\partial y_1^{p_1} \dots \partial y_d^{p_d}}$$

by the equation

$$A\, f(y(x)) = \left(\tilde{A}\, f(y) \right)_{y=y(x)}.$$

The transformation law of the principal symbol is of particular simplicity as it follows from the following

Proposition 1.6 *Let $a_m(x,\xi)$ be the principal symbol of a linear differential operator A. Denote $\tilde{a}_m(y,\tilde{\xi})$ the principal symbol of the same operator written in the coordinates y, i.e., the principal symbol of the operator $\tilde{A}$. Then*

$$a_m(y(x),\tilde{\xi}) = a_m(x,\xi) \quad \text{provided} \quad \xi_i = \sum_{k=1}^{d} \frac{\partial y_k}{\partial x_i} \tilde{\xi}_k. \tag{1.3.3}$$

Proof: Applying the formula (1.2.4) one easily derives the equality

$$\begin{aligned} &a_m(x, S_x) = \tilde{a}_m(y, S_y) \\ &y = y(x) \\ &S_x = \left(\frac{\partial S}{\partial x_1}, \dots, \frac{\partial S}{\partial x_d} \right), \quad S_y = \left(\frac{\partial S}{\partial y_1}, \dots, \frac{\partial S}{\partial y_d} \right). \end{aligned}$$

Applying the chain rule

$$\frac{\partial S}{\partial x_i} = \sum_{k=1}^{d} \frac{\partial y_k}{\partial x_i} \frac{\partial S}{\partial y_k}$$

we arrive at the transformation rule (1.3.3) for the particular case

$$\xi_i = \frac{\partial S}{\partial x_i}, \quad \tilde{\xi}_k = \frac{\partial S}{\partial y_k}.$$

This proves the proposition since the gradients can take arbitrary values. ■

1.4 Canonical form of linear differential operators of order ≤ 2 with constant coefficients

Consider a first order linear differential operator

$$A = a_1 \frac{\partial}{\partial x_1} + \cdots + a_d \frac{\partial}{\partial x_d} \tag{1.4.1}$$

with constant coefficients $a_1, \ldots, a_d$. One can find a linear transformation of the coordinates

$$\xi_i = \sum_{k=1}^{d} c_{ki} \tilde{\xi}_k, \quad i = 1, \ldots, d \tag{1.4.2}$$

that maps the vector $a = (a_1, \ldots, a_d)$ to the unit coordinate vector of the axis y_d. After such a transformation the operator A becomes the partial derivative operator

$$A = \frac{\partial}{\partial y_d}.$$

Therefore the general solution of the first order linear differential equation

$$A\,\varphi = 0$$

can be written in the form

$$\varphi(y_1, \ldots, y_d) = \varphi_0(y_1, \ldots, y_{d-1}). \tag{1.4.3}$$

Here φ_0 is an arbitrary smooth function of $(d-1)$ variables.

Exercise 1.7 *Prove that the general solution to the equation*

$$A\,\varphi + b\,\varphi = 0 \tag{1.4.4}$$

with A of the form (1.4.1) and a constant b reads

$$\varphi(y_1, \ldots, y_d) = \varphi_0(y_1, \ldots, y_{d-1}) e^{-b\,y_d}$$

for an arbitrary $\mathcal{C}^1$ function $\varphi_0(y_1, \ldots, y_{d-1})$.

Consider now a second order linear differential operator of the form

$$A = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c \tag{1.4.5}$$

with constant coefficients. Without loss of generality one can assume the coefficient matrix a_{ij} to be symmetric. Denote

$$Q(\xi) = -a_2(x,\xi) = \sum_{i,j=1}^d a_{ij}\xi_i\xi_j \tag{1.4.6}$$

the quadratic form coinciding with the principal symbol, up to a common sign. Recall the following theorem from linear algebra.

Theorem 1.8 *There exists a linear invertible change of variables of the form (1.4.2) reducing the quadratic form (1.4.6) to the form*

$$Q = \tilde\xi_1^2 + \dots + \tilde\xi_p^2 - \tilde\xi_{p+1}^2 - \dots - \tilde\xi_{p+q}^2. \tag{1.4.7}$$

The numbers $p \geq 0$, $q \geq 0$, $p+q \leq d$ do not depend on the choice of the reducing transformation.

Note that, according to the Proposition 1.6 the transformation (1.4.2) corresponds to the linear invertible change of independent variables $x \to y$ of the form

$$y_k = \sum_{i=1}^d c_{ki}x_i, \quad k = 1, \dots, d. \tag{1.4.8}$$

Invertibility means that the coefficient matrix of the transformation does not degenerate:

$$\det\left(c_{ki}\right)_{1\leq k,i\leq d} \neq 0.$$

We arrive at

Corollary 1.9 *A second order linear differential operator with constant coefficients can be reduced to the form*

$$A = \frac{\partial^2}{\partial y_1^2} + \dots + \frac{\partial^2}{\partial y_p^2} - \frac{\partial^2}{\partial y_{p+1}^2} - \dots - \frac{\partial^2}{\partial y_{p+q}^2} + \sum_{k=1}^d \tilde b_k y_k + c \tag{1.4.9}$$

by a linear transformation of the form (1.4.8). The numbers p and q do not depend on the choice of the reducing transformation.

1.5 Elliptic and hyperbolic operators. Characteristics

Let $a_m(x,\xi)$ be the principal symbol of a linear differential operator A.

Definition 1.10 *It is said that the operator $A : \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega)$ is* elliptic *if*

$$a_m(x,\xi) \neq 0 \quad \text{for any} \quad \xi \neq 0, \quad x \in \Omega. \tag{1.5.1}$$

For example the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

is elliptic on $\Omega = \mathbb{R}^d$. The *Tricomi operator*

$$A = \frac{\partial^2}{\partial x^2} + x \frac{\partial^2}{\partial y^2} \tag{1.5.2}$$

is elliptic on the right half plane $x > 0$.

Definition 1.11 *Given a point $x_0 \in \Omega$, the hypersurface in the ξ-space defined by the equation*

$$a_m(x_0, \xi) = 0 \tag{1.5.3}$$

is called characteristic cone *of the operator A at x_0. The vectors ξ satisfying (1.5.3) are called* characteristic vectors *at the point x_0.*

Observe that the hypersurface (1.5.3) is invariant with respect to rescalings

$$\xi \mapsto \lambda\xi \quad \forall\, \lambda \in \mathbb{R} \tag{1.5.4}$$

since the polynomial $a_m(x_0, \xi)$ is homogeneous of degree m:

$$a_m(x, \lambda\,\xi) = \lambda^m a_m(x, \xi).$$

The characteristic cone of an elliptic operator is one point $\xi = 0$. For the example of wave operator

$$A = \frac{\partial^2}{\partial t^2} - \Delta, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2} \tag{1.5.5}$$

the characteristic cone is given by the equation

$$\tau^2 - \xi_1^2 - \cdots - \xi_d^2 = 0. \tag{1.5.6}$$

Thus it coincides with the standard cone in the Euclidean $(d+1)$-dimensional space. The characteristic cone of the heat operator

$$\frac{\partial}{\partial t} - \Delta \tag{1.5.7}$$

is the τ-line

$$\xi_1 = \cdots = \xi_d = 0. \tag{1.5.8}$$

Definition 1.12 *The hypersurface in $\mathbb{R}^d$ is called* characteristic surface *or simply* characteristics *for the operator A if at every point x of the surface the normal vector ξ is a characteristic vector:*

$$a_m(x, \xi) = 0.$$

If the hypesurface is defined by a local equation

$$S(x) = 0 \tag{1.5.9}$$

then $S(x)$ satisfies the equation

$$a_m\left(x, S_x(x)\right) = 0 \tag{1.5.10}$$

at every point of the hypersurface (1.5.9).

As it follows from the Proposition 1.6 the characteristics do not depend on the choice of a system of coordinates.

Example. For a first order linear differential operator

$$A = a_1(x)\frac{\partial}{\partial x_1} + \cdots + a_d(x)\frac{\partial}{\partial x_d} \tag{1.5.11}$$

the function $S(x)$ defining a characteristic hypersurface must satisfy the equation

$$A\,S(x) = 0. \tag{1.5.12}$$

It is therefore a first integral of the following system of ODEs

$$\begin{aligned} \dot{x}_1 &= a_1(x_1, \dots, x_d) \\ &\dots \\ \dot{x}_d &= a_d(x_1, \dots, x_d) \end{aligned} \tag{1.5.13}$$

Indeed, the equation (1.5.12) says that the function $S(x)$ is constant along the integral curves of the system (1.5.13). It is known from the theory of ordinary differential equations that locally, near a point x^0 such that $\left(a_1(x^0), \dots, a_d(x^0)\right) \neq 0$ there exists a smooth invertible change of coordinates

$$(x_1, \dots, x_d) \mapsto (y_1, \dots, y_d), \quad y_k = y_k(x_1, \dots, x_d)$$

such that, in the new coordinates the system reduces to the form

$$\begin{aligned} \dot{y}_1 &= 0 \\ &\dots \\ \dot{y}_{d-1} &= 0 \\ \dot{y}_d &= 1 \end{aligned} \tag{1.5.14}$$

(the so-called rectification of a vector field). For the particular case of constant coefficients the needed transformation is linear (see above). In these coordinates the general solution to the equation (1.5.12) reads

$$S(y_1, \dots, y_d) = S_0(y_1, \dots, y_{d-1}). \tag{1.5.15}$$

Hyperbolic operators. Let us consider a linear differential operator A acting on smooth functions on a domain Ω in the $(d+1)$-dimensional space with Euclidean coordinates $(t, x_1, \dots, x_d)$. Denote $a_m(t, x, \tau, \xi)$ the principal symbol of this operator. Here

$$\tau \in \mathbb{R}, \quad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d.$$

Recall that the principal symbol of an operator of order m is a polynomial of degree m in τ, ξ_1, $\dots$, ξ_d.

Definition 1.13 *The linear differential operator A is called* hyperbolic *with respect to the time variable t if for any fixed $\xi \neq 0$ and any $(t, x) \in \Omega$ the equation for τ*

$$a_m(t, x, \tau, \xi) = 0 \tag{1.5.16}$$

has m pairwise distinct real roots

$$\tau_1(t, x, \xi), \dots, \tau_m(t, x, \xi).$$

For brevity we will often say that a linear differential operator is hyperbolic if all its characteristics are real and pairwise distinct. For elliptic operators the characteristics are purely imaginary.

The wave operator (1.5.5) gives a simple example of a hyperbolic operator. Indeed, the equation

$$\tau^2 = \xi_1^2 + \cdots + \xi_d^2$$

has two distinct roots

$$\tau = \pm\sqrt{\xi_1^2 + \cdots + \xi_d^2}$$

for any $\xi \neq 0$. The heat operator (1.5.7) is neither hyperbolic nor elliptic.

Finding the j-th characteristic of a hyperbolic operator requires knowledge of solutions to the following Hamilton–Jacobi equation for the functions $S = S(x, t)$

$$\frac{\partial S}{\partial t} = \tau_j \left(t, x, \frac{\partial S}{\partial x} \right). \tag{1.5.17}$$

From the course of analytical mechanics it is known that the latter problem is reduced to integrating the Hamilton equations

$$\left.\begin{array}{rcl} \dot{x}_i & = & \frac{\partial H(t,x,p)}{\partial p_i} \\ \\ \dot{p}_i & = & -\frac{\partial H(t,x,p)}{\partial x_i} \end{array}\right\} \tag{1.5.18}$$

with the time-dependent Hamiltonian $H(t, x, p) = \tau_j(t, x, p)$. In the next section we will consider the particular case $d = 1$ and apply it to the problem of canonical forms of the second order linear differential operators in a two-dimensional space.

1.6 Reduction to a canonical form of second order linear differential operators in a two-dimensional space

Consider a linear differential operator

$$A = a(x, y)\frac{\partial^2}{\partial x^2} + 2b(x, y)\frac{\partial^2}{\partial x \partial y} + c(x, y)\frac{\partial^2}{\partial y^2}, \quad (x, y) \in \Omega \subset \mathbb{R}^2. \tag{1.6.1}$$

The characteristics of these operator are curves

$$x = x(t), \quad y = y(t).$$

Here t is some parameter on the characteristic. Let (dx, dy) be the tangent vector to the curve. Then the normal vector $(-dy, dx)$ must satisfy the equation

$$a(x, y)dy^2 - 2b(x, y)dx\, dy + c(x, y)dx^2 = 0. \tag{1.6.2}$$

Assuming $a(x, y) \neq 0$ one obtains a quadratic equation for the vector dy/dx

$$a(x, y)\left(\frac{dy}{dx}\right)^2 - 2b(x, y)\frac{dy}{dx} + c(x, y) = 0. \tag{1.6.3}$$

The operator (1.6.1) is hyperbolic *iff* the discriminant of this equation is positive:

$$b^2 - a\, c > 0. \tag{1.6.4}$$

For elliptic operators the discriminant is strictly negative.

For a hyperbolic operator one has two families of characteristics to be found from the ODEs

$$\frac{dy}{dx} = \frac{b(x,y) + \sqrt{b^2(x,y) - a(x,y)\,c(x,y)}}{a(x,y)} \tag{1.6.5}$$

$$\frac{dy}{dx} = \frac{b(x,y) - \sqrt{b^2(x,y) - a(x,y)\,c(x,y)}}{a(x,y)}. \tag{1.6.6}$$

Let

$$\phi(x,y) = c_1, \quad \psi(x,y) = c_2 \tag{1.6.7}$$

be the equations of the characteristics[1]. Here c_1 and c_2 are two integration constants. Such curves pass through any point $(x,y) \in \Omega$. Moreover they are not tangent at every point. Let us introduce new local coordinates u, v by

$$u = \phi(x,y), \quad v = \psi(x,y). \tag{1.6.8}$$

Lemma 1.14 *The change of coordinates*

$$(x,y) \mapsto (u,v)$$

is locally invertible. Moreover the inverse functions

$$x = x(u,v), \quad y = y(u,v)$$

are smooth.

Proof: We have to check non-vanishing of the Jacobian

$$\det \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} = \det \begin{pmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{pmatrix} \neq 0. \tag{1.6.9}$$

By definition the first derivatives of the functions ϕ and ψ correspond to two different roots of the same quadratic equation

$$a(x,y)\phi_x^2 + 2b(x,y)\phi_x\phi_y + c(x,y)\phi_y^2 = 0, \quad a(x,y)\psi_x^2 + 2b(x,y)\psi_x\psi_y + c(x,y)\psi_y^2 = 0.$$

The determinant (1.6.9) vanishes *iff* the gradients of ϕ and ψ are proportional:

$$(\phi_x, \phi_y) \sim (\psi_x, \psi_y).$$

This contradicts the requirement to have the roots distinct. ■

Let us rewrite the linear differential operator A in the new coordinates:

$$A = \tilde a(u,v)\frac{\partial^2}{\partial u^2} + 2\tilde b(u,v)\frac{\partial^2}{\partial u \partial v} + \tilde c(u,v)\frac{\partial^2}{\partial v^2} + \dots \tag{1.6.10}$$

where the dots stand for the terms with the low order derivatives.

[1] The function $\phi(x,y)$, resp. $\psi(x,y)$, is a first integral for the ODE (1.6.5), resp. (1.6.6), that is, it takes constant values along the integral curves of this differential equation.

Theorem 1.15 *In the new coordinates the linear differential operator reads*

$$A = 2\tilde{b}(u,v)\frac{\partial^2}{\partial u\,\partial v} + \dots$$

Proof: In the new coordinates the characteristic have the form

$$u = c_1, \quad v = c_2$$

for arbitrary constants c_1 and c_2. Therefore their tangent vectors $(1,0)$ and $(0,1)$ must satisfy the equation for characteristics

$$\tilde{a}(u,v)dv^2 - 2\tilde{b}(u,v)du\,dv + \tilde{c}(u,v)dv^2 = 0.$$

This implies $\tilde{a}(u,v) = \tilde{c}(u,v) = 0$. ■

For the case of elliptic operator (1.6.1) the analogue of the differential equations (1.6.5), (1.6.6) are complex conjugated equations

$$\frac{dy}{dx} = \frac{b \pm i\sqrt{a\,c - b^2}}{a}, \quad a = a(x,y), \quad b = b(x,y), \quad c = c(x,y). \tag{1.6.11}$$

Assuming analyticity of the functions $a(x,y)$, $b(x,y)$, $c(x,y)$ one can prove existence of a complex valued first integral

$$S(x,y) = \phi(x,y) + i\,\psi(x,y) \tag{1.6.12}$$

satisfying

$$a\,S_x + \left(b - i\sqrt{a\,c - b^2}\right)S_y = 0. \tag{1.6.13}$$

Let us introduce new system of coordinates by

$$u = \phi(x,y), \quad v = \psi(x,y). \tag{1.6.14}$$

Exercise 1.16 *Prove that the transformation*

$$(x,y) \mapsto (u,v)$$

is locally smoothly invertible. Prove that the operator A in the new coordinates takes the form

$$A = \tilde{a}(u,v)\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + \dots \tag{1.6.15}$$

with some nonzero smooth function $\tilde{a}(u,v)$. Like above the dots stand for the terms with lower order derivatives.

Let us now consider the case of linear differential operators of the form (1.6.1) with identically vanishing discriminant

$$b^2(x,y) - a(x,y)\,c(x,y) \equiv 0. \tag{1.6.16}$$

Operators of this class are called *parabolic*. In this case we have only one characteristic to be found from the equation

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}. \tag{1.6.17}$$

Let $\phi(x,y)$ be a first integral of this equation

$$a\,\phi_x + b\,\phi_y = 0, \quad \phi_x^2 + \phi_y^2 \neq 0. \tag{1.6.18}$$

Choose an arbitrary smooth function $\psi(x,y)$ such that

$$\det \begin{pmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{pmatrix} \neq 0.$$

In the coordinates

$$u = \phi(x,y), \quad v = \psi(x,y)$$

the coefficient $\tilde a(u,v)$ vanishes, since the line $\phi(x,y) = \text{const}$ is a characteristic. But then the coefficient $\tilde b(u,v)$ must vanish either because of vanishing of the discriminant

$$\tilde b^2 - \tilde a\,\tilde c = 0.$$

Thus the canonical form of a parabolic operator is

$$A = \tilde c(u,v)\frac{\partial^2}{\partial v^2} + \dots \tag{1.6.19}$$

where the dots stand for the terms of lower order.

1.7 General solution of a second order hyperbolic equation with constant coefficients in the two-dimensional space

Consider a hyperbolic operator

$$A = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x\,\partial y} + c\frac{\partial^2}{\partial y^2} \tag{1.7.1}$$

with constant coefficients a, b, c satisfying the hyperbolicity condition

$$b^2 - a\,c > 0.$$

The equations for characteristics (1.6.5), (1.6.6) can be easily integrated. This gives two linear first integrals

$$\begin{gathered} u = y - \lambda_1 x, \quad v = y - \lambda_2 x \\ \lambda_{1,2} = \frac{b \pm \sqrt{b^2 - a\,c}}{a}. \end{gathered} \tag{1.7.2}$$

In the new coordinates the hyperbolic equation $A\,\varphi = 0$ reduces to

$$\frac{\partial^2\varphi}{\partial u\,\partial v} = 0. \tag{1.7.3}$$

The general solution to this equation can be written in the form

$$\varphi = f(y - \lambda_1 x) + g(y - \lambda_2 x) \tag{1.7.4}$$

where f and g are two arbitrary smooth[2] functions of one variable.

For example consider the wave equation

$$\varphi_{tt} = a^2 \varphi_{xx} \tag{1.7.5}$$

where a is a positive constant. The general solution reads

$$\varphi(x,t) = f(x - a\,t) + g(x + a\,t). \tag{1.7.6}$$

Observe that $f(x - a\,t)$ is a right-moving wave propagating with constant speed a. In a similar way $g(x + a\,t)$ is a left-moving wave. Therefore the general solution to the wave equation (1.7.5) is a superposition of two such waves.

1.8 Exercises to Section 1

Exercise 1.17 *Reduce to the canonical form the following equations*

$$u_{xx} + 2u_{xy} - 2u_{xz} + 2u_{yy} + 6u_{zz} = 0 \tag{1.8.1}$$
$$u_{xy} - u_{xz} + u_x + u_y - u_z = 0. \tag{1.8.2}$$

Exercise 1.18 *Reduce to the canonical form the following equations*

$$x^2 u_{xx} + 2x\,y\,u_{xy} - 3y^2 u_{yy} - 2x\,u_x + 4y\,u_y + 16x^4 u = 0 \tag{1.8.3}$$
$$y^2 u_{xx} + 2x\,y\,u_{xy} + 2x^2 u_{yy} + y\,u_y = 0 \tag{1.8.4}$$
$$u_{xx} - 2u_{xy} + u_{yy} + u_x + u_y = 0 \tag{1.8.5}$$

Exercise 1.19 *Find general solution to the following equations*

$$x^2 u_{xx} - y^2 u_{yy} - 2y\,u_y = 0 \tag{1.8.6}$$
$$x^2 u_{xx} - 2x\,y\,u_{xy} + y^2 u_{yy} + x\,u_x + y\,u_y = 0. \tag{1.8.7}$$

[2] It suffices to take the functions of the $\mathcal{C}^2$ class.

Chapter 2

Wave equation

2.1 Vibrating string

We consider small oscillations of an elastic string on the (x, u)-plane. Let the x-axis be the equilibrium state of the string. Denote $u(x, t)$ the displacement of the point x at a time t. It will be assumed to be orthogonal to the x-axis. Thus the shape of the string at the time t is given by the graph of the function $u(x, t)$. The velocity of the string at the point x is equal to $u_t(x, t)$. We will also assume that the only force to be taken into consideration is the tension directed along the string. In particular the string will be assumed to be totally elastic.

Consider a small interval of the string from x to $x + \Delta x$. We will write the equation of motion for this interval. Denote $T = T(x)$ the tension of the string at the point x. The horizontal and vertical components at the points x and $x + \Delta x$ are equal to

$$T_{\text{hor}}(x) = T_1 \cos\alpha, \quad T_{\text{vert}}(x) = T_1 \sin\alpha$$
$$T_{\text{hor}}(x + \Delta x) = T_2 \cos\beta, \quad T_{\text{vert}}(x + \Delta x) = T_2 \sin\beta$$

where $T_1 = T(x)$, $T_2 = T(x + \Delta x)$ (see Fig. 1).

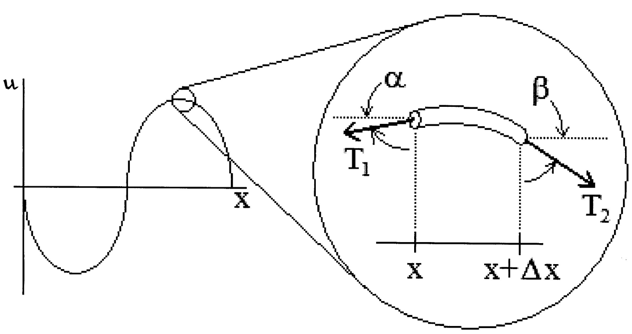

Fig. 1.

The angle α between the string and the x-axis at the point x is given by

$$\cos\alpha = \frac{1}{\sqrt{1 + u_x^2}}, \quad \sin\alpha = \frac{u_x}{\sqrt{1 + u_x^2}}.$$

The oscillations are assumed to be *small.* More precisely this means that the term u_x is small. So at the leading approximation we can neglect the square of it to arrive at

$$\cos\alpha \simeq 1, \quad \sin\alpha \simeq u_x(x)$$
$$\cos\beta \simeq 1, \quad \sin\beta \simeq u_x(x+\Delta x)$$

So the horizontal and vertical components at the points x and $x+\Delta x$ are equal to

$$T_{\text{hor}}(x) \simeq T_1, \quad T_{\text{vert}}(x) \simeq T_1 u_x(x)$$
$$T_{\text{hor}}(x+\Delta x) \simeq T_2, \quad T_{\text{vert}}(x+\Delta x) = T_2 u_x(x+\Delta(x),$$

Since the string moves in the u-direction, the horizontal components at the points x and $x+\Delta x$ must coincide:

$$T_1 = T(x) = T(x+\Delta x) = T_2.$$

Therefore $T(x) \equiv T = \text{const}$.

Let us now consider the vertical components. The resulting force acting on the piece of the string is equal to

$$f = T_2 \sin\beta - T_1 \sin\alpha = T\, u_x(x+\Delta x) - T\, u_x(x) \simeq T\, u_{xx}(x)\,\Delta x.$$

On another side the vertical component of the total momentum of the piece of the string is equal to

$$p = \int_x^{x+\Delta x} \rho(x) u_t(x,t)\, ds(x) \simeq \rho(x) u_t(x,t)\,\Delta x$$

where $\rho(x)$ is the linear mass density of the string and

$$ds(x) = \frac{dx}{\sqrt{1+u_x^2(x)}} \simeq dx$$

is the element of the length[1]. The second Newton law

$$p_t = f$$

in the limit $\Delta x \to 0$ yields

$$\rho(x) u_{tt} = T\, u_{xx}.$$

In particular in the case of constant mass density one arrives at the equation

$$u_{tt} = a^2 u_{xx} \tag{2.1.1}$$

where the constant a is defined by

$$a^2 = \frac{T}{\rho}. \tag{2.1.2}$$

[1]This means that the length s of the segment of the string between $x = x_1$ and $x = x_2$ is equal to

$$s = \int_{x_1}^{x_2} ds(x),$$

and the total mass m of the same segment is equal to

$$m = \int_{x_1}^{x_2} \rho(x)\, ds(x).$$

Exercise 2.1 *Prove that the* plane wave

$$u(x,t) = A\, e^{i(k\,x+\omega\, t)} \tag{2.1.3}$$

satisfies the wave equation (2.1.1) *if and only if the real parameters* ω *and* k *satisfy the following* dispersion relation

$$\omega = \pm a\, k. \tag{2.1.4}$$

The parameters ω and k are called resp. the *frequency*[2] and *wave number* of the plane wave. The arbitrary parameter A is called the *amplitude* of the wave. It is clear that the plane wave is periodic in x with the period

$$L = \frac{2\pi}{k} \tag{2.1.5}$$

since the exponential function is periodic with the period $2\pi\, i$. The plane wave is also periodic in t with the period

$$T = \frac{2\pi}{\omega}. \tag{2.1.6}$$

Due to linearity of the wave equation the real and imaginary parts of the solution (2.1.3) solve the same equation (2.1.1). Assuming A to be real we thus obtain the real valued solutions

$$\operatorname{Re} u = A\, \cos(k\, x + \omega\, t), \quad \operatorname{Im} u = A\, \sin(k\, x + \omega\, t). \tag{2.1.7}$$

2.2 D'Alembert formula

Let us start with considering oscillations of an *infinite string*. That is, the spatial variable x varies from $-\infty$ to ∞. The Cauchy problem for the equation (2.1.1) is formulated in the following way: find a solution $u(x,t)$ defined for $t \geq 0$ such that at $t = 0$ the initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \tag{2.2.1}$$

hold true. The solution is given by the following *D'Alembert formula*:

Theorem 2.2 (D'Alembert formula) *For arbitrary initial data* $\phi(x) \in \mathcal{C}^2(\mathbb{R})$, $\psi(x) \in \mathcal{C}^1(\mathbb{R})$ *the solution to the Cauchy problem* (2.1.1), (2.2.1) *exists and is unique. Moreover it is given by the formula*

$$u(x,t) = \frac{\phi(x - a\, t) + \phi(x + a\, t)}{2} + \frac{1}{2a}\int_{x-a\,t}^{x+a\,t} \psi(s)\, ds. \tag{2.2.2}$$

Proof: As we have proved in Section 1.7 the general solution to the equation (2.1.1) can be represented in the form

$$u(x,t) = f(x - a\, t) + g(x + a\, t). \tag{2.2.3}$$

[2] In physics literature the number $-\omega$ is called frequency.

Let us choose the functions f and g in order to meet the initial conditions (2.2.1). We obtain the following system:

$$\begin{aligned} f(x) + g(x) &= \phi(x) \\ a\,\left[g'(x) - f'(x)\right] &= \psi(x). \end{aligned} \tag{2.2.4}$$

Integrating the second equation yields

$$g(x) - f(x) = \frac{1}{a}\int_{x_0}^{x} \psi(s)\,ds + C$$

where C is an integration constant. So

$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2a}\int_{x_0}^{x}\psi(s)\,ds - \frac{1}{2}C$$

$$g(x) = \frac{1}{2}\phi(x) + \frac{1}{2a}\int_{x_0}^{x}\psi(s)\,ds + \frac{1}{2}C.$$

Thus

$$u(x,t) = \frac{1}{2}\phi(x - a\,t) - \frac{1}{2a}\int_{x_0}^{x-a\,t}\psi(s)\,ds + \frac{1}{2}\phi(x + a\,t) + \frac{1}{2a}\int_{x_0}^{x+a\,t}\psi(s)\,ds.$$

This gives (2.2.2). It remains to check that, given a pair of functions $\phi(x) \in \mathcal{C}^2$, $\psi(x) \in \mathcal{C}^1$ the D'Alembert formula yields a solution to (2.1.1). Indeed, the function (2.2.2) is twice differentiable in x and t. It remains to substitute this function into the wave equation and check that the equation is satisfied. We leave it as an exercise for the reader. It is also straightforward to verify validity of the initial data (2.2.1). ■

Example. For the constant initial data

$$u(x,0) = u_0, \quad u_t(x,0) = v_0$$

the solution has the form

$$u(x,t) = u_0 + v_0 t.$$

This solution corresponds to the free motion of the string with the constant speed v_0.

Moreover the solution to the wave equation is stable with respect to small variations of the initial data. Namely,

Exercise 2.3 *For any $\epsilon > 0$ and any $T > 0$ there exists $\delta > 0$ such that the solutions $u(x,t)$ and $\tilde{u}(x,t)$ of the two Cauchy problems with initial conditions (2.2.1) and*

$$\tilde{u}(x,0) = \tilde{\phi}(x), \quad \tilde{u}_t(x,0) = \tilde{\psi}(x) \tag{2.2.5}$$

satisfy

$$\sup_{x\in\mathbb{R},\ t\in[0,T]} |\tilde{u}(x,t) - u(x,t)| < \epsilon \tag{2.2.6}$$

provided the initial conditions satisfy

$$\sup_{x\in\mathbb{R}} |\tilde{\phi}(x) - \phi(x)| < \delta, \quad \sup_{x\in\mathbb{R}} |\tilde{\psi}(x) - \psi(x)| < \delta. \tag{2.2.7}$$

Remark 2.4 *The property formulated in the above exercise is usually referred to as* well posedness *of the Cauchy problem (2.1.1), (2.2.1). We will return later to the discussion of this important property.*

2.3 Some consequences of the D'Alembert formula

Let (x_0, t_0) be a point of the (x, t)-plane, $t_0 > 0$. As it follows from the D'Alembert formula the value of the solution at the point (x_0, t_0) depends only on the values of $\phi(x)$ at $x = x_0 \pm a\, t_0$ and value of $\psi(x)$ on the interval $[x_0 - a\, t_0, x_0 + a\, t_0]$. The triangle with the vertices (x_0, t_0) and $(x_0 \pm a\, t_0, 0)$ is called *the dependence domain* of the segment $[x_0 - a\, t_0, x_0 + a\, t_0]$. The values of the solution *inside* this triangle are completely determined by the values of the initial data on the segment.

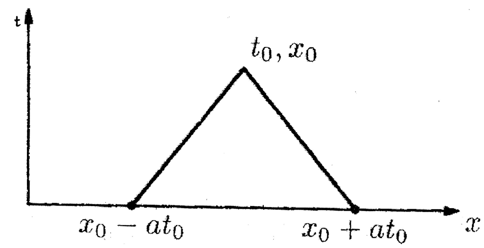

Fig. 2. The dependence domain of the segment $[x_0 - a\, t_0, x_0 + a\, t_0]$.

Another important definition is the *influence domain* for a given segment $[x_1, x_2]$ consider the domain defined by inequalities

$$x + a\, t \geq x_1, \quad x - a\, t \leq x_2, \quad t \geq 0. \tag{2.3.1}$$

Changing the initial data on the segment $[x_1, x_2]$ will not change the solution $u(x, t)$ outside the influence domain.

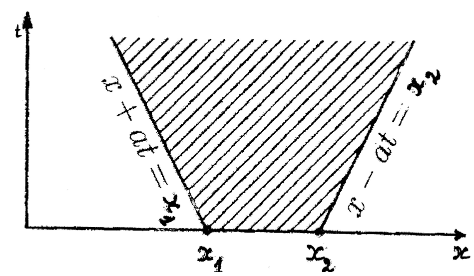

Fig. 3. The influence domain of the segment $[x_1, x_2]$.

Remark 2.5 *It will be convenient to slightly extend the class of initial data admitting piecewise smooth functions $\phi(x)$, $\psi(x)$ (all singularities of the latter must be integrable). If x_j are the singularities of these functions, $j = 1, 2, \dots$, then the solution $u(x, t)$ given by the D'Alembert formula will satisfy the wave equation outside the lines*

$$x = \pm a\, t + x_j, \quad t \geq 0, \quad j = 1, 2, \dots$$

The above formula says that the singularities of the solution propagate along the characteristics.

Example. Let us draw the profile of the string for the triangular initial data $\phi(x)$ shown on Fig. 4 and $\psi(x) \equiv 0$.

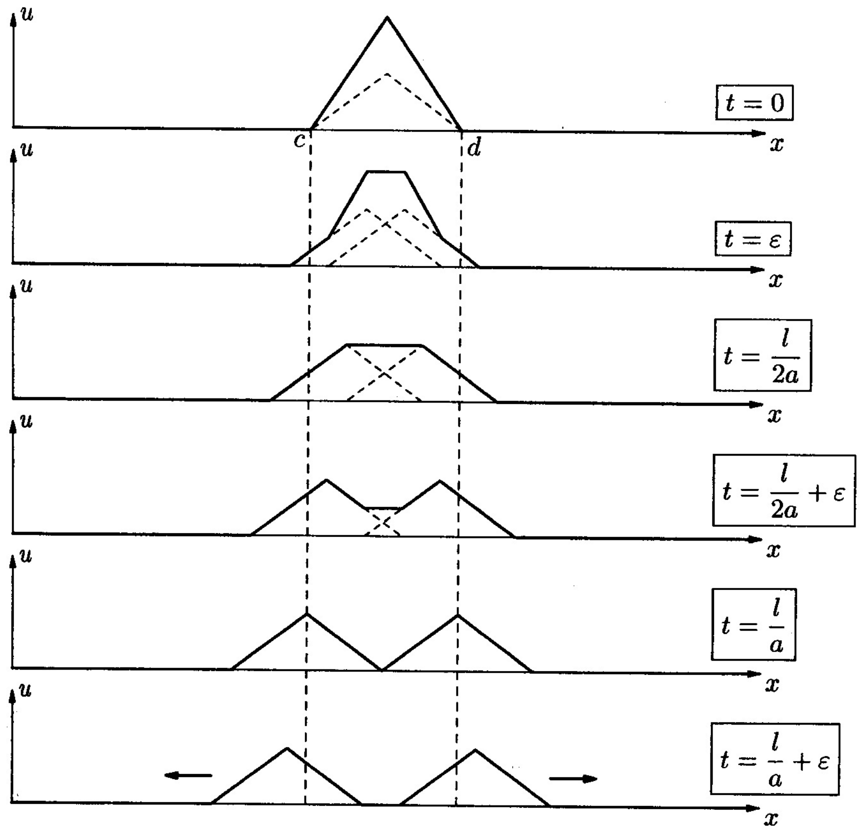

Fig. 4. The solution of the Cauchy problem for wave equation on the real line with a triangular initial profile at different instants of time.

2.4 Semi-infinite vibrating string

Let us begin with the following simple observation.

Lemma 2.6 *Let $u(x,t)$ be a solution to the wave equation. Then so are the functions*

$$\pm u(\pm x, \pm t)$$

with arbitrary choices of all three signs.

Proof: This follows from linearity of the wave equation and from its invariance with respect to the spatial reflection

$$x \mapsto -x$$

and time inversion

$$t \mapsto -t.$$

■

Let us consider oscillations of a string with a fixed point. Without loss of generality we can assume that the fixed point is at $x = 0$. We arrive at the following Cauchy problem for (2.1.1) on the half-line $x > 0$:

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x > 0. \tag{2.4.1}$$

The solution must also satisfy the boundary condition

$$u(0,t) = 0, \quad t \geq 0. \tag{2.4.2}$$

The problem (2.1.1), (2.4.1), (2.4.2) is often called *mixed problem* since we have both initial conditions and boundary conditions.

The solution to the mixed problem on the half-line can be reduced to the problem on the infinite line by means of the following trick.

Lemma 2.7 *Let the initial data $\phi(x)$, $\psi(x)$ for the Cauchy problem (2.1.1), (2.2.1) be odd functions of x. Then the solution $u(x,t)$ is an odd function for all t.*

Proof: Denote

$$\tilde{u}(x,t) := -u(-x,t).$$

According to Lemma 2.6 the function $\tilde{u}(x,t)$ satisfies the same equation. At $t = 0$ we have

$$\tilde{u}(x,0) = -u(-x,0) = -\phi(-x) = \phi(x), \quad \tilde{u}_t(x,0) = -u_t(-x,0) = -\psi(-x) = \psi(x)$$

since ϕ and ψ are odd functions. Therefore $\tilde{u}(x,t)$ is a solution to the same Cauchy problem (2.1.1), (2.2.1). Due to uniqueness $\tilde{u}(x,t) = u(x,t)$, i.e. $-u(-x,t) = u(x,t)$ for all x and t. ∎

We are now ready to present a recipe for solving the mixed problem for the wave equation on the half-line. Let us extend the initial data onto entire real line as odd functions. We arrive at the following Cauchy problem for the wave equation:

$$u(x,0) = \begin{cases} \phi(x), & x > 0 \\ -\phi(-x), & x < 0 \end{cases}, \quad u_t(x,0) = \begin{cases} \psi(x), & x > 0 \\ -\psi(-x), & x < 0 \end{cases} \tag{2.4.3}$$

According to Lemma 2.7 the solution $u(x,t)$ to the Cauchy problem (2.1.1), (2.4.3) given by the D'Alembert formula will be an odd function for all t. Therefore

$$u(0,t) = -u(0,t) = 0 \quad \text{for all} \quad t.$$

Example. Consider the evolution of a triangular initial profile on the half-line. The graph of the initial function $\phi(x)$ is non-zero on the interval $[l, 3l]$; the initial velocity $\psi(x) = 0$. The evolution is shown on Fig. 5 for few instants of time. Observe the reflected profile (the dotted line) on the negative half-line.

In a similar way one can treat the mixed problem on the half-line with a free boundary. In this case the vertical component $T\,u_x$ of the tension at the left edge must vanish at all times. Thus the boundary condition (2.4.2) has to be replaced with

$$u_x(0,t) = 0 \quad \text{for all} \quad t \geq 0. \tag{2.4.4}$$

One can solve the mixed problem (2.1.1), (2.4.1), (2.4.4) by using *even extension* of the initial data onto the negative half-line. We leave the details of the construction as an exercise for the reader.

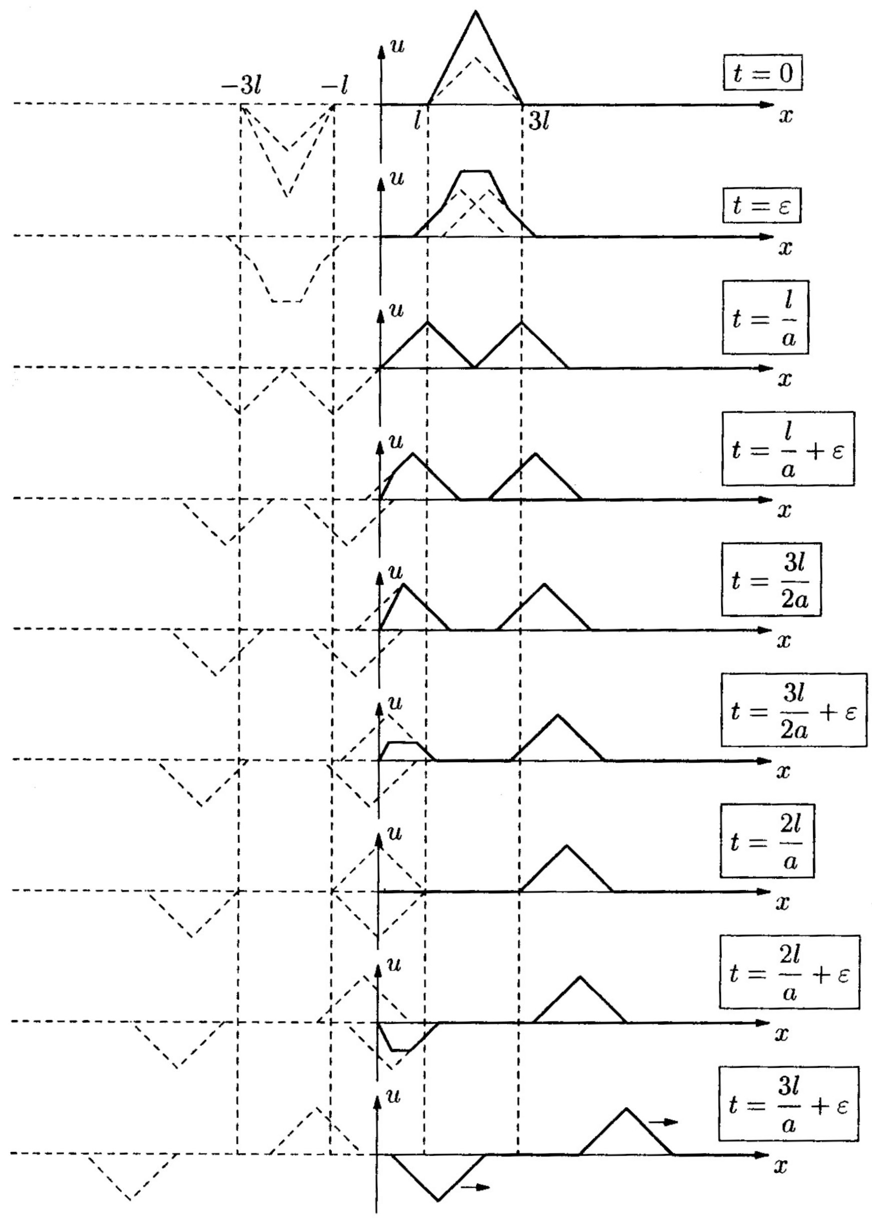

Fig. 5. The solution of the Cauchy problem for wave equation on the half-line with a triangular initial profile.

2.5 Periodic problem for wave equation. Introduction to Fourier series

Let us look for solutions to the wave equation (2.1.1) periodic in x with a given period $L > 0$. Thus we are looking for a solution $u(x,t)$ satisfying

$$u(x+L,t) = u(x,t) \quad \text{for any} \quad t \geq 0. \tag{2.5.1}$$

The initial data of the Cauchy problem

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \tag{2.5.2}$$

must also be L-periodic functions.

Theorem 2.8 *Given L-periodic initial data $\phi(x) \in \mathcal{C}^2(\mathbb{R})$, $\psi(x) \in \mathcal{C}^1(\mathbb{R})$ the periodic Cauchy problem (2.5.1), (2.5.2) for the wave equation (2.1.1) has a unique solution.*

Proof: According to the results of Section 2.2 the solution $u(x,t)$ to the Cauchy problem (2.1.1), (2.5.2) on $-\infty < x < \infty$ exists and is unique and is given by the D'Alembert formula. Denote

$$\tilde{u}(x,t) := u(x+L,t).$$

Since the coefficients of the wave equation do not depend on x the function $\tilde{u}(x,t)$ satisfies the same equation. The initial data for this function have the form

$$\tilde{u}(x,0) = \phi(x+L) = \phi(x), \quad \tilde{u}_t(x,t) = \psi(x+L) = \psi(x)$$

because of periodicity of the functions $\phi(x)$ and $\psi(x)$. So the initial data of the solutions $u(x,t)$ and $\tilde{u}(x,t)$ coincide. From the uniqueness of the solution we conclude that $\tilde{u}(x,t) = u(x,t)$ for all x and t, i.e. the function $u(x,t)$ is periodic in x with the same period L. ■

Exercise 2.9 *Prove that the complex exponential function e^{ikx} is L-periodic* iff *the wave number k has the form*

$$k = \frac{2\pi n}{L}, \quad n \in \mathbb{Z}. \tag{2.5.3}$$

In the following two exercises we will consider the particular case $L = 2\pi$. In this case the complex exponential

$$e^{\frac{2\pi i n x}{L}}$$

obtained in the previous exercise reduces to e^{inx}.

Exercise 2.10 *Prove that the solution of the periodic Cauchy problem with the Cauchy data*

$$u(x,0) = e^{inx}, \quad u_t(x,0) = 0 \tag{2.5.4}$$

is given by the formula

$$u(x,t) = e^{inx} \cos nat. \tag{2.5.5}$$

Exercise 2.11 *Prove that the solution of the periodic Cauchy problem with the Cauchy data*

$$u(x,0) = 0, \quad u_t(x,0) = e^{inx} \tag{2.5.6}$$

is given by the formula

$$u(x,t) = \begin{cases} e^{inx}\frac{\sin nat}{na}, & n \neq 0 \\ t, & n = 0. \end{cases} \tag{2.5.7}$$

Using the theory of *Fourier series* we can represent any solution to the periodic problem to the wave equation as a superposition of the solutions (2.5.5), (2.5.7). Let us first recall some basics of the theory of Fourier series.

Let $f(x)$ be a 2π-periodic continuously differentiable complex valued function on $\mathbb{R}$. The *Fourier series* of this function is defined by the formula

$$\sum_{n\in\mathbb{Z}} c_n e^{inx} \tag{2.5.8}$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx. \tag{2.5.9}$$

The following theorem is a fundamental result of the theory of Fourier series.

Theorem 2.12 *For any function $f(x)$ satisfying the above conditions the Fourier series is uniformly convergent to the function $f(x)$.*

In particular we conclude that any $\mathcal{C}^1$-smooth 2π-periodic function $f(x)$ can be represented as a sum of uniformly convergent Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx. \tag{2.5.10}$$

For completeness we remind the proof of this Theorem.

Let us introduce *Hermitean inner product* in the space of complex valued 2π-periodic continuous functions:

$$(f,g) = \frac{1}{2\pi}\int_0^{2\pi} \bar f(x)g(x)\,dx. \tag{2.5.11}$$

Here the bar stands for complex conjugation. This inner product satisfies the following properties:

$$(g,f) = \overline{(f,g)} \tag{2.5.12}$$

$$\begin{aligned} (\lambda f_1 + \mu f_2, g) &= \bar\lambda(f_1,g) + \bar\mu(f_2,g) \\ (f, \lambda g_1 + \mu g_2) &= \lambda(f,g_1) + \mu(f,g_2) \end{aligned} \quad \text{for any} \quad \lambda,\, \mu \in \mathbb{C} \tag{2.5.13}$$

$$(f,f) > 0 \quad \text{for any nonzero continuous function} \quad f(x). \tag{2.5.14}$$

The real nonnegative number (f,f) will be used for defining the L_2-norm of the function:

$$\|f\| := \sqrt{(f,f)}. \tag{2.5.15}$$

Exercise 2.13 *Prove that the L_2-norm satisfies the triangle inequality:*

$$\|f+g\| \leq \|f\| + \|g\|. \tag{2.5.16}$$

Observe that the complex exponentials e^{inx} form an orthonormal system with respect to the inner product (2.5.11):

$$\left(e^{imx}, e^{inx}\right) = \delta_{m\,n} = \begin{cases} 1, & m=n \\ 0 & m \neq n \end{cases}. \tag{2.5.17}$$

(check it!).

Let $f(x)$ be a continuous function; denote c_n its Fourier coefficients. The following formula

$$c_n = (e^{inx}, f), \quad n \in \mathbb{Z} \tag{2.5.18}$$

gives a simple interpretation of the Fourier coefficients as the coefficients of decomposition of the function f with respect to the orthonormal system made from exponentials. Moreover, the partial sum of the Fourier series

$$S_N(x) = \sum_{n=-N}^{N} c_n e^{inx} \tag{2.5.19}$$

can be interpreted as the orthogonal projection of the vector f onto the $(2N+1)$-dimensional linear subspace

$$V_N = \mathrm{span}\left(1, e^{\pm ix}, e^{\pm 2ix}, \dots, e^{\pm iNx}\right) \tag{2.5.20}$$

consisting of all *trigonometric polynomials*

$$P_N(x) = \sum_{n=-N}^{N} p_n e^{inx} \tag{2.5.21}$$

of degree N. Here p_0, $p_{\pm 1}$, $\dots p_{\pm N}$ are arbitrary complex numbers.

Lemma 2.14 *The following inequality holds true:*

$$\sum_{n=-N}^{N} |c_n|^2 \leq \|f\|^2. \tag{2.5.22}$$

The statement of this lemma is called *Bessel inequality.*

Proof: We have

$$\begin{aligned} 0 \leq \|f(x) - \sum_{n=-N}^{N} c_n e^{inx}\|^2 &= \left(f(x) - \sum_{n=-N}^{N} c_n e^{inx}, f(x) - \sum_{n=-N}^{N} c_n e^{inx}\right) \\ &= (f,f) - \sum_{n=-N}^{N} \left[c_n\left(f, e^{inx}\right) + \bar{c}_n\left(e^{inx}, f\right)\right] + \sum_{m,n=-N}^{N} \bar{c}_m c_n \left(e^{imx}, e^{inx}\right). \end{aligned}$$

Using (2.5.18) and orthonormality (2.5.17) we recast the right hand side of the last equation in the form

$$(f,f) - \sum_{n=-N}^{N} |c_n|^2.$$

This proves Bessel inequality. ∎

Geometrically the Bessel inequality says that the square length of the orthogonal projection of a vector onto the linear subspace V_N cannot be longer than the square length of the vector itself.

Corollary 2.15 *For any continuous function $f(x)$ the series of squares of absolute values of Fourier coefficients converges:*

$$\sum_{n\in\mathbb{Z}} |c_n|^2 < \infty. \tag{2.5.23}$$

The following *extremal property* says that the N-th partial sum of the Fourier series gives the best L_2-approximation of the function $f(x)$ among all trigonometric polynomials of degree N.

Lemma 2.16 *For any trigonometric polynomial $P_N(x)$ of degree N the following inequality holds true*

$$\|f(x) - S_N(x)\| \le \|f(x) - P_N(x)\|. \tag{2.5.24}$$

Here $S_N(x)$ is the N-th partial sum (2.5.19) of the Fourier series of the function f. The equality in (2.5.24) takes place iff *the trigonometric polynomial $P_N(x)$ coincides with $S_N(x)$, i.e.,*

$$p_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx, \quad n = 0, \pm1, \pm2, \dots, \pm N,$$

Proof: From (2.5.18) we derive that

$$(f(x) - S_N(x), P_N(x)) = 0 \quad \text{for any} \quad P_N(x) \in V_N.$$

Hence

$$\begin{aligned}
&\|f(x) - P_N(x)\|^2 = \|(f - S_N) + (S_N - P_N\|^2 = \\
&= (f - S_N, f - S_N) + (f - S_N, Q_N) + (Q_N, f - S_N) + (Q_N, Q_N) \\
&= (f - S_N, f - S_N) + (Q_N, Q_N) \ge (f - S_N, f - S_N) = \|f - S_N\|^2.
\end{aligned}$$

Here we denote

$$Q_N = S_N(x) - P_N(x) \in V_N.$$

Clearly the equality takes place *iff* $Q_N = 0$, i.e. $P_N = S_N$. ∎

Lemma 2.17 *For any continuous 2π-periodic function the following* Parseval equality *holds true:*

$$\sum_{n\in\mathbb{Z}} |c_n|^2 = \|f\|^2. \tag{2.5.25}$$

The Parseval equality can be considered as an infinite-dimensional analogue of the Pythagoras theorem: sum of the squares of orthogonal projections of a vector on the coordinate axes is equal to the square length of the vector.

Proof: According to Stone – Weierstrass theorem[3] any continuous 2π-periodic function can be uniformly approximated by Fourier polynomials

$$P_N(x) = \sum_{n=-N}^{N} p_n e^{inx}. \tag{2.5.26}$$

That means that for a given function $f(x)$ and any $\epsilon > 0$ there exists a trigonometric polynomial $P_N(x)$ of some degree N such that

$$\sup_{x\in[0,2\pi]} |f(x) - P_N(x)| < \epsilon.$$

Then

$$\|f - P_N\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x) - P_N(x)|^2 dx < \epsilon^2.$$

Therefore, due to the extremal property (see Lemma 2.16 above), we obtain the following inequality

$$\|f - S_N\|^2 < \epsilon^2.$$

Repeating the computation used in the proof of Bessel inequality

$$\|f - S_N\|^2 = \|f\|^2 - \sum_{n=-N}^{N} |c_n|^2 < \epsilon^2$$

we arrive at the proof of Lemma. ∎

[3] The Stone – Weierstrass theorem is a very general result about uniform approximation of continuous functions on a compact K in a metric space. Let us recall this important theorem. Let $A \subset \mathcal{C}(K)$ be a subset of functions in the space of continuous real- or complex-valued functions on a compact K. The following requirements must hold true.

1. A must be a *subalgebra* in $\mathcal{C}(K)$, i.e. for $f,\, g \in A$, $\alpha,\, \beta \in \mathbb{R}$ (or $\alpha,\, \beta \in \mathbb{C}$) the linear combination and the product belong to A:

$$\alpha\, f + \beta\, g \in A, \quad f \cdot g \in A.$$

2. The functions in A must *separate points* in K, i.e., $\forall x,\, y \in K$, $x \neq y$ there exists $f \in A$ such that

$$f(x) \neq f(y).$$

3. The subalgebra is *non-degenerate*, i.e., $\forall x \in K$ there exists $f \in A$ such that $f(x) \neq 0$.
The last condition has to be imposed in the complex situation.

4. The subalgebra A is said to be *self-adjoint* if for any function $f \in A$ the complex conjugate function $\bar{f}$ also belongs to A.

Theorem 2.18 *Given an algebra of functions $A \subset \mathcal{C}(K)$ that separates points, is non-degenerate and, for complex-valued functions, is self-adjoint then A is an everywhere dense subset in $\mathcal{C}(K)$.*

Recall that density means that for any continuous function $F \in \mathcal{C}(K)$ and an arbitrary $\epsilon > 0$ there exists $f \in A$ such that

$$\sup_{x\in K}|F(x) - f(x)| < \epsilon.$$

In the particular case of algebra of polynomials one obtains the classical Weierstrass theorem about polynomial approximations of continuous functions on a finite interval. For the needs of the theory of Fourier series one has to apply the Stone – Weierstrass theorem to the subalgebra of Fourier polynomials in the space of continuous 2π-periodic functions. We leave as an exercise to verify applicability of the Stone – Weierstrass theorem in this case.

The Parseval equality is also referred to as *completeness* of the trigonometric system of functions

$$1, e^{\pm ix}, e^{\pm 2ix}, \dots .$$

For the case of infinite-dimensional spaces equipped with a Hermitean (or Euclidean) inner product the property of completeness is the right analogue of the notion of an orthonormal *basis* of the space.

Corollary 2.19 *Two continuous 2π-periodic functions $f(x)$, $g(x)$ with all equal Fourier coefficients identically coincide.*

Proof: Indeed, the difference $h(x) = f(x) - g(x)$ is continuous function with zero Fourier coefficients. The Parseval equality implies $\|h\|^2 = 0$. So $h(x) \equiv 0$. ■

We can now prove that uniform convergence of the Fourier series of a $\mathcal{C}^1$-function. Denote c_n' the Fourier coefficients of the derivative $f'(x)$. Integrating by parts we derive the following formula:

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx = -\frac{1}{2\pi i n} f(x) e^{-inx} \,\big|_0^{2\pi} + \frac{1}{2\pi i n} \int_0^{2\pi} f'(x) e^{-inx}\, dx = -\frac{i}{n} c_n'.$$

This implies convergence of the series

$$\sum_{n\in\mathbb{Z}} |c_n|.$$

Indeed,

$$|c_n| = \frac{|c_n'|}{n} \leq \frac{1}{2}\left(|c_n'|^2 + \frac{1}{n^2}\right).$$

The series $\sum |c_n'|^2$ converges according to the Corollary 2.15; convergence of the series $\sum \frac{1}{n^2}$ is well known. Using Weierstrass theorem we conclude that the Fourier series converges absolutely and uniformly

$$\sum_{n\in\mathbb{Z}} \left| c_n e^{inx} \right| = \sum_{n\in\mathbb{Z}} |c_n| < \infty.$$

Denote $g(x)$ the sum of this series. It is a continuos function. The Fourier coefficients of g coincide with those of f:

$$\left(e^{inx}, g \right) = c_n.$$

Hence $f(x) \equiv g(x)$. ■

For the specific case of real valued function the Fourier coefficients satisfy the following property.

Lemma 2.20 *The function $f(x)$ is real valued* iff *its Fourier coefficients satisfy*

$$\bar c_n = c_{-n} \quad \text{for all} \quad n \in \mathbb{Z}. \tag{2.5.27}$$

Proof: Reality of the function can be written in the form

$$\bar f(x) = f(x).$$

Since

$$\overline{e^{inx}} = e^{-inx}$$

we have

$$\bar{c}_n = \frac{1}{2\pi}\int_0^{2\pi} \bar{f}(x) e^{inx} dx = c_{-n}.$$

■

Note that the coefficient

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx$$

is always real if $f(x)$ is a real valued function.

Let us establish the correspondence of the complex form (2.5.10) of the Fourier series of a real valued function with the real form.

Lemma 2.21 *Let $f(x)$ be a real valued 2π-periodic smooth function. Denote c_n its Fourier coefficients (2.5.9). Introduce coefficients*

$$a_n = c_n + c_{-n} \quad = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx, \quad n = 0, 1, 2, \dots \tag{2.5.28}$$

$$b_n = i(c_n - c_{-n}) = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx, \quad n = 1, 2, \dots \tag{2.5.29}$$

Then the function $f(x)$ is represented as a sum of uniformly convergent Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1}\left(a_n \cos nx + b_n \sin nx\right). \tag{2.5.30}$$

We leave the proof of this Lemma as an exercise for the reader.

Exercise 2.22 *For any real valued continuous function $f(x)$ prove the following version*[4] *of Bessel inequality (2.5.22):*

$$\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_0^{2\pi} f^2(x)\, dx \tag{2.5.31}$$

and Parseval equality (2.5.25)

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_0^{2\pi} f^2(x)\, dx. \tag{2.5.32}$$

The following statement can be used in working with functions with an arbitrary period.

Exercise 2.23 *Given an arbitrary constant $c \in \mathbb{R}$ and a solution $u(x,t)$ to the wave equation (2.1.1) then*

$$\tilde{u}(x,t) = u\left(c\, x, c\, t\right) \tag{2.5.33}$$

also satisfies (2.1.1).

[4]Notice a change in the normalization of the L_2 norm.

Note that for $c \neq 0$ the function $\tilde{u}(x,t)$ is periodic in x with the period $L = \frac{2\pi}{c}$ if $u(x,t)$ was 2π-periodic.

For non-smooth functions the problem of convergence of Fourier series is more delicate. Let us consider an example giving some idea about the convergence of Fourier series for piecewise smooth functions. Consider the function

$$\operatorname{sign} x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}. \tag{2.5.34}$$

This function will be considered on the interval $[-\pi, \pi]$ and then continued 2π-periodically onto entire real line. The Fourier coefficients of this function can be easily computed:

$$a_n = 0, \quad b_n = \frac{2}{\pi}\frac{(1-(-1)^n)}{n}.$$

So the Fourier series of this functions reads

$$\frac{4}{\pi}\sum_{k\geq 1}\frac{\sin(2k-1)x}{2k-1}. \tag{2.5.35}$$

One can prove that this series converges to the sign function at every point of the interval $(-\pi, \pi)$. Moreover this convergence is uniform on every closed subinterval non containing 0 or $\pm\pi$. However the character of convergence near the discontinuity points $x = 0$ and $x = \pm\pi$ is more complicated as one can see from the following graph of a partial sum of the series (2.5.35).

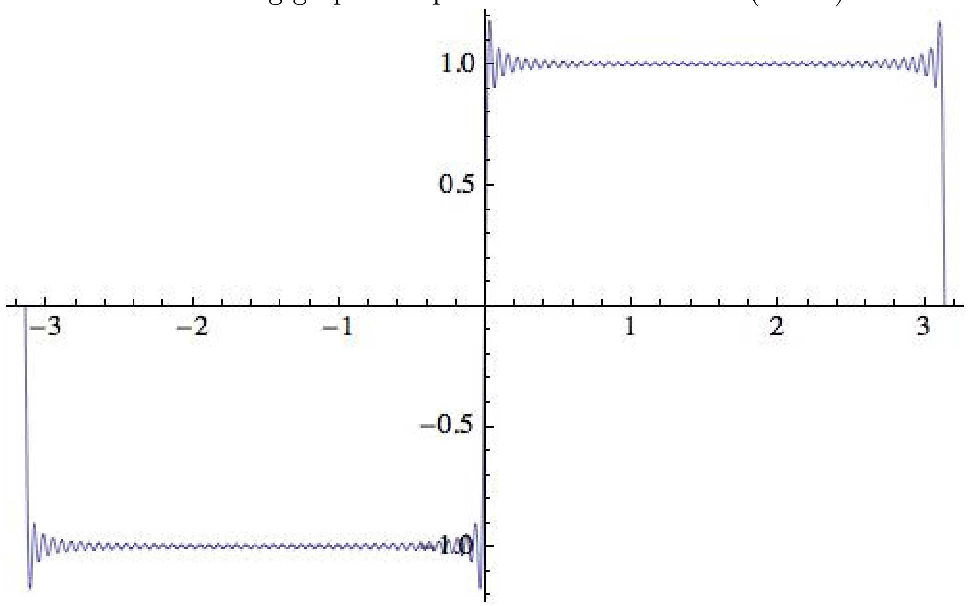

Fig. 6. Graph of the partial sum $S_n(x) = \frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin(2k-1)x}{2k-1}$ for $n = 50$.

In general for piecewise smooth functions $f(x)$ with some number of discontinuity points one can prove that the Fourier series converges to the mean value $\frac{1}{2}(f(x_0+0)+f(x_0-0))$ at every first kind discontinuity point x_0. The non vanishing oscillatory behavior of partial sums near discontinuity points is known as *Gibbs phenomenon* (see Exercise 2.51 below).

Let us return to the wave equation. Using the theory of Fourier series we can represent any periodic solution to the Cauchy problem (2.5.2) as a superposition of solutions of the form (2.5.5), (2.5.7). Namely, let us expand the initial data in Fourier series:

$$\phi(x) = \sum_{n\in\mathbb{Z}} \phi_n e^{inx}, \quad \psi(x) = \sum_{n\in\mathbb{Z}} \psi_n e^{inx}. \tag{2.5.36}$$

Then the solution to the periodic Cauchy problem reads

$$u(x,t) = \sum_{n\in\mathbb{Z}} \phi_n e^{inx} \cos ant + \psi_0 t + \frac{1}{a} \sum_{n\in\mathbb{Z}\setminus 0} \psi_n e^{inx} \frac{\sin ant}{n}. \tag{2.5.37}$$

Remark 2.24 *The formula* (2.5.37) *says that the solutions*

$$\begin{aligned} u_n^{(1)}(x,t) &= e^{inx}\cos ant \\ u_n^{(2)}(x,t) &= \begin{cases} t, & n=0 \\ e^{inx}\frac{\sin ant}{n}, & n\neq 0 \end{cases} \end{aligned} \tag{2.5.38}$$

for $n \in \mathbb{Z}$ form a basis in the space of 2π-periodic solutions to the wave equation. Observe that all these solutions can be written in the so-called separated form

$$u(x,t) = X(x)T(t) \tag{2.5.39}$$

for some smooth functions $X(x)$ and $T(t)$. A rather general method of separation of variables *for solving boundary value problems for linear PDEs has this observation as a starting point. This method will be explained later on.*

2.6 Finite vibrating string. Standing waves

Let us proceed to considering a finite string of the length l. We begin with considering the oscillations of the string with fixed endpoints. So we have to solve the following mixed problem for the wave equation (2.1.1)

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in [0,l] \tag{2.6.1}$$

$$u(0,t) = 0, \quad u(l,t) = 0 \quad \text{for all} \quad t > 0. \tag{2.6.2}$$

The idea of solution is, again, in a suitable extension of the problem onto entire line.

Lemma 2.25 *Let the initial data $\phi(x)$, $\psi(x)$ of the Cauchy problem* (2.2.1) *for the wave equation on $\mathbb{R}$ be odd $2l$-periodic functions. Then the solution $u(x,t)$ will also be an odd $2l$-periodic function for all t satisfying the boundary conditions* (2.6.2).

Proof: As we already know from Lemma 2.7 the solution is an odd function for all t. So

$$u(0,t) = 0 \quad \text{for all} \quad t > 0.$$

Next, the solution will be $2l$-periodic for all t according to Theorem 2.8 above. So

$$u(l-x,t) = -u(x-l,t) = -u(x+l,t).$$

Substituting $x = 0$ we get

$$u(l,t) = -u(l,t), \quad i.e. \quad u(l,t) = 0.$$

■

The above Lemma gives an algorithm for solving the mixed problem (2.6.1), (2.6.2) for the wave equation. Namely, we extend the initial data $\phi(x)$, $\psi(x)$ from the interval $[0,x]$ onto the real axis as odd $2l$-periodic functions. After this we apply D'Alembert formula to the extended initial data. The resulting solution will satisfy the initial conditions (2.6.1) on the interval $[0,l]$ as well as the boundary conditions (2.6.2) at the end points of the interval.

We will apply now the technique of Fourier series to the mixed problem (2.6.1), (2.6.2).

Lemma 2.26 *Let a 2π-periodic functions $f(x)$ be represented as the sum of its Fourier series*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

The function $f(x)$ is even/odd iff *the Fourier coefficients satisfy*

$$c_{-n} = \pm c_n$$

respectively.

Proof: For an even function one must have

$$\sum_{n\in\mathbb{Z}} c_n e^{inx} = f(x) = f(-x) = \sum_{n\in\mathbb{Z}} c_n e^{-inx} = \sum_{n\in\mathbb{Z}} c_{-n} e^{inx}.$$

This proves $c_{-n} = c_n$. A similar argument gives $c_{-n} = -c_n$ for the case of an odd function. ■

Corollary 2.27 *Any even/odd smooth 2π-periodic function can be expanded in Fourier series in cosines/sines:*

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos nx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx, \quad f(x) \quad \text{is even} \tag{2.6.3}$$

$$f(x) = \sum_{n\geq 1} b_n \sin nx, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx, \quad f(x) \quad \text{is odd.} \tag{2.6.4}$$

Proof: Let us consider the case of an odd function. In this case we have $c_{-n} = -c_n$, and, in particular, $c_0 = 0$, so we rewrite the Fourier series in the following form

$$\begin{aligned} f(x) &= \sum_{n\geq 1} c_n e^{inx} + \sum_{n\leq -1} c_n e^{inx} \\ &= \sum_{n\geq 1} c_n \left(e^{inx} - e^{-inx}\right) = 2i \sum_{n\geq 1} c_n \sin nx. \end{aligned}$$

Denote

$$b_n = 2ic_n, \quad n \geq 1.$$

For this coefficient we obtain

$$b_n = \frac{2i}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{i}{\pi}\int_0^{\pi} f(x)e^{-inx}dx + \frac{i}{\pi}\int_{-\pi}^{0} f(x)e^{-inx}dx.$$

In the second integral we change the integration variable $x \mapsto -x$ and use that $f(-x) = -f(x)$ to arrive at

$$b_n = \frac{i}{\pi}\int_0^{\pi} f(x)e^{-inx}dx + \frac{i}{\pi}\int_{\pi}^{0} f(x)e^{inx}dx = \frac{i}{\pi}\int_0^{\pi} f(x)\left[e^{-inx} - e^{inx}\right]dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx.$$

■

Let us return to the solution to the wave equation on the interval $[0, l]$ with fixed endpoints boundary condition. Summarizing the previous considerations we arrive at the following

Theorem 2.28 *Let $\phi(x) \in \mathcal{C}^2([0,l])$, $\psi(x) \in \mathcal{C}^1([0,l])$ be two arbitrary smooth functions. Then the solutions to the mixed problem (2.6.1), (2.6.2) for the wave equation is written in the form*

$$u(x,t) = \sum_{n\geq 1} \sin\frac{\pi n x}{l}\left(b_n \cos\frac{\pi a n t}{l} + \dot{b}_n \sin\frac{\pi a n t}{l}\right) \tag{2.6.5}$$

$$b_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{\pi n x}{l}\, dx, \quad \dot{b}_n = \frac{2}{\pi a n}\int_0^l \psi(x)\sin\frac{\pi n x}{l}\, dx.$$

Particular solutions to the wave equation giving a basis in the space of all solutions satisfying the boundary conditions (2.6.1) have the form

$$u_n^{(1)}(x,t) = \sin\frac{\pi n x}{l}\cos\frac{\pi a n t}{l}, \quad u_n^{(2)}(x,t) = \sin\frac{\pi n x}{l}\sin\frac{\pi a n t}{l}, \quad n = 1, 2, \dots \tag{2.6.6}$$

are called *standing waves*. Observe that these solutions have the separated form (2.5.39). The shape of these waves essentially does not change in time, only the size does change. In particular the location of the *nodes*

$$x_k = k\,\frac{l}{n}, \quad k = 0, 1, \dots, n \tag{2.6.7}$$

of the n-th solution $u_n^{(1)}(x,t)$ or $u_n^{(2)}(x,t)$ does not depend on time. The n-th standing waves (2.6.6) has $(n+1)$ nodes on the string. The solution takes zero values at the nodes at all times.

2.7 Energy of vibrating string

Let us consider the vibrating string with fixed points $x = 0$ and $x = l$. It is clear that the kinetic energy of the string at the moment t is equal to

$$K = \frac{1}{2}\int_0^l \rho\, u_t^2(x,t)\, dx. \tag{2.7.1}$$

Let us now compute the potential energy U of the string. By definition U is equal to the work done by the elastic force moving the string from the equilibrium $u \equiv 0$ to the actual position given by the graph $u(x)$. The motion can be described by the one-parameter family of curves

$$v(x; s) = s\, u(x) \tag{2.7.2}$$

where the parameter s changes from $s = 0$ (the equilibrium) to $s = 1$ (the position of the string). As we already know the vertical component of the force acting on the interval of the string (2.7.2) between x and $x + \Delta x$ is equal to

$$F = T\, \left(v_x(x + \Delta x; s) - v_x(x; s)\right) \simeq s\, T\, u_{xx}(x)\, \Delta x.$$

The work A to move the string from the position $v(x; s)$ to $v(x; s + \Delta s)$ is therefore equal to

$$A = -F \cdot [v(x; s + \Delta s) - v(x; s)] \simeq -s\, T\, u(x) \Delta x\, \Delta s$$

(the negative sign since the direction of the force is opposite to the direction of the displacement). The total work of the elastic forces for moving the string of length l from the equilibrium $s = 0$ to the given configuration at $s = 1$ is obtained by integration:

$$U = -\int_0^1 ds \int_0^l s\, T\, u_{xx}(x) u(x)\, dx = -\frac{1}{2} \int_0^l T\, u_{xx}(x) u(x)\, dx.$$

By definition this work is equal to the potential energy of the string. Integrating by parts and using the boundary conditions

$$u(0) = u(l) = 0$$

we finally arrive at the following expression for the potential energy:

$$U = \frac{1}{2} \int_0^l T\, u_x^2(x)\, dx. \tag{2.7.3}$$

Summarizing (2.7.1) and (2.7.3) gives the formula for the total energy $E = E(t)$ of the vibrating string at the moment t

$$E = K + U = \int_0^l \left(\frac{1}{2}\, \rho\, u_t^2(x, t) + \frac{1}{2}\, T\, u_x^2(x, t) \right) dx. \tag{2.7.4}$$

Exercise 2.29 *Prove that the same expression* (2.7.3) *holds true for the total work of elastic forces moving the string from the equilibrium to the given position* $u(x)$ *along an* arbitrary path

$$v(x; s), \quad v(x; 0) \equiv 0, \quad v(x; s) = u(x)$$

in the space of configurations.

It is understood that $v(x; t)$ *is a smooth function on* $[0, l] \times [0, 1]$.

We will now prove that the total energy E of vibrating string with fixed end points does not depend on time.

Lemma 2.30 *Let the function* $u(x, t)$ *satisfy the wave equation. Then the following identity holds true*

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\, \rho\, u_t^2(x, t) + \frac{1}{2}\, T\, u_x^2(x, t) \right) = \frac{\partial}{\partial x} \left(T\, u_x u_t \right). \tag{2.7.5}$$

Proof: A straightforward differentiation using $u_{tt} = a^2 u_{xx}$ yields

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\,\rho\, u_t^2(x,t) + \frac{1}{2}\,T\, u_x^2(x,t)\right) = \rho\, a^2 u_t u_{xx} + T\, u_x u_{xt}.$$

Since

$$a^2 = \frac{T}{\rho}$$

(see above) we rewrite the last equation in the form

$$= T\left(u_t u_{xx} + u_{tx} u_x\right) = T\, \left(u_t u_x\right)_x.$$

∎

Corollary 2.31 *Denote $E_{[a,b]}(t)$ the energy of a segment of vibrating string*

$$E_{[a,b]}(t) = \int_a^b \left(\frac{1}{2}\,\rho\, u_t^2(x,t) + \frac{1}{2}\,T\, u_x^2(x,t)\right)\, dx. \tag{2.7.6}$$

The following formula describes the dependence of this energy on time:

$$\frac{d}{dt} E_{[a,b]}(t) = T\, u_t u_x|_{x=b} - T\, u_t u_x|_{x=a}. \tag{2.7.7}$$

Remark 2.32 *In physics literature the quantity*

$$\frac{1}{2}\,\rho\, u_t^2(x,t) + \frac{1}{2}\,T\, u_x^2(x,t) \tag{2.7.8}$$

is called energy density*. It is equal to the energy of a small piece of the string from x to $x + dx$ at the moment t. The total energy of a piece of a string is obtained by integration of this density in x. Another important notion is the* flux density

$$-\,T\, u_t u_x. \tag{2.7.9}$$

The formula (2.7.7) says that the change of the energy of a given piece of the string for the time dt is given by the total flux though the boundary of the piece.

Finally we arrive at the *conservation law* of the total energy of a vibrating string with fixed end points.

Theorem 2.33 *The total energy (2.7.4) of the vibrating string with fixed end points does not depend on t:*

$$\frac{d}{dt} E = 0.$$

Proof: The formula (2.7.7) for the particular case $a = 0$, $b = l$ gives

$$\frac{d}{dt}E = T\ (u_t(l,t)u_x(l,t) - u_t(0,t)u_x(0,t)) = 0$$

since

$$u_t(0,t) = \partial_t u(0,t) = 0, \quad u_t(l,t) = \partial_t u(l,t) = 0$$

due to the boundary conditions $u(0,t) = u(l,t) = 0$. ■

The conservation law of total energy makes it evident that the vibrating string is a *conservative system.*

Exercise 2.34 *Derive the formula for the total energy and prove the conservation law for a vibrating string of finite length with free boundary conditions $u_x(0,t) = u_x(l,t) = 0$.*

Exercise 2.35 *Prove that the energy of the vibrating string represented as sum* (2.6.5) *of standing waves* (2.6.6) *is equal to the sum of energies of standing waves.*

The conservation of total energy can be used for proving uniqeness of solution for the wave equation. Indeed, if $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ are two solutions vanishing at $x = 0$ and $x = l$ with the same initial data. The difference

$$u(x,t) = u^{(2)}(x,t) - u^{(1)}(x,t)$$

solves wave equation, satisfies the same boundary conditions and has zero initial data $u(x,0) = \phi(x) = 0$, $u_t(x,0) = \psi(x) = 0$. The conservation of energy for this solution gives

$$E(t) = \int_0^l \left(\frac{1}{2}\,\rho\, u_t^2(x,t) + \frac{1}{2}\,T\, u_x^2(x,t)\right)\, dx = E(0) = \int_0^l \left(\frac{1}{2}\,\rho\,\psi^2(x) + \frac{1}{2}\,T\,\phi_x^2(x)\right)\, dx = 0.$$

Hence $u_x(x,t) = u_t(x,t) = 0$ for all x, t. Using the boundary conditions one concludes that $u(x,t) \equiv 0$,

2.8 Inhomogeneous wave equation: Duhamel principle

To give a heuristic motivation of the method we start by reminding that for solving linear first order ODEs

$$\dot{u}(t) + Lu(t) = g(t), \tag{2.8.1}$$

with L a constant (in t) we can use *variation of parameters* which gives the particular solution $u_p(t)$

$$u_p(t) = \mathrm{e}^{-Lt}\int_0^t \mathrm{e}^{Ls} g(s)\mathrm{d}s = \int_0^t \mathrm{e}^{-L(t-s)} g(s)\mathrm{d}s; \quad u_p(0) = 0. \tag{2.8.2}$$

Denoting the integrand of this latter equation by $f(t;s) = \mathrm{e}^{L(t-s)}g(s)$ we note that it is also a solution of the **homogeneous** ODE

$$\partial_t f(t;s) + Lf(t;s) = 0, \qquad f(t;s)\big|_{t=s} = g(s). \tag{2.8.3}$$

This shows that the particular solution (2.8.2) of the non-homogeneous equation (2.8.1) can be written as a superposition (integral) of *homogeneous* solutions with $g(s)$ is the initial value at $t = s$.

Similarly for second order ODEs:

$$\ddot{u}(t) + Lu(t) = g(t) \ . \tag{2.8.4}$$

A particular solution given by the variation of parameters formula appears in the form

$$u_p(t) = \int_0^t \frac{\sin(\sqrt{L}(t-s))}{\sqrt{L}} g(s)\mathrm{d}s \ , \quad u_p(0) = 0, \ \ \dot{u}_p(0) = 0. \tag{2.8.5}$$

Once more we observe that the integral of the above formula $f(t;s) = \frac{\sin(\sqrt{L}(t-s))}{\sqrt{L}} g(s)$ is the solution of the Cauchy problem

$$\partial_t^2 f(t;s) + Lf(t;s) = 0 \ ; \qquad f(t;s)\big|_{t=s} = 0, \qquad \partial_t f(t;s)\big|_{t=s} = g(s). \tag{2.8.6}$$

With appropriate interpretation, the same formulæ would hold if $u(t)$ is a function taking values in an arbitrary vector space (even infinite dimensional, formally) as long as L is a linear operator *independent of* t. Since ∂_x^2 could be construed as such, this motivates the following theorem

Theorem 2.36 (Duhamel formula (principle)) *Consider the inhomogeneous equation of the string with external forcing* $g(x,t) \in \mathcal{C}^0(\mathbb{R}^2)$*:*

$$u_{tt}(x,t) - a^2 u_{xx}(x,t) = g(x,t), \quad u(x,0) = 0 = u_t(x,0). \tag{2.8.7}$$

Then the solution is given by the formula

$$u(x,t) = \int_0^t F(x,t;s)\mathrm{d}s \tag{2.8.8}$$

where $F(x,t;s)$ *is the solution of the homogeneous wave equation with initial conditions at* $t = s$*;*

$$F_{tt} - a^2 F_{xx} = 0 \tag{2.8.9}$$

$$F(x,t;s)\big|_{t=s} = 0 \tag{2.8.10}$$

$$F_t(x,t;s)\big|_{t=s} = g(x,s) \tag{2.8.11}$$

Proof. We verify that the formula gives the solution; first of all we observe that from the conditions we deduce that (using the chain rule)

$$(F_t + F_s)\big|_{t=s} \equiv 0, \quad \forall x. \tag{2.8.12}$$

Now we can compute the derivatives of u as follows

$$u_{tt} = \partial_t \left(F(x;t,t) + \int_0^t F_t(x,t;s)\mathrm{d}s \right) \overset{(2.8.12)}{=} F_t(x,t;s)\big|_{s=t} + \int_0^t F_{tt}(x,t;s)\mathrm{d}s =$$

$$= g(x,t) + a^2 \int_0^t F_{xx}(x,t;s)\mathrm{d}x = g(x,t) + a^2 u_{xx}. \tag{2.8.13}$$

We need to verify the initial conditions: now, clearly $u(x,0)=0$ because of the integral. Secondly we have

$$u_t(x,0) = F(x,t;s)\big|_{t=s=0} + \int_0^0 F_t(x,t;s)\mathrm{d}s = 0. \tag{2.8.14}$$

This concludes the proof. ■

If we need to solve the nonhomogeneous wave equation with different initial conditions, we simply write the solution as the sum of the particular solution provided for by Duhamel's principle plus the solution of the homogeneous problem with the given initial conditions. See Problem 2.44.

Solution using D'Alembert's formula Combining Duhamel's principle (Thm. 2.36) with D'Alembert's formula (Thm. 2.2) we obtain

$$u(x,t) = \frac{1}{2a}\int_0^t \int_{x-a(t-s)}^{x+a(t-s)} g(\xi,s)\mathrm{d}\xi\mathrm{d}s. \tag{2.8.15}$$

Remark 2.37 *The integral in (2.8.15) has the following nice interpretation: the value of u at (x,t) in the spacetime plane, is the area integral of $g(x',t')$ over the whole characteristic cone at (x,t) up to $t=0$. (Picture on board!)*

2.9 The weak solutions of the wave equation

In some applications (and some exercises) it is convenient to extend the meaning of the wave equation to a larger class. As one can plainly see, the D'Alembert equation (Thm. (2.2)) is rather "agnostic" regarding the regularity class of the functions ϕ, ψ, as long as the integration makes sense. However it is not immediately clear what meaning to attribute to the differential equation itself if -say- ϕ is a piecewise continuous function.

For this reason we introduce the notion *weak solutions*, while we refer to the $\mathcal{C}^2$ solutions as *classical solutions.*

Definition 1 (Weak solutions of the wave equation) *A function $u(x,t)$ is called a* **weak solution** *of the wave equation $u_{tt} - a^2 u_{xx} = 0$ on $(x,t) \in \mathbb{R}\times\mathbb{R}$ if, for every $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ the following holds:*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} u(x,t)\big(\varphi_{tt}(x,t) - a^2\varphi_{xx}(x,t)\big)\mathrm{d}x\mathrm{d}t = 0 \tag{2.9.1}$$

This is accompanied with the definition of weak solution subject to IC and also external forge

Definition 2 *A function $u(x,t)$ is called a* **weak solution** *of the wave equation $u_{tt} - a^2 u_{xx} = g(x,t)$ on $(x,t) \in \mathbb{R} \times [0,\infty)$ subject to the initial conditions (IC)*

$$u(x,0) = \phi(x); \qquad u_t(x,0) = \psi(x) \tag{2.9.2}$$

if [5] *for every $\varphi \in \mathcal{C}_0^\infty\big(\mathbb{R} \times [0,\infty)\big)$ the following holds*

$$\int_0^\infty \int_{\mathbb{R}} u(\varphi_{tt} - a^2 \varphi_{xx}) \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}} \Big(\varphi_t(x,0)\phi(x) - \varphi(x,0)\psi(x)\Big) = \int_0^\infty \int_{\mathbb{R}} g\varphi \mathrm{d}x\mathrm{d}t \tag{2.9.3}$$

The motivation of these definitions relies on the notion of "distribution" that the reader may have already encountered. It is motivated by the following

Proposition 2.38 *If $u(x,t)$ is a classical solution of the forced DE + IC, then it is also a weak solution in the sense of Def 2.*

Proof. The proof consists of the following chain of identities. For an arbitrary $\varphi \in \mathcal{C}_0^\infty\big(\mathbb{R} \times [0,\infty)\big)$ let $R > 0$ be sufficiently large so that $\mathrm{supp}\varphi \subset [-R,R] \times [0,R]$. The value R is understood to be such that the support of φ does not intersect the left, right and top sides of the boundary of the rectangle $[-R,R] \times [0,R]$ but, of course, it may intersect the segment $(x,t) \in (-R,R) \times \{0\}$ (picture on board!)

$$\iint_{\mathbb{R}_+ \times \mathbb{R}} u\varphi = \iint_{\mathbb{R}_+ \times \mathbb{R}} \Big(u_{tt} - a^2 u_{xx}\Big)\varphi = \int_{\mathbb{R}} \mathrm{d}x \int_0^\infty \mathrm{d}t \, u_{tt}\varphi - a^2 \int_0^\infty \mathrm{d}t \int_{-R}^R \mathrm{d}x\mathrm{d}t \, u_{xx}\varphi \tag{2.9.4}$$

The inner integral in the second term can be integrated by parts twice without contribution from the boundary $x = -R, R$; the inner integral in the first term, on the other hand should be handled with some care:

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{d}x \int_0^\infty \mathrm{d}t u_{tt}\varphi = \int_{\mathbb{R}} \mathrm{d}x \int_0^R \mathrm{d}t \; u_{tt}\varphi = \int \mathrm{d}x (u_t\varphi)\Big|_{t=0}^{t=R} - \int_{\mathbb{R}} \mathrm{d}x \int_0^R \mathrm{d}t \; u_t\varphi_t = \\
= -\int \mathrm{d}x \, \psi\varphi\big|_{t=0} - \int \mathrm{d}x (u\varphi_t)\Big|_{t=0}^{t=R} + \int_{\mathbb{R}} \mathrm{d}x \int_0^R \mathrm{d}t \; u\varphi_{tt} = \\
= -\int \mathrm{d}x \, \psi\varphi\big|_{t=0} + \int \mathrm{d}x (\phi\varphi_t)\big|_{t=0} + \int_{\mathbb{R}} \mathrm{d}x \int_0^\infty \mathrm{d}t \; u\varphi_{tt}.
\end{aligned} \tag{2.9.5}$$

Recombining the terms yields

$$\iint_{\mathbb{R}_+ \times \mathbb{R}} u\varphi = -\int \mathrm{d}x \, \psi\varphi\big|_{t=0} + \int \mathrm{d}x (\phi\varphi_t)\big|_{t=0} + \int_{\mathbb{R}} \mathrm{d}x \int_0^\infty \mathrm{d}t \; u(\varphi_{tt} - a^2\varphi_{xx}) \tag{2.9.6}$$

This proves the statement. ∎

2.10 Exercises to Section 2

Exercise 2.39 *We know that a solution (weak or classical) of $u_{tt} - u_{xx} = 0$ is the sum of a left and right traveling waves: $u(x,t) = f(x-t) + g(x+t)$. Suppose now that f, g are only $\mathcal{C}^1_0(\mathbb{R})$ so that u is a weak(er) solution.*

1. *Show that for any $t \in \mathbb{R}$ fixed, the function $u(x,t)$ is compactly supported with respect to x.*
2. *Show that the energy*
$$E = \frac{1}{2}\int_{\mathbb{R}} (u_t^2 + u_x^2)\mathrm{d}x \tag{2.10.1}$$
is well defined (i.e. not infinite).
3. *Show that the energy is still conserved. Show also that the energy is the sum of the energy of the left and right traveling waves. Note that f, g are not assumed to be twice differentiable and hence you cannot use this for showing the conservation of energy.*

Exercise 2.40 *Prove a similar statement as Prop. 2.38 for the first definition of weak solution, Def. 1*

Exercise 2.41 *Give an appropriate definition of the notion of weak solution for the following DE+IC+BC for the finite string $x \in [0, \ell]$*

$$\begin{aligned}
&(DE) \;\; u_{tt} - u_{xx} = g,\\
&(IC) \;\; u(x,0) = \phi(x),\;\; u_t(x,0) = \psi(x),\\
&(BC) \;\; u(x,0) = 0 = u(\ell, 0)
\end{aligned} \tag{2.10.2}$$

Exercise 2.42 *Let $f(x)$ be a piecewise continuous function on $\mathbb{R}$. Show that $u(x,t) = f(x-t)$ is a weak solution of $u_{tt} - u_{xx} = 0$.*

Exercise 2.43 *For few instants of time $t \geq 0$ make a graph of the solution $u(x,t)$ to the wave equation with the initial data*

$$u(x,0) = 0, \quad u_t(x,0) = \begin{cases} 1, & x \in [x_0, x_1] \\ 0 & \text{otherwise} \end{cases}, \quad -\infty < x < \infty.$$

Exercise 2.44 *Solve the following DE + IC on the whole line $x \in \mathbb{R}$:*

$$u_{tt} - u_{xx} = x - t \tag{2.10.3}$$
$$u(x,0) = x^4 \tag{2.10.4}$$
$$u_t(x,0) = \sin(x) \tag{2.10.5}$$

Exercise 2.45 *For few instants of time $t \geq 0$ make a graph of the solution $u(x,t)$ to the wave equation on the half line $x \geq 0$ with the free boundary condition*

$$u_x(0,t) = 0$$

and with the initial data

$$u(x,0) = \phi(x), \quad u_t(x,0) = 0, \quad x > 0$$

where the graph of the function $\phi(x)$ is an isosceles triangle of height 1 and the base $[l, 3l]$.

Exercise 2.46 *For few instants of time $t \geq 0$ make a graph of the solution $u(x,t)$ to the wave equation on the half line $x \geq 0$ with the fixed point boundary condition*

$$u(0,t) = 0$$

and with the initial data

$$u(x,0) = 0, \quad u_t(x,0) = \begin{cases} 1, & x \in [l, 3l] \\ 0, & \text{otherwise} \end{cases}, \quad x > 0.$$

Exercise 2.47 *Prove that*

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2} \quad \text{for} \quad 0 < x < 2\pi.$$

Compute the sum of the Fourier series for all other values of $x \in \mathbb{R}$.

Exercise 2.48 *Compute the sums of the following Fourier series:*

$$\sum_{n=1}^{\infty} \frac{\sin 2nx}{2n}, \quad 0 < x < \pi;$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx, \quad |x| < \pi.$$

Exercise 2.49 *Prove that*

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx, \quad |x| < \pi.$$

Exercise 2.50 *Compute the sums of the following Fourier series:*

$$\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$$

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

Exercise 2.51 *Denote*

$$S_n(x) = \frac{4}{\pi} \sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}$$

the n-th partial sum of the Fourier series (2.5.35). Prove that

1) for any $x \in (-\pi, \pi)$

$$\lim_{n \to \infty} S_n(x) = \operatorname{sign} x.$$

2) Verify that the n-th partial sum has a maximum at

$$x_n = \frac{\pi}{2n}.$$

Hint: derive the following expression for the derivative

$$S_n'(x) = \frac{2}{\pi}\frac{\sin 2nx}{\sin x}.$$

3) Prove that

$$S_n(x_n) = \frac{2}{\pi}\sum_{k=1}^{n}\frac{\pi}{n}\cdot\frac{\sin\frac{(2k-1)\pi}{2n}}{\frac{(2k-1)\pi}{2n}} \to \frac{2}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx \simeq 1.17898$$

for $n \to \infty$.

Thus for the trigonometric series (2.5.35)

$$\limsup_{n\to\infty} S_n(x) > 1 \quad \text{for} \quad x > 0.$$

In a similar way one can prove that

$$\liminf_{n\to\infty} S_n(x) < -1 \quad \text{for} \quad x < 0.$$

Exercise 2.52 *Consider the DE $u_{tt} - u_{xx} = 0$ on the semi-infinite axis $x \in [0, \infty)$ with Neumann boundary conditions and the following IC:*

$$u(x,0) = \phi(x); \quad u_t(x,0) = \phi'(x) \tag{2.10.6}$$

where ϕ is the smooth compactly supported function

$$\phi(x) = \begin{cases} (x-1)^3(2-x)^3 & x = \in [1,2] \\ 0 & x \notin [1,2]. \end{cases} \tag{2.10.7}$$

Give a sketch of ϕ and describe the evolution of the string in the following three intervals of time:

$$t \in [0,1], \quad t \in [1,2], \quad t \geq 2. \tag{2.10.8}$$

Also answer the same question where the Neumann condition is replaced with a Dirichlet condition.

Chapter 3

Laplace equation

3.1 Ill-posedness of the Cauchy problem for the Laplace equation

In the study of various classes of solutions to the Cauchy problem for the wave equation we were able to establish

- *existence* of the solution in a suitable class of functions;
- *uniqueness* of the solution;
- *continuous dependence* of the solution on the initial data (see Exercise 2.3 above) with respect to a suitable topology.

One may ask whether these properties remain valid for all evolutionary PDEs satisfying conditions of the Cauchy – Kovalevskaya theorem?

Let us consider a counterexample found by J.Hadamard (1922). Changing the sign in the wave equation one arrives at an equation of *elliptic* type

$$u_{tt} + a^2 u_{xx} = 0. \tag{3.1.1}$$

(The equation (3.1.1) is usually called *Laplace equation*.) Does the change of the type of equation affect seriously the properties of solutions?

To be more specific we will deal with the periodic Cauchy problem

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \tag{3.1.2}$$

with two 2π-periodic smooth initial functions $\phi(x)$, $\psi(x)$. For simplicity let us choose $a = 1$. We will see that the solution to this Cauchy problem *does not* depend continuously on the initial data. To do this let us consider the following sequence of initial data: for any integer $k > 0$ denote $u_k(x,t)$ solution to the Cauchy problem

$$u(x,0) = 0, \quad u_t(x,0) = \frac{\sin k\,x}{k}. \tag{3.1.3}$$

The 2π-periodic solution can be expanded in Fourier series

$$u_k(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} [a_n(t)\cos nx + b_n(t)\sin nx]$$

with some coefficients $a_n(t)$, $b_n(t)$. Substituting the series into equation

$$u_{tt} + u_{xx} = 0$$

we obtain an infinite system of ODEs

$$\begin{aligned} \ddot{a}_n &= n^2 a_n \\ \ddot{b}_n &= n^2 b_n, \end{aligned}$$

$n = 0, 1, 2, \ldots$. The initial data for this infinite system of ODEs follow from the Cauchy problem (3.1.2):

$$\begin{aligned} &a_n(0) = 0, \quad \dot{a}_n = 0 \quad \forall n, \\ &b_n(0) = 0, \quad \dot{b}_n(0) = \begin{cases} 1/k, & n = k \\ 0, & n \neq k. \end{cases} \end{aligned}$$

The solution has the form

$$\begin{aligned} &a_n(t) = 0 \quad \forall n, \quad b_n(t) = 0 \quad \forall n \neq k \\ &b_k(t) = \frac{1}{k^2} \sinh kt. \end{aligned}$$

So the solution to the Cauchy problem (3.1.2) reads

$$u_k(x,t) = \frac{1}{k^2} \sin kx \, \sinh kt. \tag{3.1.4}$$

Using this explicit solution we can prove the following

Theorem 3.1 *For any positive ϵ, M, t_0 there exists an integer K such that for any $k \geq K$ the initial data (3.1.3) satisfy*

$$\sup_{x \in [0,2\pi]} \left(|u_k(x,0)| + |\partial_t u_k(x,0)| \right) < \epsilon \tag{3.1.5}$$

but the solution $u_k(x,t)$ at the moment $t = t_0 > 0$ satisfies

$$\sup_{x \in [0,2\pi]} \left(|u_k(x,t_0)| + |\partial_t u_k(x,t_0)| \right) \geq M. \tag{3.1.6}$$

Proof: Choosing an integer K_1 satisfying

$$K_1 > \frac{1}{\epsilon}$$

we will have the inequality (3.1.5) for any $k \geq K_1$. In order to obtain a lower estimate of the form (3.1.6) let us first observe that

$$\sup_{x \in [0,2\pi]} \left(|u_k(x,t)| + |\partial_t u_k(x,t)| \right) = \frac{1}{k^2} \sinh kt + \frac{1}{k} \cosh kt > \frac{e^{kt}}{k^2}$$

where we have used an obvious inequality

$$\frac{1}{k} > \frac{1}{k^2} \quad \text{for} \quad k > 1.$$

The function

$$y = \frac{e^x}{x^2}$$

is monotone increasing for $x > 2$ and

$$\lim_{x \to +\infty} \frac{e^x}{x^2} = +\infty.$$

Hence for any $t_0 > 0$ there exists x_0 such that

$$\frac{e^x}{x^2} > \frac{M}{t_0^2} \quad \text{for} \quad x > x_0.$$

Let K_2 be a positive integer satisfying

$$K_2 > \frac{x_0}{t_0}.$$

Then for any $k > K_2$

$$\frac{e^{k\,t_0}}{k^2} = t_0^2 \frac{e^{k\,t_0}}{(k\,t_0)^2} > t_0^2 \frac{e^{x_0}}{x_0^2} > M.$$

Choosing

$$K = \max(K_1, K_2)$$

we complete the proof of the Theorem. ■

The statement of the Theorem is usually referred to as *ill-posedness* of the Cauchy problem (3.1.1), (3.1.2).

A natural question arises: what kind of initial or boundary conditions can be chosen in order to uniquely specify solutions to Laplace equation without violating the continuous dependence of the solutions on the boundary/initial conditions?

3.2 Dirichlet and Neumann problems for Laplace equation on the plane

The *Laplace operator* in the d-dimensional Euclidean space is defined by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}. \tag{3.2.1}$$

The symbol (coinciding with the principal symbol) of this operator is equal to

$$-\left(\xi_1^2 + \cdots + \xi_d^2\right) < 0 \quad \text{for all} \quad \xi \neq 0.$$

So Laplace operator is an example of an *elliptic* operator.

In this section we will formulate the two main boundary value problems (b.v.p.'s) for the *Laplace equation*

$$\Delta u = 0, \quad u = u(x), \quad x \in \Omega \subset \mathbb{R}^d. \tag{3.2.2}$$

The solutions to the Laplace equation are called *harmonic functions* in the domain Ω.

We will assume that the boundary $\partial\Omega$ of the domain Ω is a smooth hypersurface. Moreover we assume that the domain Ω does not go to infinity, i.e., Ω belongs to some ball in $\mathbb{R}^d$. Denote $n = n(x)$ the unit external normal vector at every point $x \in \partial\Omega$ of the boundary.

Problem 1 (*Dirichlet problem*). Given a function $f(x)$ defined at the points of the boundary find a function $u = u(x)$ satisfying the Laplace equation on the internal part of the domain Ω and the boundary condition

$$u(x)|_{x\in\partial\Omega} = f(x) \tag{3.2.3}$$

on the boundary of the domain.

Problem 2 (*Neumann problem*). Given a function $g(x)$ defined at the points of the boundary find a function $u = u(x)$ satisfying the Laplace equation on the internal part of the domain Ω and the boundary condition

$$\left(\frac{\partial u(x)}{\partial n}\right)_{x\in\partial\Omega} = g(x) \tag{3.2.4}$$

on the boundary of the domain.

Example 1. For $d = 1$ the Laplace operator is just the second derivative

$$\Delta = \frac{d^2}{dx^2}.$$

The Dirichlet b.v.p. in the domain $\Omega = (a, b)$

$$u''(x) = 0, \quad u(a) = f_a, \quad u(b) = f_b$$

has an obvious unique solution

$$u(x) = \frac{f_b - f_a}{b - a}(x - a) + f_a.$$

The Neumann b.v.p. in the same domain

$$u''(x) = 0, \quad -u'(a) = g_a, \quad u'(b) = g_b$$

has solution only if

$$g_a + g_b = 0. \tag{3.2.5}$$

Example 2. In two dimensions the Laplace operator reads.

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{3.2.6}$$

Exercise 3.2 *Prove that in the polar coordinates*

$$\left.\begin{array}{rl} x = & r\cos\phi \\ y = & r\sin\phi \end{array}\right\} \tag{3.2.7}$$

the Laplace operator takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}. \tag{3.2.8}$$

In the particular case

$$\Omega = \{(x, y) \,|\, x^2 + y^2 < \rho^2\} \tag{3.2.9}$$

(a circle of radius ρ) the Dirichlet b.v.p. is formulated as follows: find a solution to the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u = u(x, y), \quad \text{for} \quad x^2 + y^2 < \rho^2 \tag{3.2.10}$$

satisfying the boundary condition

$$u\,|_{r=\rho} = f(\phi). \tag{3.2.11}$$

Here we represent the boundary condition defined on the boundary of the circle as a function depending only on the polar angle ϕ. Similarly, the Neumann problem consists of finding a solution to the Laplace equation satisfying

$$\left(\rho\,\frac{\partial u}{\partial r}\right)_{r=\rho} = g(\phi) \tag{3.2.12}$$

for a given function $g(\phi)$. The factor ρ in the left side is only a convenient normalization of the boundary data.

Let us return to the general d-dimensional case. The following identity will be useful in the study of harmonic functions.

Theorem 3.3 (Green's formula) . *For arbitrary smooth functions u, v on the* **closed and bounded** *domain $\bar{\Omega}$ with a piecewise smooth boundary $\partial\Omega$ the following identity holds true*

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}V + \int_\Omega u\,\Delta v \,\mathrm{d}V = \int_{\partial\Omega} u\,\frac{\partial v}{\partial n}\,\mathrm{d}S. \tag{3.2.13}$$

where $\partial v/\partial n$ denotes the directional derivative of v along the outer normal vector $\mathbf{n}$

Here

$$\nabla u \cdot \nabla v = \sum_{i=1}^d \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$$

is the inner product of the gradients of the functions,

$$dV = dx_1 \dots dx_d$$

is the Euclidean volume element, n the external normal and dS is the area element on the hypersurface $\partial\Omega$.

This identity is a consequence of another identity known as the **Divergence Theorem**:

Theorem 3.4 (Divergence theorem) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain (open and connected set) with piecewise smooth boundary $\partial\Omega$. Let $\vec{F} : \Omega \to \mathbb{R}^d$ be a vector field of class $\mathcal{C}^1(\Omega)$ and $\mathcal{C}^0(\overline{\Omega})$. Then*

$$\int_\Omega \operatorname{div} \vec{F} \, \mathrm{d}V = \int_{\partial\Omega} \vec{F} \cdot \mathbf{n} \, \mathrm{d}S \tag{3.2.14}$$

Example 1. For $d = 1$ and $\Omega = (a, b)$ the Green's formula reads

$$\int_a^b u_x \, v_x \, dx + \int_a^b u \, v_{xx} \, dx = u \, v_x|_a^b$$

since the oriented boundary of the interval consists of two points $\partial[a, b] = b - a$. This is an easy consequence of integration by parts.

Example 2. For $d = 2$ and a rectangle $\Omega = (a, b) \times (c, d)$ the Green's formula becomes

$$\int_\Omega (u_x v_x + u_y v_y) \, dx \, dy + \int_\Omega u \, (v_{xx} + v_{yy}) \, dx \, dy = \int_a^b (u \, v_y)_c^d \, dx + \int_c^d (u \, v_x)_a^b \, dy$$

(the sum of integrals over four pieces of the boundary $\partial\Omega$ stands in the right hand side of the formula).

Let us return to the general discussion of Laplace equation. The following corollary follows immediately from the Green's formula.

Corollary 3.5 *For a function u harmonic in a* bounded *domain Ω with a piecewise smooth boundary the following identity holds true*

$$\int_\Omega (\nabla u)^2 = \int_{\partial\Omega} \frac{1}{2} \partial_n u^2 \, dS. \tag{3.2.15}$$

Proof: This is obtained from (3.2.13) by choosing $u = v$. ■

Using this identity we can easily derive uniqueness of solution to the Dirichlet problem.

Theorem 3.6 *1) Let u_1, u_2 be two functions harmonic in the bounded domain Ω and smooth in the closed domain $\bar{\Omega}$ coinciding on the boundary $\partial\Omega$. Then $u_1 \equiv u_2$.*

2) Under the same assumptions about the functions u_1, u_2, if the normal derivatives on the boundary coincide

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n}$$

then the functions differ by a constant.

Proof: Applying to the difference $u = u_2 - u_1$ the identity (3.2.15) one obtains

$$\int_\Omega (\nabla u)^2 dV = 0$$

since the right hand side vanishes. Hence $\nabla u = 0$, and thus the function u is equal to a constant. The value of this constant on the boundary is zero. Therefore $u \equiv 0$. The second statement has a similar proof. ■

The following counterexample shows that the uniqueness does not hold true for *infinite* domains. Let Ω be the upper half plane:

$$\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, y > 0.\}.$$

The linear function $u(x, y) = y$ is harmonic in Ω and vanishes on the boundary. Clearly $u \neq 0$ on Ω.

Our goal is to solve the Dirichlet and Neumann boundary value problems. The first result in this direction is the following

Theorem 3.7 (Solution of the Laplace equation on a disk: Dirichlet problem) *For an arbitrary $\mathcal{C}^1$-smooth 2π-periodic function $f(\phi)$ the solution to the Dirichlet b.v.p.* (3.2.10), (3.2.11) *exists and is unique. Moreover it is given by the following formula*

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho\, r\, \cos(\phi - \psi) + r^2} f(\psi)\, d\psi. \tag{3.2.16}$$

The expression (3.2.16) for the solution to the Dirichlet b.v.p. in the circle is called *Poisson formula.*

Proof: We will first use the method of separation of variables in order to construct particular solutions to the Laplace equation. At the second step we will represent solutions to the Dirichlet b.v.p. as a linear combination of the particular solutions.

The method of *separation of variables* starts from looking for solutions to the Laplace equation in the form

$$u = R(r)\Phi(\phi). \tag{3.2.17}$$

Here r, ϕ are the polar coordinates on the plane (see Exercise 3.2 above). Using the form (3.2.8) we reduce the Laplace equation $\Delta u = 0$ to

$$R''(r)\Phi(\phi) + \frac{1}{r}R'(r)\Phi(\phi) + \frac{1}{r^2}R(r)\Phi''(\phi) = 0.$$

After division by $\frac{1}{r^2}R(r)\Phi(\phi)$ we can rewrite the last equation in the form

$$\frac{R''(r) + \frac{1}{r}R'(r)}{\frac{1}{r^2}R(r)} = -\frac{\Phi''(\phi)}{\Phi(\phi)}.$$

The left hand side of this equation depends on r while the right hand side depends on ϕ. The equality is possible only if both sides are equal to some constant λ. In this way we arrive at two ODEs for the functions $R = R(r)$ and $\Phi = \Phi(\phi)$

$$R'' + \frac{1}{r}R' - \frac{\lambda}{r^2}R = 0 \tag{3.2.18}$$

$$\Phi'' + \lambda\Phi = 0. \tag{3.2.19}$$

We have now to determine the admissible values of the parameter λ. To this end let us begin from the second equation (3.2.19). Its solutions have the form

$$\Phi(\phi) = \begin{cases} A\, e^{\sqrt{-\lambda}\,\phi} + B\, e^{-\sqrt{-\lambda}\,\phi}, & \lambda < 0 \\ A + B\,\phi, & \lambda = 0 \\ A\, \cos\sqrt{\lambda}\,\phi + B\, \sin\sqrt{\lambda}\,\phi, & \lambda > 0 \end{cases}.$$

Since the pairs of polar coordinates (r, ϕ) and $(r, \phi + 2\pi)$ correspond to the same point on the Euclidean plane the solution $\Phi(\phi)$ must be a 2π-periodic function. Hence we must discard the negative values of λ. Moreover λ must have the form

$$\lambda = n^2, \quad n = 0,\, 1,\, 2, \dots. \tag{3.2.20}$$

This gives

$$\Phi(\phi) = A\, \cos n\phi + B\, \sin n\phi. \tag{3.2.21}$$

The first ODE (3.2.18) for $\lambda = n^2$ becomes

$$R'' + \frac{1}{r}R' - \frac{n^2}{r^2}R = 0.$$

This is a particular case of Euler equation. One can look for solutions in the form

$$R(r) = r^k.$$

The exponent k has to be determined from the characteristic equation

$$k(k-1) + k - n^2 = 0$$

obtained by the direct substitution of $R = r^k$ into the equation. The roots of the characteristic equation are $k = \pm n$. For $n > 0$ this gives the general solution of the equation (3.2.18) in the form

$$R = a\, r^n + \frac{b}{r^n}$$

with two integration constants a and b. For $n = 0$ the general solution is

$$R = a + b\, \log r.$$

As the solution must be smooth at $r = 0$ one must always choose $b = 0$ for all n. In this way we arrive at the following family of particular solutions to the Laplace equation

$$u_n = r^n \left(a_n \cos n\phi + b_n \sin n\phi\right), \quad n = 0,\, 1,\, 2, \dots \tag{3.2.22}$$

We want now to represent any solution to the Dirichlet b.v.p. in the circle of radius ρ as a linear combination of these solutions:

$$
\begin{aligned}
& u = \frac{A_0}{2} + \sum_{n\geq 1} r^n \left(A_n \cos n\phi + B_n \sin n\phi\right) \\
& u|_{r=\rho} = f(\phi).
\end{aligned}
\tag{3.2.23}
$$

The boundary data function $f(\phi)$ must be a 2π-periodic function. Assuming this function to be $\mathcal{C}^1$-smooth let us expand it in Fourier series

$$
\begin{aligned}
f(\phi) &= \frac{a_0}{2} + \sum_{n\geq 1} \left(a_n \cos n\phi + b_n \sin n\phi\right) \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(\phi)\cos n\phi \, d\phi, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sin n\phi \, d\phi.
\end{aligned}
\tag{3.2.24}
$$

Comparison of (3.2.23) with (3.2.24) yields

$$
A_n = \frac{a_n}{\rho^n}, \quad B_n = \frac{b_n}{\rho^n},
$$

or, equivalently

$$
u = \frac{a_0}{2} + \sum_{n\geq 1} \left(\frac{r}{\rho}\right)^n \left(a_n \cos n\phi + b_n \sin n\phi\right). \tag{3.2.25}
$$

Recall that this formula holds true on the circle of radius ρ, i.e., for

$$
r \leq \rho.
$$

The last formula can be rewritten as follows:

$$
\begin{aligned}
u &= \frac{1}{\pi}\int_0^{2\pi} \left[\frac{1}{2} + \sum_{n\geq 1}\left(\frac{r}{\rho}\right)^n \left(\cos n\phi \cos n\psi + \sin n\phi \sin n\psi\right)\right] f(\psi)\, d\psi \\
&= \frac{1}{\pi}\int_0^{2\pi} \left[\frac{1}{2} + \sum_{n\geq 1}\left(\frac{r}{\rho}\right)^n \cos n(\phi-\psi)\right] f(\psi)\, d\psi.
\end{aligned}
$$

To compute the sum in the square bracket we represent it as a geometric series converging for $r < \rho$:

$$
\begin{aligned}
& \frac{1}{2} + \sum_{n\geq 1}\left(\frac{r}{\rho}\right)^n \cos n(\phi-\psi) = \frac{1}{2} + \mathrm{Re}\sum_{n\geq 1}\left(\frac{r}{\rho}\right)^n e^{in(\phi-\psi)} \\
& = \frac{1}{2} + \mathrm{Re}\,\frac{r\, e^{i(\phi-\psi)}}{\rho - r\, e^{i\,(\phi-\psi)}} = \frac{1}{2} + \frac{1}{2}\left(\frac{r\, e^{i(\phi-\psi)}}{\rho - r\, e^{i\,(\phi-\psi)}} + \frac{r\, e^{-i(\phi-\psi)}}{\rho - r\, e^{-i\,(\phi-\psi)}}\right) \\
& = \frac{1}{2}\frac{\rho^2 - r^2}{\rho^2 - 2\rho r\, \cos(\phi-\psi) + r^2}.
\end{aligned}
$$

■

In a similar way one can treat the Neumann boundary problem. However in this case one has to impose an additional constraint for the boundary value of the normal derivative (cf. (3.2.5) above in dimension 1).

Lemma 3.8 *Let v be a smooth function on the closed domain $\bar{\Omega}$ harmonic inside the domain. Then the integral of the normal derivative of v over the boundary $\partial\Omega$ vanishes:*

$$\int_{\partial\Omega} \frac{\partial v}{\partial n}\, dS = 0. \tag{3.2.26}$$

Proof: Applying the Green formula to the pair of functions $u \equiv 1$ and v one obtains

$$\int_{\Omega} \Delta v\, dV = \int_{\partial\Omega} \frac{\partial v}{\partial n}\, dS.$$

The left hand side of the equation vanishes since $\Delta v = 0$ in Ω. ■

Corollary 3.9 *The Neumann problem (3.2.4) can have a solution only if the boundary function g satisfies*

$$\int_{\partial\Omega} g\, dS = 0. \tag{3.2.27}$$

We will now prove, for the particular case of a circle domain in the dimension $d = 2$ that this necessary condition of solvability is also a sufficient one.

Theorem 3.10 (Solution of the Laplace equation on a disk: Neumann problem.) *For an arbitrary $\mathcal{C}^1$-smooth 2π-periodic function $g(\phi)$ satisfying*

$$\int_0^{2\pi} g(\phi)\, d\phi = 0 \tag{3.2.28}$$

the Neumann b.v.p. (3.2.10), (3.2.12) has a solution unique up to an additive constant. This solution can be represented by the following integral formula

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\rho^2}{\rho^2 - 2\rho\, r\, \cos(\phi-\psi) + r^2}\, g(\psi)\, d\psi. \tag{3.2.29}$$

Proof: Repeating the above arguments one arrives at the following expression for the solution $u = u(r,\phi)$:

$$\begin{aligned} u &= \frac{A_0}{2} + \sum_{n\geq 1} r^n \left(A_n \cos n\phi + B_n \sin n\phi\right) \\ \left(\rho \frac{\partial u}{\partial r}\right)_{r=\rho} &= g(\phi). \end{aligned} \tag{3.2.30}$$

Let us consider the Fourier series of the function $g(\phi)$

$$g(\phi) = \frac{a_0}{2} + \sum_{n\geq 1} \left(a_n \cos n\phi + b_n \sin n\phi \right).$$

Due to the constraint (3.2.28) the constant term vanishes:

$$a_0 = 0.$$

Comparing this series with the boundary condition (3.2.30) we find that

$$u(r,\phi) = \frac{A_0}{2} + \sum_{n\geq 1} \frac{1}{n} \left(\frac{r}{\rho}\right)^n \left(a_n \cos n\phi + b_n \sin n\phi \right)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \cos n\psi \, g(\psi)\, d\psi, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} \sin n\psi \, g(\psi)\, d\psi.$$

Here A_0 is an arbitrary constant. Combining the two last equations we arrive at the following expression:

$$u(r,\phi) = \frac{1}{\pi} \int_0^{2\pi} \sum_{n\geq 1} \frac{1}{n} \left(\frac{r}{\rho}\right)^n \cos n(\phi - \psi) g(\psi)\, d\psi. \tag{3.2.31}$$

It remains to compute the sum of the trigonometric series in the last formula.

Lemma 3.11 *Let R and θ be two real numbers, $R < 1$. Then*

$$\sum_{n=1}^{\infty} \frac{1}{n} R^n \cos n\theta = \frac{1}{2} \log \frac{1}{1 - 2R\cos\theta + R^2}. \tag{3.2.32}$$

Proof: The series under consideration can be represented as the real part of a complex series

$$\sum_{n=1}^{\infty} \frac{1}{n} R^n \cos n\theta = \mathrm{Re} \sum_{n=1}^{\infty} \frac{1}{n} R^n e^{in\theta}.$$

The latter can be written as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n} R^n e^{in\theta} = \int_0^R \sum_{n=1}^{\infty} \frac{1}{R} R^n e^{in\theta}\, dR.$$

We can easily compute the sum of the geometric series with the denominator $Re^{i\theta}$. Integrating we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} R^n e^{in\theta} = \int_0^R \frac{e^{i\theta}}{1 - Re^{i\theta}}\, dR = -\log\left(1 - Re^{i\theta}\right).$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n} R^n \cos n\theta = \frac{1}{2} \left[\log \frac{1}{1 - Re^{i\theta}} + \log \frac{1}{1 - Re^{-i\theta}} \right] = \frac{1}{2} \log \frac{1}{1 - 2R\cos\theta + R^2}.$$

■

Applying the formula of the Lemma to the series (3.2.31) we complete the proof of the Theorem.
■

3.3 Properties of harmonic functions: mean value theorem, the maximum principle

In this section we will establish, for the specific case of dimension $d = 2$, the two fundamental properties of harmonic functions.

Let $\Omega \subset \mathbb{R}^d$ be a domain. Recall that a point $x_0 \in \Omega$ is called *internal* if there exists a ball of some radius $R > 0$ with the centre at x_0 entirely belonging to Ω. For an internal point $x_0 \in \Omega$ denote

$$S^{d-1}(x_0, R) = \{x \in \mathbb{R}^d \,|\, |x - x_0| = R\}$$

a sphere of radius $R > 0$ with the center at x_0.

Remark 3.12 *The area a_{d-1} of the unit sphere in $\mathbb{R}^d$ can be computed with the following "trick": we start from the d–dimensional Gaussian integral*

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\|x\|^2} \mathrm{d}V = \pi^{\frac{d}{2}}. \tag{3.3.1}$$

Rewriting it in "spherical" coordinates it reads

$$\int_0^\infty r^{d-1} \mathrm{e}^{-r^2} \mathrm{d}r \int_{S^{d-1}} \mathrm{d}S = a_{d-1} \frac{1}{2} \Gamma\left(\frac{d}{2}\right) \tag{3.3.2}$$

Comparing the two formulas we obtain

$$a_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{3.3.3}$$

The radius is chosen small enough to guarantee that the sphere belongs to the domain Ω. Denote a_{d-1} the area of the unit sphere in $\mathbb{R}^d$. For any continuous function $f(x)$ on the sphere the *mean value* is defined by the formula

$$\bar{f} = \frac{1}{a_{d-1} R^{d-1}} \int_{S^{d-1}(x_0,R)} f(x)\, dS. \tag{3.3.4}$$

In the particular case of a constant function the mean value coincides with the value of the function.

For example, in dimension $d = 1$ the "sphere" consists of two points $x_0 \pm R$. The formula (3.3.3) for the area of the zero-dimensional sphere gives

$$a_0 = \frac{\pi^{1/2}}{\Gamma\left(\frac{3}{2}\right)} = 2.$$

So the mean value of a function is just the arithmetic mean value of the two numbers $f(x_0 \pm R)$:

$$\bar{f} = \frac{f(x_0 + R) + f(x_0 - R)}{2}.$$

In the next case $d = 2$ the sphere is just a circle of radius R with the centre at x_0. The area (i.e., the length) element is $dS = R\, d\phi$. The restriction of f to the circle is a 2π-periodic function $f(\phi)$. So the mean value on this circle is given by

$$\bar{f} = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi.$$

Theorem 3.13 *Let $u = u(x)$ be a function harmonic in a domain Ω. Then the mean value of u over a small sphere centered at a point $x_0 \in \Omega$ is equal to the value of the function at this point:*

$$u(x_0) = \frac{1}{a_{d-1}R^{d-1}} \int_{S^{d-1}(x_0,R)} u(x)\, dS. \tag{3.3.5}$$

Moreover we also have

$$u(x_0) = \frac{1}{V_d(R)} \int_{B_R(x_0)} u \, \mathrm{d}V \tag{3.3.6}$$

where $V_d(R) = a_{d-1}\frac{R^d}{d}$ is the volume of the ball of radius R.

Proof. We start with $d = 2$. Denote $f(\phi)$ the restriction of the harmonic function u onto the small circle $|x - x_0| = R$. By definition the function $u(x)$ satisfies the Dirichlet b.v.p. inside the circle:

$$\begin{aligned} &\Delta u(x) = 0, \quad |x - x_0| < R \\ &u(x)|_{|x-x_0|=R} = f(\phi). \end{aligned}$$

As we already know from the proof of Theorem 3.7 the solution to this b.v.p. can be represented by the Fourier series

$$u(r,\phi) = \frac{a_0}{2} + \sum_{n\geq 1} \left(\frac{r}{R}\right)^n (a_n \cos n\phi + b_n \sin n\phi) \tag{3.3.7}$$

for $r := |x - x_0| < R$ (cf. (3.2.25) above). In this formula a_n and b_n are the Fourier coefficients of the boundary function

$$f(\phi) = u(x)|_{|x-x_0|=R}.$$

In particular

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi$$

is the mean value of the function u on the circle. On the other side the value of the function u at the center of the circle can be evaluated substituting $r = 0$ in the formula (3.3.7):

$$u(x_0) = \frac{a_0}{2}.$$

Comparing the last two equations we arrive at (3.3.5).

For general dimension we can proceed as follows: Let $B_r(x_0)$ be the ball of radius r centered at $x_0 \in \Omega \subset \mathbb{R}^d$. Then

$$\begin{aligned} 0 = \int_{B_r(x_0)} \triangle u \overset{\text{Div. Thm.}}{=} \int_{\partial B_r(x_0)} \nabla_{\mathbf{n}} u \, \mathrm{d}S = r^{d-1} \int_{S^{d-1}} \frac{\partial}{\partial r} u(x_0 + ry) \mathrm{d}S(y) = \\ = r^{d-1} \frac{\partial}{\partial r} \int_{S^{d-1}} u(x_0 + ry) \mathrm{d}S(y) \end{aligned} \tag{3.3.8}$$

Now divide by the volume of the sphere $V_d = a_{d-1}\frac{r^d}{d}$ so that (denoting by $\fint$ the average)

$$0 = \fint_{B_r(x_0)} \triangle u = \frac{d}{r}\frac{\partial}{\partial r}\fint_{S^{d-1}} u(x_0 + ry)\mathrm{d}S(y) \tag{3.3.9}$$

The integral under differentiation is the average of u over the surface of the ball $B_r(x_0)$. Thus we conclude that

$$\fint_{\partial B_r(x_0)} u\mathrm{d}S = C(x_0) \tag{3.3.10}$$

is a constant independent of the radius of the ball (within the domain Ω). Since $u \in \mathcal{C}^2(\Omega)$ we know that it takes a maximum and minimum on $\partial B_d(x_0)$ (which is compact), and a simple continuity argument shows that, as $r \to 0$ the average must converge to $u(x_0)$. Thus $\fint_{\partial B_r(x_0)} u\mathrm{d}S = u(x_0)$.

The second formula is proven by integration of the first:

$$\int_{B_R(x_0)} u\mathrm{d}V = \int_0^R \left(\int_{S^{d-1}} u(x_0 + ry)\mathrm{d}S(y)\right) r^{d-1}\mathrm{d}r =$$
$$= a_{d-1}u(x_0)\int_0^R r^{d-1}\mathrm{d}r = V_d(R)u(x_0). \tag{3.3.11}$$

Dividing by the volume $V_d(R)$ concludes the proof. ■

Using the mean value theorem we will now prove another important property of harmonic functions, namely the *maximum principle.* Recall that a function $u(x)$ defined on a domain $\Omega \subset \mathbb{R}^d$ is said to have a *local maximum* at the point x_0 if the inequality

$$u(x) \leq u(x_0) \tag{3.3.12}$$

holds true for any $x \in \Omega$ sufficiently close to x_0. A *local minimum* is defined in a similar way.

Theorem 3.14 *Let a function $u(x)$ be harmonic in a bounded connected domain Ω and continuous in a closed domain $\bar{\Omega}$. Denote*

$$M = \sup_{x\in\partial\Omega} u(x), \quad m = \inf_{x\in\partial\Omega} u(x).$$

Then

1) $m \leq u(x) \leq M$ for all $x \in \Omega$;

2) if $u(x) = M$ or $u(x) = m$ for some internal point $x \in \Omega$ then the function u is constant.

Proof: It is based on the following Main Lemma.

Lemma 3.15 *Let the harmonic function $u(x)$ have a local maximum/minimum at an internal point $x_0 \in \Omega$. Then $u(x) \equiv u(x_0)$ on some neighborhood of the point x_0.*

Proof: Let us consider the case of a local maximum. Choosing a sufficiently small sphere with the centre at x_0 we obtain, according to the mean value theorem, that

$$u(x_0) = \frac{1}{a_{d-1}R^{d-1}} \int_{|x-x_0|=R} u(x)\, dS.$$

We can assume the inequality (3.3.12) holds true for all x on the sphere. So

$$u(x_0) = \frac{1}{a_{d-1}R^{d-1}} \int_{|x-x_0|=R} u(x)\, dS \le \frac{1}{a_{d-1}R^{d-1}} \int_{|x-x_0|=R} u(x_0)\, dS = u(x_0). \tag{3.3.13}$$

If there exists a point x sufficiently close to x_0 such that $u(x) < u(x_0)$ then also the inequality (3.3.13) is strict. Such a contradiction shows that the function $u(x)$ takes constant values on some ball with the centre at x_0. The case of a local minimum can be treated in a similar way. ∎

Let us return to the proof of the Theorem. Denote

$$M' = \sup_{x\in\bar\Omega} u(x)$$

the maximum of the function u continuous on the compact $\bar\Omega$. We want to prove that $M' \le M$. Indeed, if $M' > M$ then there exists an internal point $x_0 \in \Omega$ such that $u(x_0) = M'$. Denote $\Omega' \subset \Omega$ the set of points x of the domain where the function u takes the same value M'. According to the Main Lemma this subset is open. Clearly it is also closed and nonempty. Hence $\Omega' = \Omega$ since the domain is connect. In other words the function is constant everywhere in Ω. Because of continuity it takes the same value M' at the points of the boundary $\partial\Omega$. Hence $M' \le M$. The contradiction we arrived at shows that the value of a harmonic function at an internal point of the domain cannot be bigger than the value of this function on the boundary of the domain. Moreover if the harmonic function takes the value M at an internal point then it is constant. In a similar way we prove that a non-constant harmonic function cannot have a minimum outside the boundary of the domain. ∎

Corollary 3.16 *Given two functions $u_1(x)$, $u_2(x)$ harmonic in a bounded domain Ω and continuous in the closed domain $\bar\Omega$. If*

$$|u_1(x) - u_2(x)| \le \epsilon \quad \text{for} \quad x \in \partial\Omega$$

then

$$|u_1(x) - u_2(x)| \le \epsilon \quad \text{for any} \quad x \in \Omega$$

Proof: Denote

$$u(x) = u_1(x) - u_2(x).$$

The function u is harmonic in Ω and continuous in $\bar\Omega$. By assumption we have $-\epsilon \le u(x) \le \epsilon$ for any $x \in \partial\Omega$. So

$$-\epsilon \le \inf_{x\in\partial\Omega} u(x), \quad \sup_{x\in\partial\Omega} u(x) \le \epsilon.$$

According to the maximum principle it must be also

$$-\epsilon \le \inf_{x\in\Omega} u(x), \quad \sup_{x\in\Omega} u(x) \le \epsilon.$$

∎

The Corollary implies that the the solution to the Dirichlet boundary value problem, if exists, *depends continuously* on the boundary data.

3.3.1 Boundary problem on annuli

The Poisson kernels for the Dirichlet and Neumann boundary conditions on circles does not work for other domains. We consider here an annulus $\Omega := \{\rho^2 < x^2 + y^2 < R^2\}$.

Since the domain does not contain the origin, the same considerations already used allow us to say that any harmonic function on Ω must take the form

$$u(r,\theta) = \frac{A_0}{2} + \frac{C_0}{2}\ln r + \sum_{n\geq 1} r^n \Big(A_n \cos(n\theta) + B_n \sin(n\theta)\Big) + \frac{C_n \cos(n\theta) + D_n \sin(n\theta)}{r^n} \tag{3.3.14}$$

There are clearly four types of boundary conditions: D-D, D-N, N-D, N-N, where D stands for Dirichlet and N for Neumann. Here we consider only D-D.

Suppose we want to find the kernel for D-D BCs

$$\triangle u = 0, \quad (x,y) \in \Omega \tag{3.3.15}$$

$$u\Big|_{r=\rho} = f(\theta); \quad u\Big|_{r=R} = g(\theta). \tag{3.3.16}$$

Let the Fourier expansion of f, g be

$$f = \frac{\alpha_0}{2} + \sum_{n\geq 1} \alpha_n \cos(n\theta) + \beta_n \sin(n\theta); \tag{3.3.17}$$

$$g = \frac{\gamma_0}{2} + \sum_{n\geq 1} \gamma_n \cos(n\theta) + \delta_n \sin(n\theta). \tag{3.3.18}$$

The coefficients A_n, B_n, C_n, D_n must solve the system

$$\begin{cases} A_0 + C_0 \ln(\rho) = \alpha_0 \\ A_0 + C_0 \ln(R) = \gamma_0 \\ A_n \rho^n + \frac{C_n}{\rho^n} = \alpha_n \\ B_n \rho^n + \frac{D_n}{\rho^n} = \beta_n \\ A_n R^n + \frac{C_n}{R^n} = \beta_n \\ B_n R^n + \frac{D_n}{R^n} = \delta_n \end{cases} \tag{3.3.19}$$

It is more practical, in concrete problems, to solve directly the system rather than writing a kernel.

3.3.2 Laplace equation on rectangles

Consider the equation

$$D.E.: \qquad \triangle u = 0, \quad (x,y) \in [0,L] \times [0,M] \tag{3.3.20}$$

$$B.C.: \qquad \begin{cases} u(x,0) = f(x) \\ u(0,y) = h(y) \end{cases} \qquad \begin{cases} u(x,M) = g(x) \\ u(L,y) = k(y) \end{cases} \tag{3.3.21}$$

The B.C. are assumed to be continuous; so, for example $f(0) = h(0)$ and so on. We consider here the simpler case where $f(0) = f(M) = h(0) = h(L) = g(0) = g(L) = k(0) = k(M) = 0$ so that each of the functions f, h, g, k admits periodic odd extensions to continuous functions of periods $2L$ or $2M$.

Namely we assume that all of them have a sin Fourier series representation:

$$f(x) = \sum_{n\geq 1} a_n \sin\left(n\pi \frac{x}{L}\right) \qquad g(x) = \sum_{n\geq 1} b_n \sin\left(n\pi \frac{x}{L}\right) \tag{3.3.22}$$

$$h(x) = \sum_{n\geq 1} c_n \sin\left(n\pi \frac{y}{M}\right) \qquad k(x) = \sum_{n\geq 1} d_n \sin\left(n\pi \frac{y}{M}\right) \tag{3.3.23}$$

Consider first the problem where $g = h = k \equiv 0$; if we solve this BVP, then we can analogously solve the others and the complete solution will simply be the sum of the various solutions.

First we look for factorized solutions $u(x, y) = X(x)Y(y)$; plugging into the equation yields separation of variables

$$X''Y + XY'' = 0 \qquad X'' = -\lambda X; \qquad Y'' = \lambda Y. \tag{3.3.24}$$

Depending on the sign of λ we have various possibilities. Since we must have $u(0, y) = u(L, y) = 0$ we quickly conclude that $-\lambda = \frac{n^2\pi^2}{L}$ with $n \in \mathbb{N}$, and we arrive at possible solutions

$$u_n(x, y) = \sin\left(\frac{n\pi x}{L}\right)\left(A_n \mathrm{e}^{\frac{n\pi y}{L}} + B_n \mathrm{e}^{-\frac{n\pi y}{L}}\right) \tag{3.3.25}$$

Imposing also that $u_n(x, M) = 0$ gives

$$u_n(x, y) = \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi(M - y)}{L}\right) \tag{3.3.26}$$

so that

$$u(x, y) = \sum_{n\geq 1} A_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi(M - y)}{L}\right) \tag{3.3.27}$$

Finally, imposing $u(x, 0) = f(x)$ yields:

$$u(x, y) = \sum_{n\geq 1} \frac{a_n}{\sinh\left(\frac{n\pi M}{L}\right)} \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi(M - y)}{L}\right) \tag{3.3.28}$$

Solution of the full problem. Therefore we have the solution of the full problem as follows:

$$\begin{aligned}
u(x, y) = &\sum_{n\geq 1} \frac{a_n}{\sinh\left(\frac{n\pi M}{L}\right)} \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi(M - y)}{L}\right) + && (3.3.29)\\
&+ \sum_{n\geq 1} \frac{b_n}{\sinh\left(\frac{n\pi M}{L}\right)} \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right) + && (3.3.30)\\
&+ \sum_{n\geq 1} \frac{c_n}{\sinh\left(\frac{n\pi L}{M}\right)} \sin\left(\frac{n\pi y}{M}\right) \sinh\left(\frac{n\pi(L - x)}{M}\right) + && (3.3.31)\\
&+ \sum_{n\geq 1} \frac{d_n}{\sinh\left(\frac{n\pi L}{M}\right)} \sin\left(\frac{n\pi y}{M}\right) \sinh\left(\frac{n\pi x}{M}\right) && (3.3.32)
\end{aligned}$$

3.3.3 Poisson equation

The Poisson equation is the non-homogeneous version of the Laplace equation

$$\triangle u(x) = g(x) \tag{3.3.33}$$

possibly subject to some boundary conditions.

Note that if $\Omega = \mathbb{R}^d$ typically one requires g to be either compactly supported or decaying at infinity. Uniqueness of a solution is then based on the following Lemma left as exercise

Lemma 3.17 *Let $u \in \mathcal{C}^2(\mathbb{R}^d)$ be harmonic. If $\lim_{|x|\to\infty} u(\vec{x}) = 0$ then u vanishes identically.*

The Lemma 3.17 allows to replace the Dirichlet conditions on a finite domain with an "asymptotic" Dirichlet condition.

The solution can be found according to the general philosophy of finding a particular solution of the non-homogeneous equation and then adding a suitable solution of the homogeneous equation that also takes care of the boundary conditions.

We start with the Lemma

Lemma 3.18 *The functions*

$$G_1(x) = \frac{1}{2}|x|, \qquad x \in \mathbb{R}^1 \tag{3.3.34}$$

$$G_2(\vec{x}) = \frac{1}{2\pi}\ln(|\vec{x}|), \qquad \vec{x} \in \mathbb{R}^2 \setminus \{\vec{0}\} \tag{3.3.35}$$

$$G_d(\vec{x}) = -\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}|\vec{x}|^{d-2}}, \qquad \vec{x} \in \mathbb{R}^d \setminus \{\vec{0}\},\ d \geq 3 \tag{3.3.36}$$

are all harmonic in $\mathbb{R}^d \setminus \{\vec{0}\}$. Here the multiplicative constants are chosen for later convenience.

Observe that all G_d's are functions only of the distance from the origin; furthermore the formula for G_d gives the same result for $d = 1$. For $d = 2$ the function G_2 is the limit

$$G_2(r) = \lim_{d=2}\left(G_d(r) + \frac{1}{2\pi(d-2)} + \frac{\gamma + \ln(\pi)}{4\pi}\right) \tag{3.3.37}$$

where $\gamma \simeq 0.5772...$ here is the Euler–Mascheroni constant (this is an example of *renormalization*).

Exercise 3.19 *Prove Lemma 3.18.*

Definition 3 *The functions G_d are called "Green functions" for the Laplace operator in d–dimensions.*

Remark 3.20 *For the readers who know what the Dirac delta distribution is, we can say that the Green's functions of the Laplace operator are function that satisfy the following equation in the distributional sense (which is precisely what we prove below):*

$$\triangle_y G_d(y-x) = \delta_x^d(y) \tag{3.3.38}$$

where $\delta_x^d(y)$ denotes the Dirac distribution in the variable y in d–dimensions supported at $y = x$.

Remark 3.21 (Connection with Maxwell's equations of electromagnetism) *Maxwell's equations are a set of PDEs for two 3–dimensional vector-fields $\mathbb{E}(\vec{x},t), \mathbb{B}(\vec{x},t)$. (electric/magnetic fields). They read:*

$$\operatorname{div} \mathbb{E} = \frac{\rho(\vec{x},t)}{\epsilon_0} \tag{3.3.39}$$

$$\operatorname{div} \mathbb{B} = 0 \tag{3.3.40}$$

$$\operatorname{curl}\mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t} \tag{3.3.41}$$

$$\operatorname{curl}\mathbb{B} = \mu_0 \left(\epsilon_0 \frac{\partial \mathbb{E}}{\partial t} + \mathbb{J}(x,t)\right) \tag{3.3.42}$$

where ρ is the density of charge per unit volume, $\mathbb{J}$ is the electric current, ϵ_0 is the permittivity of space (dielectric constant) and μ_0 the permeability of space (magnetic constant).

If the sources $\rho, \mathbb{J}$ are independent of time and we seek for static solutions (independent of time) we see that $\operatorname{curl} \mathbb{E} = 0$ and hence (in $\mathbb{R}^3$) we can write $\mathbb{E} = -\nabla V$ (the sign is conventional), where V is the electrostatic potential.

Thus the potential solves the Poisson equation $\triangle V = -\frac{\rho(\vec{x})}{\epsilon_0}$.

You may also notice that $G_3(\vec{x})$ is (up to a suitable constant) the Coulomb potential for an isolated point-like charge placed at the origin.

For these reasons, the study of the Laplace/Poisson equation is usually part of the branch of mathematics called potential theory.

Proposition 3.22 *Let $g(\vec{x})$ be $\mathcal{C}_0^1(\mathbb{R}^d)$. Then*

$$u(\vec{x}) := \int_{\mathbb{R}^d} G_d(\vec{y}-\vec{x}) g(\vec{y}) \mathrm{d}V(\vec{y}) \tag{3.3.43}$$

is a solution of the Poisson equation $\triangle u(\vec{x}) = g(\vec{x})$.

Proof. We sketch the proof (we discount some analytical details for simplicity).

First of all we observe that the integral is well defined; this is seen by passing to polar coordinates centered at x and writing $\vec{y} = \vec{x} + \rho \mathbf{n}$, $\mathrm{d}V(y) = \rho^{d-1}\mathrm{d}\rho \mathrm{d}S(\mathbf{n})$. One can also see that it is possible to differentiate G_d with respect to $\vec{x}$ **once** and still have a convergent integral.

With that in mind we can integrate by parts

$$\nabla_x u(x) = \nabla_x \int_{\mathbb{R}^d} G_d(\vec{y}-\vec{x})g(\vec{y})\mathrm{d}V(\vec{y}) = -\int_{\mathbb{R}^d} \nabla_y G_d(\vec{y}-\vec{x})g(\vec{y})\mathrm{d}V(\vec{y}) = \\ = \int_{\mathbb{R}^d} G_d(\vec{y}-\vec{x})\nabla_y g(\vec{y})\mathrm{d}V(\vec{y}) \tag{3.3.44}$$

Now we can compute the divergence:

$$\triangle u = \int_{\mathbb{R}^d} \nabla_x G_d(\vec{y}-\vec{x})\nabla_y g(\vec{y})\mathrm{d}V(\vec{y}) = -\int_{\mathbb{R}^d} \nabla_y G_d(\vec{y}-\vec{x})\nabla_y g(\vec{y})\mathrm{d}V(\vec{y}) \tag{3.3.45}$$

Now the integral can be split into $\mathbb{R}^d \setminus B_\epsilon(\vec{x})$ and $B_\epsilon(\vec{x})$; since the value is independent of ϵ, we are allowed to take the limit as $\epsilon \to 0^+$:

$$-\int_{\mathbb{R}^d} \nabla_y G_d(\vec{y}-\vec{x})\nabla_y g(\vec{y})\mathrm{d}V(\vec{y}) = \\ = \lim_{\epsilon\to 0^+}\left(\int_{\mathbb{R}^d\setminus B_\epsilon(x)} + \int_{B_\epsilon(\vec{x})}\right)\nabla_y G_d(\vec{y}-\vec{x})\nabla_y g(\vec{y})\mathrm{d}V(\vec{y}) \tag{3.3.46}$$

Since the integrand is integrable, the second limit tends to zero and we reach the conclusion that

$$\triangle u(\vec{x}) = -\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^d\setminus B_\epsilon(x)} \nabla_y G_d(\vec{y}-\vec{x})\nabla_y g(\vec{y})\mathrm{d}V(\vec{y}) \tag{3.3.47}$$

Applying Thm. 3.3 again to the first integral and keeping in mind that $\triangle_y G_d(y-x) = 0$ for $y \in \mathbb{R}^d \setminus B_\epsilon(x)$, we get

$$\triangle u(x) = -\lim_{\epsilon\to 0^+}\int_{\partial B_\epsilon(x)} \nabla_{\mathbf{n}} G_d(\vec{y}-\vec{x})\mathfrak{g}(\vec{y})\mathrm{d}S(\vec{y}) \tag{3.3.48}$$

The normal $\mathbf{n}$ is the normal pointing *towards* x (the outer normal of $\mathbb{R}^d \setminus B_\epsilon(x)$) and the gradient is with respect to y

$$-\nabla_{\mathbf{n}} G(\vec{y}-\vec{x})\big|_{|\vec{y}-\vec{x}|=\epsilon|} = \begin{cases} \partial_r G_d|_{r=\epsilon} = \dfrac{(d-2)\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}\epsilon^{d-1}} = \dfrac{1}{\alpha_{d-1}\epsilon^{d-1}} & d \geq 3 \\ \partial_r G_d|_{r=\epsilon} = \dfrac{1}{2\pi\epsilon} & d = 2 \\ \partial_r G_d|_{r=\epsilon} = \dfrac{1}{2} & d = 1 \end{cases} \tag{3.3.49}$$

In all cases $d = 1, 2, \ldots$ we have

$$\triangle u(x) = \lim_{\epsilon\to 0^+}\frac{1}{\alpha_{d-1}\epsilon^{d-1}}\int_{\partial B_\epsilon(x)} g(y)\mathrm{d}S(y) \tag{3.3.50}$$

Since $g(y)$ is continuous at $y = x$, its average on the surface of the ϵ–sphere at x tends to $g(x)$ as ϵ tends to zero. ■

Green's functions for domains

The Green functions presented in Lemma 3.18 allow to solve the Poisson equation on $\mathbb{R}^d$ (i.e. unbounded domains). For domains Ω with boundary the corresponding Green's function is, using the distributional notation,

$$\triangle_y G_\Omega(y,x) = \delta_x(y) \qquad G_\Omega(y,x)\Big|_{y\in\partial\Omega} = 0 \tag{3.3.51}$$

for the Dirichlet problem (analogous formulation for the Neumann problem). In other words, they allow to solve the Poisson equation

$$\triangle u = g; \qquad u\Big|_{\partial\Omega} = 0, \tag{3.3.52}$$

for $g \in \mathcal{C}_0^\infty(\Omega)$. In general these Green functions are not invariant under translations.

While the general theory is beyond the scope of the present course, we present here a simple example of the Green functions of special domains in $\mathbb{R}^2$.

It is convenient to identify $\mathbb{R}^2 \simeq \mathbb{C}$ and write a point in complex notation (see also next section)

$$z = x + iy. \tag{3.3.53}$$

Definition 4 *Given a domain $\Omega \subset \mathbb{C} \simeq \mathbb{R}^2$ the* **Green's function** *$G_\Omega(z;w)$ is a function defined for $w \in \Omega$, $z \in \Omega \setminus \{w\}$ satisfying the following properties:*

1. *$G_\Omega(z;w)$ is harmonic with respect to z in $\Omega \setminus \{w\}$;*
2. *$G_\Omega(z;w) - \frac{1}{2\pi}\ln|z-w|$ extends to a harmonic function with respect to z in the whole Ω.*
3. *$G_\Omega(z;w)$ extends to a continuous function for $z \in \overline{\Omega}\setminus\{w\}$ and $G_\Omega(z;w) = 0$ for $z \in \partial\Omega$.*

The complex conjugation geometrically represents the reflection around the real axis. Let $\mathbb{H} = \{z;\ \ \Im(z) > 0\}$ and define

$$G_\mathbb{H}(z,w) = G(z-w) - G(z-w^\star) = \frac{1}{2\pi}\ln\frac{|z-w|}{|z-w^\star|}. \tag{3.3.54}$$

Note that if $z \in \mathbb{R}$ then $G_\mathbb{H}(z,w) = 0$ for any $w \in \mathbb{H}$.

Proposition 3.23 *The function $G_\mathbb{H}$ is the Green's function of the upper half plane with Dirichlet boundary conditions; namely, for any $g \in \mathcal{C}_0^\infty(\mathbb{H})$, the solution of the Poisson-Dirichlet problem*

$$\triangle u = g \quad u(x,0) = 0 \tag{3.3.55}$$

is given by

$$u(z) = \int_{\mathbb{H}} \frac{g(w)}{2\pi} \ln \frac{|z-w|}{|z-\overline{w}|} \mathrm{d}^2 w \tag{3.3.56}$$

where $\mathrm{d}^2 w$ *denotes the Lebesgue area measure in* $\mathbb{C} \simeq \mathbb{R}^2$*.*

Similarly one can write the Green's function for the unit disk (or any other disk by simple arguments)

Proposition 3.24 *Let* $\mathbb{D} = \{|z| < 1\} \subset \mathbb{C} \simeq \mathbb{R}^2$ *and define*

$$G_{\mathbb{D}}(z,w) := \frac{1}{2\pi} \ln \frac{|z-w|}{|w| \left| z - \frac{1}{\overline{w}} \right|}, \qquad z, w \in \mathbb{D}. \tag{3.3.57}$$

Then $G_{\mathbb{D}}$ *is the Green's function of* $\mathbb{D}$ *with Dirichlet boundary conditions.*

3.4 Harmonic functions on the plane and complex analysis

In solving the wave equation $u_{xx} - u_{yy}$ we have factorized the wave operator into two derivatives along the characteristic directions:

$$\partial_x^2 - c^2 \partial_y^2 = (\partial_x - c\partial_y)(\partial_x + c\partial_y) \tag{3.4.1}$$

so that one easily concludes that the solutions are sums of functions of $x + cy$ and $x - cy$.

On a formalistic level we may assume that $c = i = \sqrt{-1}$ and proceed in the same way:

$$\partial_x^2 + \partial_y^2 = (\partial_x - i\partial_y)(\partial_x + i\partial_y) \tag{3.4.2}$$

This (heuristic) observation ushers the methods of complex analysis into the study of the Laplace equation in two-dimensions. In a certain sense, as we shall see momentarily, this is also correct.

First of all we identify $\mathbb{R}^2 \simeq \mathbb{C}^2$ via the obvious map $(x,y) \mapsto z = x + iy$. Then we recall that a differentiable complex valued function $f(x,y) = u(x,y) + iv(x,y)$ on a domain in $\mathbb{R}^2$ is called *holomorphic* if it satisfies the following system of *Cauchy – Riemann equations*

$$\left.\begin{aligned} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} &= 0 \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= 0 \end{aligned}\right\} \tag{3.4.3}$$

or, in complex form

$$\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0. \tag{3.4.4}$$

Introducing complex combinations of the Euclidean coordinates

$$z = x + iy \qquad \bar{z} = x - iy$$

we can also introduce the following two vector fields.

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right). \tag{3.4.5}$$

Note that, by construction

$$\frac{\partial}{\partial z} z = 1 = \frac{\partial}{\partial \bar{z}}\bar{z}, \qquad \frac{\partial}{\partial z}\bar{z} = 0 = \frac{\partial}{\partial z}\bar{z}. \tag{3.4.6}$$

so that, in a certain sense, we can view z and $\bar{z}$ as independent coordinates.

With the aid of the vectors (3.4.5) the Cauchy – Riemann equations can be rewritten in the form

$$\frac{\partial f}{\partial \bar{z}} = 0. \tag{3.4.7}$$

Example 3.25 Let $f(x, y)$ be a polynomial

$$f(x, y) = \sum_{k,l} a_{kl} x^k y^l.$$

It is a holomorphic function *iff*, after the substitution

$$x = \frac{z + \bar{z}}{2}$$
$$y = \frac{z - \bar{z}}{2i}$$

there will be no dependence on $\bar{z}$:

$$\sum_{k,l} a_{kl} \left(\frac{z + \bar{z}}{2}\right)^k \left(\frac{z - \bar{z}}{2i}\right)^l = \sum_m c_m z^m.$$

In that case the result will be a polynomial in z. For example a quadratic polynomial

$$f(x, y) = ax^2 + 2bxy + cy^2$$

is holomorphic *iff* $a + c = 0$ and $b = \frac{i}{2}(a - c)$.

More generally holomorphic functions are denoted $f = f(z)$. The partial derivative $\partial/\partial z$ of a holomorphic function is denoted df/dz or $f'(z)$. One can also define *antiholomorphic* functions $f = f(\bar{z})$ satisfying equation

$$\frac{\partial f}{\partial z} = 0. \tag{3.4.8}$$

Notice that the complex conjugate $\overline{f(z)}$ to a holomorphic function is an antiholomorphic function.

From complex analysis it is known that any function f holomorphic on a neighborhood of a point z_0 is also a *complex analytic* function, i.e., it can be represented as a sum of a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{3.4.9}$$

convergent uniformly and absolutely for sufficiently small $|z - z_0|$. In particular it is continuously differentiable any number of times. Its real and imaginary parts $u(x, y)$ and $v(x, y)$ are infinitely smooth functions of x and y.

Theorem 3.26 *The real and imaginary parts of a function holomorphic in a domain Ω are harmonic functions on the same domain.*

Proof: Differentiating the first equation in (3.4.3) in x and the second one in y and adding we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly, differentiating the second equation in x and subtracting the first one differentiated in y gives

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

■

Corollary 3.27 *For any integer $n \geq 1$ the functions*

$$\operatorname{Re} z^n \quad \text{and} \quad \operatorname{Im} z^n \tag{3.4.10}$$

are polynomial solutions to the Laplace equation.

Polynomial solutions to the Laplace equation are called *harmonic polynomials.* We obtain a sequence of harmonic polynomials

$$x,\ y,\ x^2 - y^2,\ xy,\ x^3 - 3xy^2,\ 3x^2 y - y^3, \dots.$$

Observe that the harmonic polynomials of degree n can be represented in the polar coordinates r, ϕ as

$$\operatorname{Re} z^n = r^n \cos n\phi, \quad \operatorname{Im} z^n = r^n \sin n\phi.$$

These are exactly the same functions we used to solve the main boundary value problems for the circle.

Exercise 3.28 *Prove that the Laplace operator*

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

in the coordinates z, $\bar{z}$ becomes

$$\Delta = 4\, \frac{\partial^2}{\partial z \partial \bar{z}}. \tag{3.4.11}$$

To a certain extent the converse of Theorem 3.26 holds as well

Theorem 3.29 *Let $u(x, y)$ be a harmonic function in a* **simply connected** *domain $\Omega \subset \mathbb{R}^2 \simeq \mathbb{C}$. Then there is a holomorphic function $f(z)$ such that $u = \Re f$. The function $v = \Im f$ is called the* **harmonic conjugate function** *to u.*

Proof. Consider the total differential of u,

$$\mathrm{d}u = u_x \mathrm{d}x + u_y \mathrm{d}y \tag{3.4.12}$$

This is clearly an exact form; now consider the "Hodge dual"

$$\star \mathrm{d}u := -u_y \mathrm{d}x + u_x \mathrm{d}y. \tag{3.4.13}$$

Due to the fact that u is harmonic, this form is also closed: $u_{yy} = -u_{xx}$. Now, since Ω is simply connected, we can define

$$v(x, y) := \int_{(x_0,y_0)}^{(x,y)} \big(- u_y \mathrm{d}x + u_x \mathrm{d}y \big) \quad \Rightarrow \quad \mathrm{d}v = \star \mathrm{d}u. \tag{3.4.14}$$

and the integration is independent of the path (here (x_0, y_0) is some choice of point in Ω).

By construction we have $v_x = -u_y$ and $v_y = u_x$; namely the function $f(x, y) = u(x, y)+iv(x, y)$ satisfies the Cauchy–Riemann equations in the domain Ω and hence it is holomorphic. ■

Remark 3.30 If we lift the assumption that Ω is simply connected, then we can only assert the local existence of v, but in general f will not be single valued. The prototypical example is $u(x, y) = \ln(\sqrt{x^2 + y^2})$ on $\mathbb{C} \setminus \{0\}$. In this case $f(z)$ is the complex logarithm, which is not single–valued.

Using the representation (3.4.11) of the two-dimensional Laplace operator one can describe all complex valued solutions to the Laplace equation.

Theorem 3.31 *Any complex valued solution u to the Laplace equation $\Delta u = 0$ on the plane can be represented as a sum of a holomorphic and an antiholomorphic function:*

$$u(x, y) = f(z) + g(\bar{z}). \tag{3.4.15}$$

Proof: Let the $\mathcal{C}^2$-smooth function $u(x, y)$ satisfy the Laplace equation

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$

Denote

$$F = \frac{\partial u}{\partial z}.$$

The Laplace equation implies that this function is holomorphic, $F = F(z)$. From complex analysis it is known that any holomorphic function admits a holomorphic primitive,

$$F(z) = f'(z).$$

Consider the difference $g := u - f$. It is an antiholomorphic function, $g = g(\bar{z})$. Indeed,

$$\frac{\partial g}{\partial z} = \frac{\partial u}{\partial z} - f' = 0.$$

So $u = f(z) + g(\bar{z})$. ■

Corollary 3.32 *Any harmonic function on the plane can be represented as the real part of a holomorphic function.*

Notice that the imaginary part of a holomorphic function $f(z)$ is equal to the real part of the function $-i\, f(z)$ that is holomorphic as well.

Corollary 3.33 *Any harmonic function on the plane is* $\mathcal{C}^\infty$*-smooth.*

Another important consequence of the complex representation (3.4.11) of the Laplace operator on the plane is invariance of the Laplace equation under conformal transformation. Recall that a smooth map

$$f : \Omega \to \Omega'$$

is called *conformal* if it preserves the angles between smooth curves. The dilatations

$$(x, y) \mapsto (k\, x, k\, y)$$

with $k \neq 0$, rotations by the angle ϕ

$$(x, y) \mapsto (x \cos\phi - y \sin\phi,\ x \sin\phi + y \sin\phi)$$

and reflections

$$(x, y) \mapsto (x, -y)$$

are examples of linear conformal transformations. These examples and their superpositions exhaust the class of linear conformal maps. The general description of conformal maps on the plane are given by

Lemma 3.34 *Let $f(z)$ be a function holomorphic in the domain Ω with never vanishing derivative:*

$$\frac{df(z)}{dz} \neq 0 \quad \forall z \in \Omega.$$

Then the map

$$z \mapsto f(z)$$

of the domain Ω to $\Omega' = f(\Omega$ is conformal. Same for antiholomorphic functions. Conversely, if the smooth map $(x, y) \mapsto (u(x, y), v(x, y))$ is conformal then the function $f = u + iv$ is holomorphic or antiholomorphic with nonvanishing derivative.

Proof: Let us consider the differential of the map $(x, y) \mapsto (u(x, y), v(x, y))$ given by the real $u = \operatorname{Re} f$ and imaginary $v = \operatorname{Im} f$ parts of the holomorphic function f. It is a linear map defined by the Jacobi matrix

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} = \begin{pmatrix} \partial u/\partial x & -\partial v/\partial x \\ \\ \partial v/\partial x & \partial u/\partial x \end{pmatrix}$$

(we have used the Cauchy – Riemann equations). Since

$$0 \neq |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2,$$

we can introduce the numbers $r > 0$ and ϕ by

$$r = |f'(z)|, \quad \cos\phi = \frac{\partial u/\partial x}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}}, \quad \sin\phi = \frac{\partial v/\partial x}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}}.$$

The Jacobi matrix then becomes a composition of the rotation by the angle ϕ and a dilatation with the coefficient r:

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} = r \begin{pmatrix} \cos\phi & -\sin\phi \\ \\ \sin\phi & \cos\phi \end{pmatrix}.$$

This is a linear conformal transformation preserving the angles. A similar computation works for an antiholomorphic map with nonvanishing derivatives $f'(\bar{z}) \neq 0$.

Conversely, the Jacobi matrix of a conformal transformation must have the form

$$r \begin{pmatrix} \cos\phi & -\sin\phi \\ \\ \sin\phi & \cos\phi \end{pmatrix}$$

or

$$r \begin{pmatrix} \cos\phi & \sin\phi \\ \\ \sin\phi & -\cos\phi \end{pmatrix}.$$

In the first case one obtains the differential of a holomorphic map while the second matrix corresponds to the antiholomorphic map. ■

We are ready to prove

Theorem 3.35 *Let*

$$f : \Omega \to \Omega'$$

be a conformal map. Then the pull-back (composition with f) of any function harmonic in Ω' will be harmonic in Ω.

Proof: According to the Lemma the conformal map is given by a holomorphic or an antiholomorphic function. Let us consider the holomorphic case,

$$z \mapsto w = f(z).$$

The transformation law of the Laplace operator under such a map is clear from the following formula:

$$\frac{\partial^2}{\partial z \partial \bar{z}} = |f'(z)|^2 \frac{\partial^2}{\partial w \partial \bar{w}}. \tag{3.4.16}$$

Thus any function U on Ω' satisfying

$$\frac{\partial^2 U}{\partial w \partial \bar{w}} = 0$$

will also satisfy

$$\frac{\partial^2 U}{\partial z \partial \bar{z}} = 0.$$

The case of an antiholomorphic map can be considered in a similar way. ■

A conformal map

$$f : \Omega \to \Omega'$$

is called *conformal transformation* if it is one-to-one. In that case the inverse map

$$f^{-1} : \Omega' \to \Omega$$

exists and is also conformal. The following fundamental *Riemann theorem* is the central result of the theory of conformal transformations on the plane.

Theorem 3.36 (Riemann uniformization theorem) *For any connected and simply connected domain Ω on the plane not coinciding with the plane itself there exists a conformal transformation of Ω to the unit circle $f : \Omega \to \mathbb{D}$.*

There is an interesting application of the Riemann Uniformization Theorem.

Theorem 3.37 *For an arbitrary simply connected domain Ω not coinciding with the plane, the Green's function for the Dirichlet problem is given by*

$$G_\Omega(z, w) = \frac{1}{2\pi} \ln \left| \frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}} \right| \tag{3.4.17}$$

The Riemann theorem, together with conformal invariance of the Laplace equation gives a possibility to reduce the main boundary value problems for any connected simply connected domain to similar problems for the unit circle.

The proof of Riemann's theorem belongs to an advanced course in complex analysis and will not be reported here.

3.4.1 Conformal maps in fluid-dynamics

Suppose $\vec{V}(x, y, z, t)$ is a vector field representing the motion of a fluid.

The fluid is called **incompressible** if $\operatorname{div} \vec{V} \equiv 0$.

Suppose that $\vec{V}$ represents a stationary flow (i.e. independent of time) and bi-dimensional (i.e. independent of z and with zero z–component). Denote by $\mathbf{v}(x, y)$ the projection of $\vec{V}$ to the x, y components.

Proposition 3.38 *If* $\mathbf{v}$ *is incompressible and irrotational, then there are two functions* Φ, Ω *such that*

$$\mathbf{v} = \Phi_x \mathbf{i} + \Phi_y \mathbf{j} = \Omega_y \mathbf{i} - \Omega_x \mathbf{j}. \tag{3.4.18}$$

These two functions are called the **velocity potential** *and the* **stream function** *respectively.*

The naming stems from the observation that the level-sets of Ω are stream lines.

A simple consequence of the above proposition is that the function

$$F(z) = \Phi(x, y) + i\Omega(x, y) \tag{3.4.19}$$

is analytic. It is called the **complex velocity potential**. Since $F'(z) = \Phi_x + i\Omega_x = v_1 - iv_2$ we see that $|F'(x)|$ is the speed of the flow. I.e. the complex conjugate of $F'(z)$ represents the velocity field $\mathbf{v}$ interpreted as a complex number.

3.5 Exercises to Section 4

Exercise 3.39 *Prove Lemma 3.17.*

Exercise 3.40 *Let* Ω *be a bounded open set in* $\mathbb{R}^d$ *with p.w. smooth boundary. Let* $u_1, u_2, u_3 \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ *be harmonic functions. Show that if* $u_1 \le u_2 \le u_3$ *when restricted to the boundary of* Ω *then the same inequality holds throughout* Ω*.*

Exercise 3.41 *Let* Ω *be the semi-infinite strip* $(0, \pi) \times (0, \infty)$*. Consider the Laplace problem on* Ω *with B.C.*

$$u(x, 0) = 0, \qquad u(0, y) = 0 \quad x \in [0, \pi], \quad y \in [0, \infty) \tag{3.5.1}$$

Find more than one harmonic solution to this problem. Explain how that does not contradict the uniqueness theorem for harmonic functions.

Exercise 3.42 *Prove Proposition 3.24.*

Exercise 3.43 *Prove Proposition 3.23.*

Exercise 3.44 *Prove that any harmonic polynomial is a linear combination of the polynomials* (3.4.10). *[Hint: if p is a harmonic polynomial, then solve the Laplace equation on the disk with BC $u\big|_{|z|=1} = p\big|_{|z|=1}$.]*

Exercise 3.45 *Find a Green function for the upper half plane $\mathbb{H}$ but with Neumann conditions on $\mathbb{R}$.*

Exercise 3.46 *Suppose that we have the Poisson equation $\triangle u = g$ where g is only an L^1 function. Show that*

$$u(x) = \int_{\mathbb{R}^d} G_d(y-x)g(y)\mathrm{d}V(y) \tag{3.5.2}$$

is a solution in the weak sense, namely show that for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} u\triangle\varphi\,\mathrm{d}V(x) = \int_{\mathbb{R}^d} g\varphi\,\mathrm{d}V(x). \tag{3.5.3}$$

Exercise 3.47 *Let $\Omega \subset \mathbb{C} \simeq \mathbb{R}^2$ be an open domain and $a \in \Omega$. Recalling Def.* **??** *prove that*

1. *If a bounded open set Ω admits a Green's function then $G_\Omega(z;w)$ is unique.*
2. *With Ω as above plus connected, prove that $G_\Omega(z;w)$ is negative on $\Omega \setminus \{a\}$.*

Exercise 3.48 *Find a function $u(x,y)$ satisfying*

$$\Delta u = x^2 - y^2$$

for $r < a$ and the boundary condition $u|_{r=a} = 0$.

Exercise 3.49 *Let $\chi_{\mathbb{D}}(z)$ be the characteristic function of the unit disk $\mathbb{D} := \{z \in \mathbb{C} : \quad |z| < 1\}$. Compute*

$$U(z) := \int_{\mathbb{C}} \chi_{\mathbb{D}}(w)\frac{\ln|z-w|}{2\pi}\mathrm{d}^2 w. \tag{3.5.4}$$

Exercise 3.50 *Find a harmonic function on the annular domain*

$$a < r < b$$

with the boundary conditions

$$u|_{r=a} = 1, \quad \left(\frac{\partial u}{\partial r}\right)_{r=b} = \cos^2\phi.$$

Exercise 3.51 *Find a harmonic function $u(x,y)$ solving the the Dirichlet b.v.p. in the rectangle*

$$0 \le x \le a, \quad 0 \le y \le b$$

satisfying the boundary conditions

$$u(0,y) = A\,y(b-y), \quad u(a,y) = 0$$
$$u(x,0) = B\,\sin\frac{\pi x}{a}, \quad u(x,b) = 0.$$

Hint: use separation of variables in Euclidean coordinates.