20
Form Factors and Correlation Functions

Elementary, my dear Watson

Arthur Conan Doyle

One of the fundamental problems of statistical mechanics and its quantum field theory formulation is the characterization of the order parameters and the computation of their correlation functions. Besides the intrinsic interest of this problem, the correlation functions are the key quantities in the determination of the universal ratios of the renormalization group and therefore they can have direct experimental confirmation (see Chapter 8). In general, the computation of correlation functions is a difficult task, usually achieved with partial success through perturbative methods.

As we saw in the previous chapters devoted to conformal field theories, an exact determination of the operator content and the correlation functions of a two-dimensional theory can be obtained only when the model is at its critical point. In this case, in fact, one has a classification of the order parameters in terms of the irreducible representation of the Virasoro algebra and, moreover, one can get an exact expression of the correlators by solving the linear differential equations that they satisfy.

Unfortunately, the simple theoretical scheme of the critical points cannot be generalized once we move away from criticality. In this case, the problem has to be faced with different techniques. As shown in this chapter, significant progress can be made when we deal with integrable theories, characterized by their elastic S-matrix and the spectrum of the asymptotic states. The central quantities are in this case the matrix elements of the various operators on the asymptotic states of the theory, called the *form factors*. The precise definition of these quantities is given below. The general properties related to the unitarity and crossing symmetry lead to a set of functional equations for the form factors that can be explicitly solved in many interesting cases. Once the matrix elements of the operators are known, their correlation functions can be recovered in terms of spectral representation series. It is worth mentioning that these series present remarkable convergence properties.

Hence, the success of the form factor method relies on two points: (a) the possibility of determining exactly the matrix elements of the order parameters on the asymptotic states of the theory, identified by scattering theory; (b) the fast convergence properties of the spectral series. These two steps lead to the determination of the correlation functions away from criticality with a precision that cannot be obtained by other methods.

20.1 General Properties of the Form Factors

An essential quantity for the computation of the matrix elements is the S-matrix of the problem. As shown in the previous chapters, the S-matrix of many two-dimensional systems is particularly simple and can be explicitly found. For an infinite number of conservation laws, the scattering processes of integrable systems are purely elastic and the n-particle S-matrix can be factorized in terms of the $n(n-1)/2$ two-body scattering amplitudes. In the following, for simplicity, we mainly focus our attention on diagonal scattering theories with a non-degenerate spectrum. To characterize the kinematic state of the particles we use the rapidities θ_i that enter the dispersion relations

$$p_i^0 \,=\, m_i \cosh \theta_i, \quad p_i^1 \,=\, m_i \sinh \theta_i. \tag{20.1.1}$$

The two-body S-matrix amplitudes depend on the difference of the rapidities $\theta_{ij} = \theta_i - \theta_j$ and satisfy the unitary and crossing symmetry equations

$$\begin{aligned} S_{ij}(\theta_{ij}) &= S_{ji}(\theta_{ij}) = S_{ij}^{-1}(-\theta_{ij}), \\ S_{i\bar{j}}(\theta_{ij}) &= S_{ij}(i\pi - \theta_{ij}). \end{aligned} \tag{20.1.2}$$

Possible bound states correspond to simple poles (or higher order odd poles) of these amplitudes, placed at imaginary values of θ_{ij} in the physical strip $0 < \mathrm{Im}\theta < \pi$.

Let's see how the S-matrix allows us to compute the matrix elements of the (semi)-local operators on the asymptotic states. To this end, it is useful to introduce an algebraic formalism.

20.1.1 Faddeev–Zamolodchikov Algebra

A key assumption of the form factor theory is that there exist some operators, both of creation and annihilation type, $V_{\alpha_i}^\dagger(\theta_i)$, $V_{\alpha_i}(\theta_i)$, that implement a generalization of the usual bosonic and fermionic algebraic relations. Let's call them *vertex operators*. Denoting by α_i the quantum number that distinguishes the different types of particles of the theory, these operators satisfy the associative algebra in which enters the S-matrix

$$V_{\alpha_i}(\theta_i)V_{\alpha_j}(\theta_j) = S_{ij}(\theta_{ij})\, V_{\alpha_j}(\theta_j)V_{\alpha_i}(\theta_i) \tag{20.1.3}$$

$$V_{\alpha_i}^\dagger(\theta_i)V_{\alpha_j}^\dagger(\theta_j) = S_{ij}(\theta_{ij})\, V_{\alpha_j}^\dagger(\theta_j)V_{\alpha_i}^\dagger(\theta_i) \tag{20.1.4}$$

$$V_{\alpha_i}(\theta_i)V_{\alpha_j}^\dagger(\theta_j) = S_{ij}(\theta_{ji})\, V_{\alpha_j}^\dagger(\theta_j)V_{\alpha_i}(\theta_i) + 2\pi\delta_{\alpha_i\alpha_j}\delta(\theta_{ij}). \tag{20.1.5}$$

Any commutation of these operators can be interpreted as a scattering process. The Poincaré group, generated by the Lorentz transformations $L(\epsilon)$ and the translations T_y, acts on the operators as

$$U_L V_\alpha(\theta) U_L^{-1} = V_\alpha(\theta + \epsilon) \tag{20.1.6}$$

$$U_{T_y} V_\alpha(\theta) U_{T_y}^{-1} = e^{ip_\mu(\theta)y^\mu} V_\alpha(\theta). \tag{20.1.7}$$

Obviously the explicit form of the creation and annihilation operators depends crucially on the theory in question and their construction is an open problem for most models. This difficulty does not stop us, however, from deriving the fundamental equations for the matrix elements starting from the algebraic equations given above.

The vertex operators define the space of physical states. The vacuum $|0\rangle$ is the state annihilated by $V_\alpha(\theta)$,

$$V_\alpha(\theta)|0\rangle = 0 = \langle 0|V_\alpha^\dagger(\theta),$$

while the Hilbert space is constructed by applying the various vertex operators $V_\alpha^\dagger(\theta)$ on $|0\rangle$:

$$|V_{\alpha_1}(\theta_1)\dots V_{\alpha_n}(\theta_n)\rangle \equiv V_{\alpha_1}^\dagger(\theta_1)\dots V_{\alpha_n}^\dagger(\theta_n)|0\rangle. \tag{20.1.8}$$

From eqn. (20.1.5), the one-particle states have the normalization

$$\langle V_{\alpha_i}(\theta_i)|V_{\alpha_j}(\theta_j)\rangle = 2\pi\,\delta_{\alpha_i\alpha_j}\delta(\theta_{ij}).$$

The algebra of the vertex operators implies that the vectors (20.1.8) are not all linearly independent. To select a basis of linearly independent vectors we need an additional requirement: for the initial states, the rapidites must be ordered in a decreasing way:

$$\theta_1 > \theta_2 > \cdots > \theta_n$$

while, for the final states in an increasing way:

$$\theta_1 < \theta_2 < \cdots < \theta_n.$$

These orderings select a set of linearly independent vectors that form a basis in the Hilbert space.

20.1.2 Form Factors

In this section we describe the principles of the theory. Unless explicitly stated, in the following we consider the matrix elements between the *in* and *out* states of the particle with the lowest mass of local, scalar, and hermitian operators $\mathcal{O}(x)$

$${}_{\rm out}\langle V(\theta_{m+1})\dots V(\theta_n)|\mathcal{O}(x)|V(\theta_1)\dots V(\theta_m)\rangle_{\rm in}. \tag{20.1.9}$$

We can always place the operator at the origin by using the translation operator, $U_{T_y}\mathcal{O}(x)U_{T_y}^{-1} = \mathcal{O}(x+y)$, and using eqn. (20.1.7), the matrix elements above are given by

$$\exp\left[i\left(\sum_{i=m+1}^{n} p_\mu(\theta_i) - \sum_{i=1}^{m} p_\mu(\theta_i)\right)x^\mu\right] \tag{20.1.10}$$
$$\times\,{}_{\rm out}\langle V(\theta_{m+1})\dots V(\theta_n)|\mathcal{O}(0)|V(\theta_1)\dots V(\theta_m)\rangle_{\rm in}.$$

It is convenient to define the functions

$$F_n^{\mathcal{O}}(\theta_1,\theta_2,\dots,\theta_n) = \langle 0 \mid \mathcal{O}(0) \mid \theta_1,\theta_2,\dots,\theta_n\rangle_{in} \tag{20.1.11}$$

called the *Form Factors* (FF), whose graphical representation is shown in Fig. 20.1: they are the matrix elements of an operator placed at the origin between the n-particle state and the vacuum.[1]

[1]From now on we use the simplified notation $|\dots V(\theta_n)\dots\rangle \equiv |\dots\theta_n\dots\rangle$ to denote the physical states of the particle with the lowest mass.

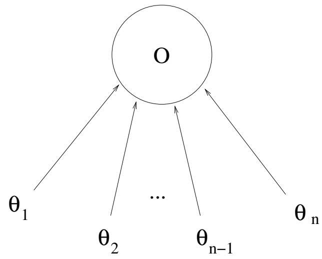


Fig. 20.1 *Form factor of the operator $\mathcal{O}$.*

For local and scalar operators, the relativistic invariance of the theory implies that the FF are functions of the differences of the rapidities θ_{ij}

$$F_n^{\mathcal{O}}(\theta_1, \theta_2, \ldots, \theta_n) \,=\, F_n^{\mathcal{O}}(\theta_{12}, \theta_{13}, \ldots, \theta_{ij}, \ldots), i < j. \tag{20.1.12}$$

The invariance under crossing symmetry permits us to recover the most general matrix elements by an analytic continuation of the functions (20.1.11)

$$F_{n+m}^{\mathcal{O}}(\theta_1, \theta_2, \ldots, \theta_m, \theta_{m+1} - i\pi, \ldots, \theta_n - i\pi) \,=\, F_{n+m}^{\mathcal{O}}(\theta_{ij}, i\pi - \theta_{sr}, \theta_{kl}) \tag{20.1.13}$$

where $1 \leq i < j \leq m$, $1 \leq r \leq m < s \leq n$, and $m < k < l \leq n$.

Apart from the poles corresponding to the bound states present in all possible channels of this amplitude, the form factors $F_n^{\mathcal{O}}$ are expected to be analytic functions in the strips $0 < \mathrm{Im}\theta_{ij} < 2\pi$.

20.2 Watson's Equations

The FF of a scalar and hermitian operator $\mathcal{O}$ satisfy a set of equations, known as *Watson's equations*, that assume a particularly simple form for the integrable systems

$$F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n) = F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n) S(\theta_i - \theta_{i+1}), \tag{20.2.14}$$
$$F_n^{\mathcal{O}}(\theta_1 + 2\pi i, \ldots, \theta_{n-1}, \theta_n) = e^{2\pi i \gamma} F_n^{\mathcal{O}}(\theta_2, \ldots, \theta_n, \theta_1) = \prod_{i=2}^{n} S(\theta_i - \theta_1) F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n),$$

where γ is the semilocal index of the operator $\mathcal{O}$ with respect to the operator that creates the particles. The first equation is a simple consequence of eqn (20.1.3), because a commutation of two operators is equivalent to a scattering process. Concerning the second equation, it states the nature of the discontinuity of these functions at the cuts $\theta_{1i} = 2\pi i$. The graphical representation of these equations is shown in Fig. 20.2. When $n = 2$, eqns (20.2.14) reduce to

$$\begin{aligned} F_2^{\mathcal{O}}(\theta) \quad &= F_2^{\mathcal{O}}(-\theta)\, S_2(\theta), \\ F_2^{\mathcal{O}}(i\pi - \theta) &= F_2^{\mathcal{O}}(i\pi + \theta). \end{aligned} \tag{20.2.15}$$

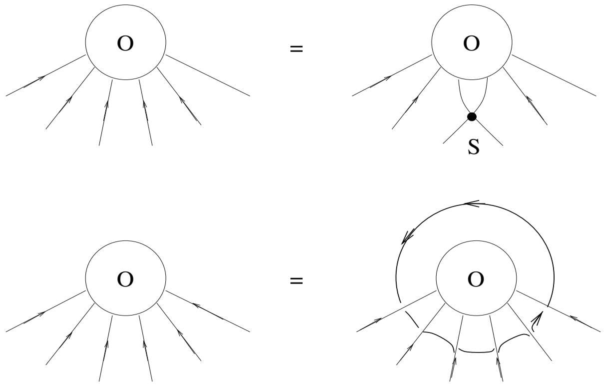


Fig. 20.2 *Graphical form of the Watson equations.*

The most general solution of the Watson equations (20.2.14) is given by

$$F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n) \,=\, K_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n) \prod_{i<j} F_{\min}(\theta_{ij}). \tag{20.2.16}$$

Let's discuss the various terms entering this expression.

Minimal two-particle form factor. $F_{\min}(\theta)$ is an analytic function in the region $0 \leq \text{Im}\ \theta \leq \pi$, the solution of the two equations (20.2.15), with neither zeros nor poles in the strip $0 < \text{Im}\theta < \pi$, and with the mildest behavior at $|\theta| \to \infty$. These requirements determine uniquely this function, up to a normalization factor $\mathcal{N}$. Its explicit expression can be found by writing the S-matrix as

$$S(\theta) \,=\, \exp\left[\int_0^\infty \frac{dt}{t}\, f(t)\, \sinh\frac{t\theta}{i\pi}\right].$$

In fact, it is easy to see that $F_{\min}(\theta)$ is given by

$$F_{\min}(\theta) \,=\, \mathcal{N}\, \exp\left[\int_0^\infty \frac{dt}{t}\, \frac{f(t)}{\sinh t}\, \sin^2\left(\frac{t\pi\hat{\theta}}{2\pi}\right)\right], \qquad \hat{\theta} = i\pi - \theta. \tag{20.2.17}$$

Note that for interacting theories, $S(0) \,=\, -1$, and therefore the first equation in (20.2.15) forces $F_{\min}(\theta)$ to have a zero at the two-particle threshold

$$F(\theta) \,\simeq\, \theta, \quad\ \theta \to 0. \tag{20.2.18}$$

$K_n^{\mathcal{O}}$ **factors**. The remaining factors $K_n^{\mathcal{O}}$ in (20.2.16) satisfy the Watson equation but with $S_2 \,=\, 1$: this implies that they are completely symmetric functions in the variables θ_{ij}, periodic with period $2\pi i$. Therefore they can be considered as functions of the variables $\cosh\theta_{ij}$. Let's investigate other properties of the functions $K_n^{\mathcal{O}}$. They must have all physical poles expected for the form factors. We recall that, in general,

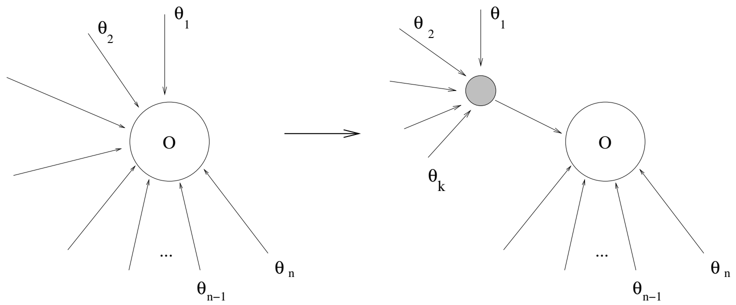


Fig. 20.3 *Kinematic configuration of a k-particle responsable for a pole in the form factors.*

there is a simple pole in the form factors when a cluster made of k particles can reach a kinematical configuration that is equivalent to that of a single particle, as shown in Fig. 20.3, with the pole given just by the propagator of the latter particle. If this is the general situation, for the integrable theories there is however an important simplification. In fact, by the factorization property of the S-matrix, it is sufficient to consider only the cases in which the clusters are made of $k = 2$ or $k = 3$: the poles coming from the two-particle clusters are dictated uniquely by the bound states of the S-matrix, while those coming from the three-particle clusters are determined by the crossing processes, although they are also related to the S-matrix (see the discussion in the next section). In conclusion, all the poles of the form factors are determined by the underlying scattering theory and they do not depend on the operator! In the light of this analysis, the functions $K_n^{\mathcal{O}}$ can be parameterized as follows

$$K_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n) \,=\, \frac{Q_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n)}{D_n(\theta_1, \ldots, \theta_n)}, \tag{20.2.19}$$

where the denominator D_n is a polynomial in $\cosh \theta_{ij}$ that is fixed only by the pole structure of the S-matrix while the information on the operator $\mathcal{O}$ is enclosed in the polynomial $Q_n^{\mathcal{O}}$ of the variables $\cosh \theta_{ij}$ placed at the *numerator*. We will come back to this important point in later sections.

Symmetric polynomials. As shown above, the functions $K_n^{\mathcal{O}}$ are symmetric under the permutation of the rapidities of the various particles. In many case it is convenient to change variables, introducing the parameters $x_i \equiv e^{\theta_i}$, so that both numerator and denominator become symmetric polynomials in the x_i variables. A basis in the functional space of the symmetric polynomials in n variables is given by the *elementary symmetric polynomials* $\sigma_k^{(n)}(x_1, \ldots, x_n)$, whose generating function is

$$\prod_{i=1}^{n}(x + x_i) \,=\, \sum_{k=0}^{n} x^{n-k}\, \sigma_k^{(n)}(x_1, x_2, \ldots, x_n). \tag{20.2.20}$$

Conventionally all $\sigma_k^{(n)}$ with $k > n$ and with $n < 0$ are zero. The explicit expressions for the other cases are

$$
\begin{aligned}
&\sigma_0 = 1, \\
&\sigma_1 = x_1 + x_2 + \ldots + x_n, \\
&\sigma_2 = x_1 x_2 + x_1 x_3 + \ldots x_{n-1} x_n, \\
&\vdots \qquad\quad \vdots \\
&\sigma_n = x_1 x_2 \ldots x_n.
\end{aligned}
\tag{20.2.21}
$$

The $\sigma_k^{(n)}$ are homogeneous polynomials in x_i, of total degree k but linear in each variable.

Total and partial degrees of the polynomials. The polynomials $Q_n^{\mathcal{O}}(x_1, \ldots, x_n)$ in the numerator of the factor $K_n^{\mathcal{O}}$ satisfy additional conditions coming from the asymptotic behavior of the form factors. The first condition simply comes from relativistic invariance: in fact, for a simultaneous translation of all the rapidities, the form factors of a scalar operator[2] satisfy

$$
F_n^{\mathcal{O}}(\theta_1 + \Lambda, \theta_2 + \Lambda, \ldots, \theta_n + \Lambda) \;=\; F_n^{\mathcal{O}}(\theta_1, \theta_2, \ldots, \theta_n). \tag{20.2.22}
$$

This implies the equality of the total degrees of the polynomials $Q_n^{\mathcal{O}}(x_1, \ldots, x_n)$ and $D_n(x_1, \ldots, x_n)$. Concerning the partial degree with respect to each variable, it is worth anticipating a result discussed in Section 20.8: in order to have a power law behavior of the two-point correlation function of the operator $\mathcal{O}(x)$, its form factors must behave for $\theta_i \to \infty$ at most as $\exp(k\theta_i)$, where k is a constant (independent of i), related to the conformal weight of the operator $\mathcal{O}$.

20.3 Recursive Equations

The poles in the FF induce a set of recursive equations that are crucial for the explicit determination of these functions. As a function of the difference of the rapidities θ_{ij}, the FF have two kinds of simple pole.[3]

Kinematical poles. The first kind of singularity does not depend on whether the model has bound states. It is in fact associated to the kinematical poles at $\theta_{ij} = i\pi$ that come from the one-particle state realized by the three-particle clusters. In turn, these processes correspond to the crossing channels of the S-matrix, as shown in Fig. 20.4. The residues at these poles give rise to a recursive equation that links the n-particle and the $(n-2)$-particle form factors

$$
-i \lim_{\tilde{\theta} \to \theta} (\tilde{\theta} - \theta) F_{n+2}^{\mathcal{O}}(\tilde{\theta} + i\pi, \theta, \theta_1, \theta_2, \ldots, \theta_n) = \left(1 - e^{2\pi i \gamma} \prod_{i=1}^{n} S(\theta - \theta_i) \right) F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n). \tag{20.3.23}
$$

[2]For the form factors of an operator $\mathcal{O}(x)$ of spin s, the equation generalizes to $F_n^{\mathcal{O}}(\theta_1 + \Lambda, \theta_2 + \Lambda, \ldots, \theta_n + \Lambda) \;=\; e^{s\Lambda}\, F_n^{\mathcal{O}}(\theta_1, \theta_2, \ldots, \theta_n)$.

[3]There could also be higher order poles, corresponding to the higher order poles of the S-matrix. Their discussion is however beyond the scope of this book.

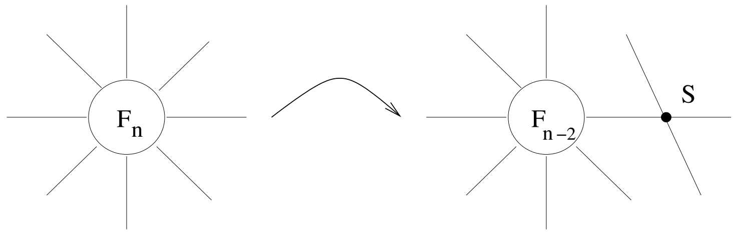


Fig. 20.4 *Recursive equation of the kinematic poles.*

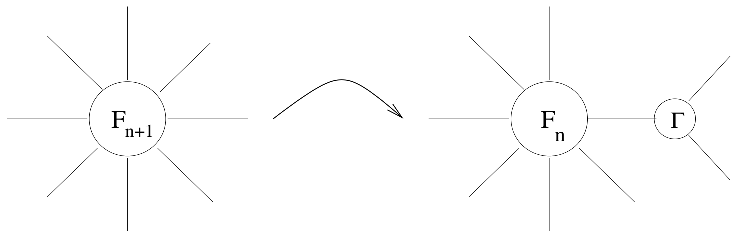


Fig. 20.5 *Recursive equation of the bound state poles.*

Let's denote concisely by $\mathcal{C}$ the map between $F_{n+2}^{\mathcal{O}}$ and $F_n^{\mathcal{O}}$ established by the recursive equation

$$F_{n+2}^{\mathcal{O}} = \mathcal{C}\, F_n^{\mathcal{O}}. \tag{20.3.24}$$

Bound state poles. There is another family of poles in F_n if the S-matrix has simple poles related to the bound states. These poles are at the values of θ_{ij} corresponding to the resonance angles. Let $\theta_{ij} = iu_{ij}^k$ be one of these poles, associated to the bound state A_k present in the channel $A_i \times A_j$. For the S-matrix we have

$$-i \lim_{\theta\to iu_{ij}^k} (\theta - iu_{ij}^k)\, S_{ij}(\theta) = \left(\Gamma_{ij}^k\right)^2 \tag{20.3.25}$$

where Γ_{ij}^k is the on-shell three-particle vertex and for the residue of the form factor F_{n+1} involving the particles A_i and A_j we have

$$-i \lim_{\epsilon\to 0} \epsilon\, F_{n+1}^{\mathcal{O}}(\theta + i\overline{u}_{ik}^j - \epsilon, \theta - i\overline{u}_{jk}^i + \epsilon, \theta_1, \ldots, \theta_{n-1}) = \Gamma_{ij}^k\, F_n^{\mathcal{O}}(\theta, \theta_1, \ldots, \theta_{n-1}), \tag{20.3.26}$$

where $\overline{u}_{ab}^c \equiv (\pi - u_{ab}^c)$. This equation sets up a recursive structure between the $(n+1)$- and the n-particle form factors, as shown in Fig. 20.5. Let's denote by $\mathcal{B}$ the map between F'_{n+1} and $F_n^{\mathcal{O}}$ set by this recursive equation

$$F_{n+1}^{\mathcal{O}} = \mathcal{B}\, F_n^{\mathcal{O}}. \tag{20.3.27}$$

When the theory presents bound states, it is possible to show that the two kinds of recursive equation are compatible, so that it is possible to reach the $(n+2)$-particle FF by the n-particle FF either using directly the recursive equation (20.4) or applying the recursive equation (20.5) twice. In terms of the mappings $\mathcal{B}$ and $\mathcal{C}$ we have $\mathcal{C} = \mathcal{B}^2$.

20.4 The Operator Space

At the critical point, one can identify the operator space of a quantum field theory in terms of the irreducible representations of the Virasoro algebra. An extremely interesting point is the characterization of the operator content also away from criticality. As argued below, this can be achieved by means of the form factor theory: although this identification is based on different principles than conformal theories, nevertheless it allows us to shed light on the classification problem of the operators.

Let's start our discussion with some general considerations. In the form factor approach, an operator $\mathcal{O}$ is defined once all its matrix elements $F_n^{\mathcal{O}}$ are known. Notice the particular nature of all the functional equations – the recursive and Watson's equations – satisfied by the form factors: (i) they are all linear; (ii) they do not refer to any particular operator! This implies that, given a fixed number n of asymptotic particles, the solutions of the form factor equations form a linear space. The classification of the operator content is then obtained by putting the vectors of this linear space in correspondence with the operators.

Kernel solutions. Among the functions of these linear spaces, there are those belonging to the kernel of the operators $\mathcal{B}$ and $\mathcal{C}$: these are the functions $\hat{F}_n^{(i)}$ and $\hat{F}_n^{(j)}$ that satisfy

$$\begin{aligned} \mathcal{B}\, \hat{F}_n^{(i)} &= 0 \\ \mathcal{C}\, \hat{F}_n^{(j)} &= 0. \end{aligned} \tag{20.4.28}$$

Their general expression is given in eqn (20.2.16) but, in this case, the function K_n does not contain poles that give rise to the recursive equations. Hence each of the functions $\hat{F}_n^{(i)}$ and $\hat{F}_n^{(j)}$ is simply a symmetric polynomial in the x_i variables. The vector space of the form factors that belong to the kernels can be further specified by assigning the total and partial degrees of these polynomials.

A non-vanishing kernel of the operators $\mathcal{B}$ and $\mathcal{C}$ has the important consequence that at each level n, if $\tilde{F}_n$ is a reference solution of the recursive equation and $\hat{F}_n$ a function of any of the two kernels, the most general form factor can be written as

$$F_n = \tilde{F}_n + \sum_i \alpha_i\, \hat{F}_n. \tag{20.4.29}$$

Therefore the identification of each operator is obtained by specifying at each level n the constants α_i. If we graphically represent by dots the linearly independent solutions at the level n of the form factor equations, we have the situation of Fig. 20.6. In this graphical representation, each operator is associated to a well-defined path on this lattice, with each step $(n+1) \rightarrow n$ (or $(n+2) \rightarrow n$) ruled by the operator $\mathcal{B}$ (or $\mathcal{C}$). We will see explicit examples of this operator structure when we discuss the form factors of the Ising and the Sinh–Gordon models.

20.5 Correlation Functions

Once we have determined the form factors of a given operator, its correlation functions can be written in terms of the spectral representation series using the completeness

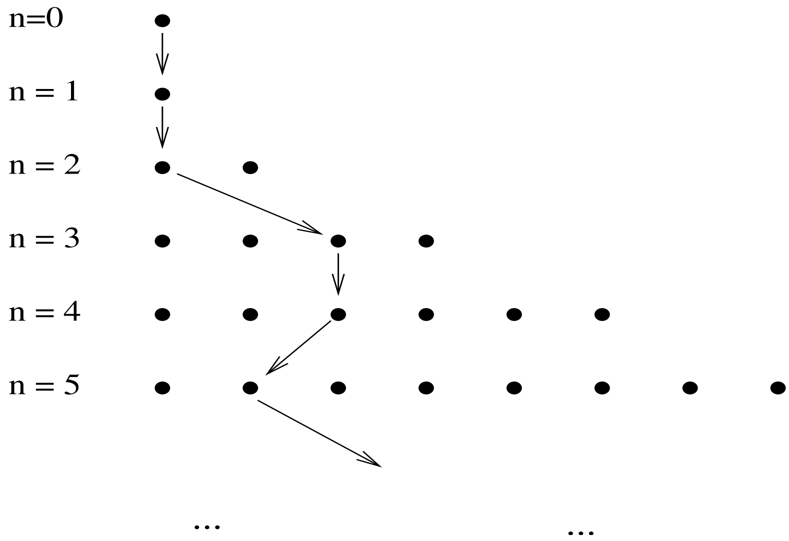


Fig. 20.6 *Vector spaces of the solutions of the form factor equations (the number of dots at each level is purely indicative). An operator is associated to the sequence of its matrix elements* F_n.

relation of the multiparticle states

$$1 = \sum_{n=0}^{\infty} \int \frac{d\theta_1 \dots d\theta_n}{n!(2\pi)^n} \, |\theta_1, \dots, \theta_n\rangle \, \langle\theta_1, \dots, \theta_n|. \tag{20.5.30}$$

For instance, for the two-point correlation function of the operator $\mathcal{O}(x)$ in euclidean space, we have

$$\begin{aligned} \langle \mathcal{O}(x) \, \mathcal{O}(0) \rangle &= \sum_{n=0}^{\infty} \int \frac{d\theta_1 \dots d\theta_n}{n!(2\pi)^n} \langle 0|\mathcal{O}(x)|\theta_1, \dots, \theta_n\rangle_{\mathrm{in\,in}}\langle\theta_1, \dots, \theta_n|\mathcal{O}(0)|0\rangle \\ &= \sum_{n=0}^{\infty} \int \frac{d\theta_1 \dots d\theta_n}{n!(2\pi)^n} \mid F_n(\theta_1 \dots \theta_n) \mid^2 \exp\left(-mr \sum_{i=1}^{n} \cosh\theta_i\right) \end{aligned} \tag{20.5.31}$$

where r is the radial distance $r = \sqrt{x_0^2 + x_1^2}$ (Fig. 20.7). Similar expressions, although more complicated, hold for the n-point correlation functions. It is worth making some comments to clarify the nature of these expressions and their advantage.

- The integrals that enter the spectral series are all convergent. This is in sharp contrast with the formalism based on the Feynman diagrams, in which one encounters the divergences of the various perturbative terms. In a nutshell, the deep reason of this difference between the two formalisms can be expressed as follows. The Feynman formalism is based on the quantization of a *free* theory and on the *bare* unphysical parameters of the lagrangian. What the renormalization

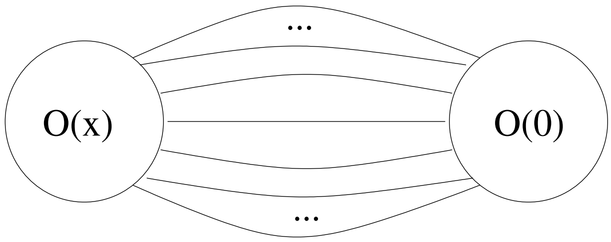


Fig. 20.7 *Spectral representation of the two-point correlation functions.*

procedure does is to implement the change from the bare to the physical parameters (such as the physical value of the mass of the particle). But the form factors employ *ab initio* all the physical parameters of the theory and for this reason the divergences of the perturbative series are absent.

- If the S-matrix depends on a coupling constant, as it happens in the Sinh–Gordon model or in other Toda field theories, each matrix element provides the exact resummation of all terms of perturbation theory.
- If the correlation functions do not have particularly violent ultraviolet singularities (this is the case, for instance, of the correlation functions of the relevant fields), the corresponding spectral series has an extremely fast convergent behavior for all values of mr. In the infrared region, that is for large values of mr, this is pretty evident from the nature of the series, because its natural parameter of expansion is e^{-mr}. The reason of the fast convergent behavior also in the ultraviolet region $mr \to 0$ is twofold: the peculiar behavior of the n-particle phase space in two-dimensional theories (see Appendix C of Chapter 17) and a further enhancement of the convergence provided by the form factors. To better understand this aspect, consider the Fourier transform of the correlator

$$G(x) \,=\, \langle \mathcal{O}(x)\,\mathcal{O}(0)\rangle \,=\, \int \frac{d^2p}{(2\pi)^2}\, e^{ip\cdot x}\,\hat{G}(p). \tag{20.5.32}$$

The function $\hat{G}(p)$ can be written as

$$\hat{G}(p) \,=\, \int_0^\infty d\mu^2\,\rho(\mu^2)\,\frac{1}{p^2+\mu^2}, \tag{20.5.33}$$

where $\rho(k^2)$ is a relativistically invariant function called the *spectral density*

$$\rho(k^2) \,=\, 2\pi \sum_{n=0}^{\infty} \int d\Omega_1 \ldots d\Omega_n\, \delta^2(k - P_n)\, |\langle 0\,|\mathcal{O}(0)\,|\theta_1,\ldots,\theta_n\rangle|^2$$

$$d\Omega \,=\, \frac{dp}{2\pi E} \,=\, \frac{d\theta}{2\pi}, \quad P_n^{(0)} = \sum_{k=0}^{n} \cosh\theta_k, \quad P_n^{(1)} = \sum_{k=0}^{n} \sinh\theta_k.$$

Since $1/(p^2+\mu^2)$ is the two-point correlation function of the euclidean free theory with mass μ, i.e. the propagator, eqn. (20.5.33) shows that the two-point correlation function can be regarded as a linear superposition of the free propagators weighted with the spectral density $\rho(\mu^2)$. Notice that the contribution given by the single-particle state of mass m in the spectral density is given by

$$\rho_{1part}(k^2) \;=\; \frac{1}{2\pi}\delta(k^2 - m^2). \tag{20.5.34}$$

To analyze the behavior of $\rho(k^2)$ by varying k^2, let's make the initial approximation to take equal to 1 all the matrix elements. In this way, each term of the spectral series coincides with the n-particle phase space

$$\Phi_n(k^2) \equiv \int \prod_{k=1}^{n} d\Omega_k \, \delta^2(k - P_n). \tag{20.5.35}$$

As shown in Appendix C of Chapter 17, in two dimensions the space goes to zero when $k^2 \to \infty$ as

$$\Phi_n(k^2) \simeq \frac{1}{(2\pi)^{n-2}} \frac{1}{(n-2)!} \frac{1}{k^2} \left(\log \frac{k^2}{m^2}\right)^{n-2}, \tag{20.5.36}$$

whereas for $d > 2$ it diverges as

$$\Phi_n(k^2) \;\sim\; k^{\frac{n(d-2)-d}{2}}. \tag{20.5.37}$$

On the other hand, $\Phi_n(k^2) = 0$ if $k^2 < (n\,m)^2$ and near the threshold values we have

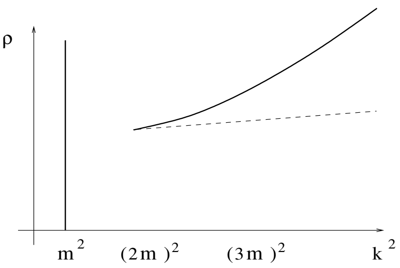
$$\Phi(k^2) \;\simeq\; A_n \left(\sqrt{k^2} - n\,m\right)^{\frac{n-3}{2}}. \tag{20.5.38}$$

Hence, we see that for pure reasons related to the phase space we have two different scenarios for the quantum field theories in two dimensions and in higher dimensions: while in $d > 2$ surpassing the various thresholds the spectral density receives contributions that are more divergent, in $d = 2$ they are all of the same order and all go to zero at large values of the energy. Hence, for $d > 2$ it is practically impossible to approximate the spectral density for large values of k^2 by using the first terms of the series, relative to the states with few particles, whereas in $d = 2$ this is perfectly plausible. If we now include in the discussion also the form factors, one realizes that the situation is even better in $d = 2$! In fact, from the general expression (20.2.16) and for the vanishing of $F_{\min}(\theta_{ij})$ at the origin (eqn 20.2.18), the form factors vanish at the n-particle thresholds as

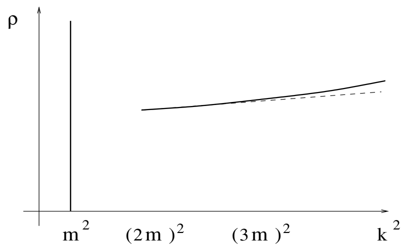
$$|\langle 0\,|\mathcal{O}(0)\,|\theta_1, \ldots, \theta_n\rangle|^2 \;\simeq\; \left(\sqrt{k^2} - n\,m\right)^{n(n-1)}, \qquad \theta_1 \simeq \ldots \simeq \theta_n \simeq 0 \tag{20.5.39}$$

while, for large values of their rapidities, they typically tend to a constant.[4] This scenario implies that the spectral density of the correlation functions of the two-dimensional integrable models usually flatten more at the thresholds and therefore becomes a very smooth function varying as k^2 (See Fig. 20.8). For all these reasons, the spectral density can be approximated with great accuracy just by taking the first terms of the series, even for large values of k^2, therefore leading to fast convergent behavior also in the ultraviolet region.

[4]This is what usually happens for the form factors of the strongly relevant operators.



(a)



(b)

Fig. 20.8 *Plot of the spectral series in a model in $d = 4$ (a) and in $d = 2$ (b). The contribution of the two-particle state is given by the dashed line. In $d = 4$ this does not provide a good approximation of $\rho(k^2)$ for large values of k^2 while in $d = 2$ it very often gives an excellent approximation of this quantity.*

20.6 Form Factors of the Stress–Energy Tensor

The stress–energy tensor is an operator that plays an important role in quantum field theory and its form factors have special properties. From its conservation law $\partial_\mu T^{\mu\nu}(x) = 0$, this operator can be written in terms of an auxiliary scalar field $A(x)$ as

$$T_{\mu\nu}(x) = (\partial_\mu \partial_\nu - g_{\mu\nu} \Box)\, A(x). \tag{20.6.40}$$

In light-cone coordinates, $x^{\pm} = x^0 \pm x^1$, its components are

$$T_{++} = \partial_+^2 A, T_{--} = \partial_-^2 A,$$

$$\Theta = T_\mu^\mu = -\Box A = -4\,\partial_+ \partial_- A.$$

Introducing the variables $x_j = e^{\theta_j}$ and the elementary symmetric polynomials $\sigma_i^{(n)}$, it is easy to see that

$$\begin{aligned} F_n^{T_{++}}(\theta_1, \ldots, \theta_n) &= -\frac{1}{4} m^2 \left(\frac{\sigma_{n-1}^{(n)}}{\sigma_n^{(n)}} \right)^2 F_n^A(\theta_1, \ldots, \theta_n), \\ F_n^{T_{--}}(\theta_1, \ldots, \theta_n) &= -\frac{1}{4} m^2 \left(\sigma_1^{(n)} \right)^2 F_n^A(\theta_1, \ldots, \theta_n), \\ F_n^{\Theta}(\theta_1, \ldots, \theta_n) &= m^2 \frac{\sigma_1^{(n)} \sigma_{n-1}^{(n)}}{\sigma_k} F_n^A(\theta_1, \ldots, \theta_n). \end{aligned} \tag{20.6.41}$$

Solving for F_n^A, we have

$$F_n^{T_{++}}(\theta_1,\ldots,\theta_n) \;=\; -\,\frac{1}{4}\,\frac{\sigma_{n-1}^{(n)}}{\sigma_1^{(n)}\sigma_n^{(n)}}\,F_n^{\Theta}(\theta_1,\ldots,\theta_n),$$
$$F_n^{T_{--}}(\theta_1,\ldots,\theta_n) \;=\; -\,\frac{1}{4}\,\frac{\sigma_1^{(n)}\sigma_n^{(n)}}{\sigma_{n-1}^{(n)}}\,F_n^{\Theta}(\theta_1,\ldots,\theta_n). \tag{20.6.42}$$

Hence, the whole set of form factors of $T_{\mu\nu}$ can be recovered by the form factors of the trace Θ. This is a scalar operator and therefore its form factors depend on the differences of the rapidities $\theta_{ij} = \theta_i - \theta_j$. Moreover, since the form factors of T_{--} and T_{++} must have the same singularities as those of Θ, $F_n^{\Theta}(\theta_1,\ldots,\theta_n)$ (for $n > 2$) has to be proportional to the combination $\sigma_1^{(n)}\sigma_{n-1}^{(n)}$ of the elementary symmetric polynomials. This combination corresponds to the relativistic invariant given by the total energy and momentum of the system.

For the normalization of these matrix elements, the recursive structure reduces the problem of finding the normalization of the form factors of $\Theta(x)$ on the one and two-particle states, i.e. $F_1^{\Theta}(\theta)$ and $F_2^{\Theta}(\theta_{12})$. The normalization of $F_2^{\Theta}(\theta_{12})$ can be determined by using the total energy of the system

$$E \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx^1\, T^{00}(x). \tag{20.6.43}$$

Computing the matrix element of both terms of this equation on the asymptotic states $\langle\theta'|$ and $|\theta\rangle$, for the left-hand side we have

$$\langle\theta'|\, E\,|\theta\rangle \;=\; 2\pi\, m\, \cosh\theta\,\delta(\theta'-\theta).$$

On the other hand, taking into account that $T^{00} = \partial_1^2 A$ and using the relation

$$\langle\theta'|\,\mathcal{O}(x)|\theta\rangle \;=\; e^{i\left(p^{\mu}(\theta') - p^{\mu}(\theta)\right)x_{\mu}}\, F_2^{\mathcal{O}}(\theta,\,\theta' - i\pi)$$

which holds for any hermitian operator $\mathcal{O}$, we obtain

$$F_2^{\partial_1^2 A}(\theta_1,\theta_2) \;=\; -\,m^2\,(\sinh\theta_1 + \sinh\theta_2)^2\, F_2^A(\theta_{12}).$$

From eqns (20.6.41) and (20.6.43) it follows that the normalization of F_2^{Θ} is given by

$$F_2^{\Theta}(i\pi) \;=\; 2\,\pi\, m^2. \tag{20.6.44}$$

However, there is no particular constraint on the one-particle form factor of $\Theta(x)$ coming from general considerations

$$F_1^{\Theta} \;=\; \langle 0 \mid \Theta(0) \mid \theta\rangle. \tag{20.6.45}$$

This is a free parameter of the theory, related to the intrinsic ambiguity of $T^{\mu\nu}(x)$, since this tensor can always be modified by adding a total divergence (see Problem 1).

20.7 Vacuum Expectation Values

The recursive equations enable us to determine the form factors $F_n^{\mathcal{O}}$ in terms of the previous $F_{n-1}^{\mathcal{O}}$ or $F_{n-2}^{\mathcal{O}}$. At the bottom of this iterative structure there are, as its initial seeds, the lowest quantities $F_0^{\mathcal{O}}$, i.e. the vacuum expectation value of the operator $\mathcal{O}$ and F_1, i.e. its matrix elements on one-particle states. Presently it is not known how to determine in general all the one-particle matrix elements. However, the situation is much better for the vacuum expectation values: they can be computed exactly for several operators both of the Sine–Gordon and Bullogh–Dodd models, as well as of RSOS restrictions thereof. The theoretical steps that lead to these results are quite technical but well described in the series of papers quoted at the end of the chapter and will not be reviewed here. In this section we will simply present the most relevant formulas, in particular, the exact vacuum expectation values of primary fields in integrable perturbed conformal field theories, with respect to the deformations $\Phi_{1,3}$, $\Phi_{1,2}$, and $\Phi_{2,1}$. In the following to denote such theories we use the notation

$$\mathcal{S}_m^{(k,l)\pm} \;=\; \mathcal{S}_m^{(CFT)} \pm \lambda \int d^2x\, \Phi_{k,l}(x), \tag{20.7.46}$$

where $\mathcal{S}_m$ is the action of the m-th conformal minimal model, $\Phi_{r,s}$ is the relevant primary field that leads to an integrable model, and $\lambda > 0$ is its dimensional coupling constant setting the scale of the quantum field theory (the sign of the coupling only makes sense after fixing the normalization of the fields $\Phi_{r,s}$). Hereafter

$$x \equiv (m+1)k - ml.$$

Integrable theory $\mathcal{S}_m^{(1,3)-}$. For $\lambda > 0$, $\Phi_{1,3}$ induces a massless flow between the minimal models $\mathcal{M}_m \to \mathcal{M}_{m-1}$ (see Section 15.6). For $\lambda < 0$, $\Phi_{1,3}$ drives instead the conformal model into a massive phase where there are kinks interpolating the $(m-1)$ RSOS degenerate vacua labeled as

$$\mathbf{a} = 0,\, \frac{1}{2}, \ldots, \frac{(m-2)}{2}.$$

For the vacuum expectation values of the primary fields on the various vacua we have

$$\langle \mathbf{a}|\Phi_{k,l}|\mathbf{a}\rangle^{(1,3)-} \;=\; \frac{\sin\left(\frac{\pi(2a+1)}{m}((m+1)k - ml)\right)}{\sin\frac{\pi(2a+1)}{m}}\, F_{k,l}^m(x) \tag{20.7.47}$$

where

$$F_{k,l}^m(x) \;=\; \left(M\frac{\sqrt{\pi}\,\Gamma\left(\frac{m+3}{2}\right)}{2\Gamma\left(\frac{m}{2}\right)}\right)^{2\Delta_{k,l}} \mathcal{Q}_{1,3}(x)$$

and

$$\mathcal{Q}_{1,3}(\eta) = \exp\left\{\int_0^\infty \frac{dt}{t}\left[\frac{\cosh(2t)\sinh((\eta-1)t)\sinh((\eta+1)t)}{2\cosh(t)\sinh(mt)\sinh((1+m)t)} - \frac{\eta^2-1}{2m(m+1)}\mathrm{e}^{-4t}\right]\right\}.$$

In the formula above

$$M = \frac{2\Gamma\left(\frac{m}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{m+1}{2}\right)} \left[\frac{\pi\lambda(1-m)(2m-1)}{(1+m)^2}\sqrt{\frac{\Gamma\left(\frac{1}{m+1}\right)\Gamma\left(\frac{1-2m}{m+1}\right)}{\Gamma\left(\frac{m}{m+1}\right)\Gamma\left(\frac{3m}{m+1}\right)}}\right]^{\frac{1+m}{4}}$$

is the common mass of the kinks expressed in term of the coupling constant λ.

Integrable theory $\mathcal{S}_m^{(1,2)}$. For the integrable model $\mathcal{S}_m^{(1,2)}$, the vacuum structure of the theory depends on whether m is odd or even.

- m **even.** When m is even, the field $\Phi_{1,2}$ is even under the Z_2 spin symmetry and the two theories $\mathcal{S}_m^{(1,2)\pm}$ are different although related by duality. The number of RSOS vacua of $\mathcal{S}_m^{(1,2)+}$ is equal to $(m-2)/2$, while the number of vacua of $\mathcal{S}_m^{(1,2)-}$ is equal to $m/2$. Their label is

$$\begin{aligned} \mathbf{a} &= \frac{1}{2}, \frac{3}{2}, \dots, \frac{m-3}{2}, \quad \lambda > 0 \\ \mathbf{a} &= 0, 1, \dots, \frac{m-2}{2}, \quad \lambda < 0. \end{aligned}$$

- m **odd**. In this case the field $\Phi_{1,2}$ is odd under the Z_2 symmetry and the two theories $\mathcal{S}_m^{(1,2)\pm}$ are equal. There are $(m-1)/2$ degenerate vacua in both theories that we label as

$$\begin{aligned} a &= \frac{1}{2}, \frac{3}{2}, \dots, \frac{m-2}{2}, \quad \lambda > 0 \\ a &= 0, 1, \dots, \frac{m-3}{2}, \quad \lambda < 0. \end{aligned}$$

The vacuum expectation values of the primary fields on the various vacua are:

$$\langle \mathbf{a}|\Phi_{k,l}|\mathbf{a}\rangle^{(1,2)} = \frac{\sin\left(\frac{\pi(2a+1)}{m}((m+1)k - ml)\right)}{\sin\frac{\pi(2a+1)}{m}} G_{k,l}^m(x) \tag{20.7.48}$$

where

$$G_{k,l}^m(x) = \left(M\frac{\pi(m+1)\Gamma\left(\frac{2m+2}{3m+6}\right)}{2^{\frac{2}{3}}\sqrt{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{m}{3m+6}\right)}\right)^{2\Delta_{k,l}} \mathcal{Q}_{1,2}(x)$$

and

$$\begin{aligned} \mathcal{Q}_{1,2}(\eta) = \exp\Bigg\{\int_0^\infty \frac{dt}{t}\Bigg[&\frac{\sinh((m+2)t)\sinh((\eta-1)t)\sinh((\eta+1)t)}{\sinh(3(m+2)t)\sinh(2(m+1)t)\sinh(mt)} \\ &\times(\cosh(3(m+2)t) + \cosh((m+4)t) - \cosh((3m+4)t) + \cosh(mt) + 1) \\ &-\frac{\eta^2-1}{2m(m+1)}\mathrm{e}^{-4t}\Bigg]\Bigg\}. \end{aligned}$$

In the formula above

$$M = \frac{2^{\frac{m+5}{3m+6}}\sqrt{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{m}{3m+6}\right)}{\pi\Gamma\left(\frac{2m+2}{3m+6}\right)}\left[\frac{\pi^2\lambda^2\Gamma\left(\frac{3m+4}{4m+4}\right)\Gamma\left(\frac{1}{2}+\frac{1}{m+1}\right)}{\Gamma\left(\frac{m}{4m+4}\right)\Gamma\left(\frac{1}{2}-\frac{1}{m+1}\right)}\right]^{\frac{m+1}{3m+6}}$$

is the mass of the kinks expressed in terms of the coupling constant λ.

Integrable theory $\mathcal{S}_m^{(2,1)}$. For this theory the situation is reversed with respect to the previous one: $\Phi_{2,1}$ is odd under the Z_2 symmetry when m is even (and the theory is independent of the sign of its coupling), while it is a Z_2 even field when m is odd (and the theories with $\lambda > 0$ and $\lambda < 0$ are related by duality). For the RSOS degenerate vacua, in this case we have the following labels:

- when m is even, both for $\lambda > 0$ and $\lambda < 0$, their number is $m/2$ and

$$\begin{aligned} a &= \frac{1}{2}, \frac{3}{2}, \ldots, \frac{m-1}{2}, \quad \lambda > 0 \\ a &= 0, 1, \ldots, \frac{m-2}{2}, \quad \lambda < 0; \end{aligned}$$

- when m is odd, their number is $(m-1)/2$ for $\lambda > 0$, and $(m+1)/2$ for $\lambda < 0$, with

$$\begin{aligned} a &= \frac{1}{2}, \frac{3}{2}, \ldots, \frac{m-2}{2}, \quad \lambda > 0 \\ a &= 0, 1, \ldots, \frac{m-1}{2}, \quad \lambda < 0. \end{aligned}$$

The vacuum expectation values of the primary fields on the various vacua are the expectation values

$$\langle \mathbf{a}|\Phi_{k,l}|\mathbf{a}\rangle^{(2,1)} = \frac{\sin\left(\frac{\pi(2a+1)}{m+1}((m+1)k - ml)\right)}{\sin\frac{\pi(2a+1)}{m+1}} H_{k,l}^m(x) \tag{20.7.49}$$

where

$$H_{k,l}^m(x) = \left(M\frac{\pi m\Gamma\left(\frac{2m}{3m-3}\right)}{2^{\frac{2}{3}}\sqrt{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{m+1}{3m-3}\right)}\right)^{2\Delta_{k,l}} \mathcal{Q}_{2,1}(x)$$

and

$$\begin{aligned} \mathcal{Q}_{2,1}(\eta) = \exp\Bigg\{ & \int_0^\infty \frac{dt}{t}\Big[\frac{\sinh((m-1)t)\sinh((\eta-1)t)\sinh((\eta+1)t)}{\sinh(3(m-1)t)\sinh(2mt)\sinh((m+1)t)} \\ & \times(\cosh(3(m-1)t) + \cosh((m-3)t) - \cosh((3m-1)t) \\ & + \cosh((m+1)t) + 1) - \frac{\eta^2-1}{2m(m+1)}\mathrm{e}^{-4t}\Big]\Bigg\}. \end{aligned} \tag{20.7.50}$$

The mass of the kinks, as a function of the coupling constant λ, is expressed by

$$M = \frac{2^{\frac{m-4}{3m-3}}\sqrt{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{m+1}{3m-3}\right)}{\pi\Gamma\left(\frac{2m}{3m-3}\right)}\left[\frac{\pi^2\lambda^2\Gamma\left(\frac{3m-1}{4m}\right)\Gamma\left(\frac{1}{2}-\frac{1}{m}\right)}{\Gamma\left(\frac{m+1}{4m}\right)\Gamma\left(\frac{1}{2}+\frac{1}{m}\right)}\right]^{\frac{m}{3m-3}}.$$

20.8 Ultraviolet Limit

In the ultraviolet limit, the correlation functions of the scaling operators has a power law behavior, dictated by the conformal weight of the operator

$$G(r) = \langle \mathcal{O}(r)\,\mathcal{O}(0)\rangle \simeq \frac{1}{r^{4\Delta}}, \qquad r \to 0. \tag{20.8.51}$$

One may wonder how the spectral series (20.5.31), which is based on the exponential terms $e^{-k\,mr}$, is able to reproduce a power law in the limit $r \to 0$. The answer to this question comes from an interesting analogy.

Feynman gas. Note that the formula (20.5.31) is formally similar to the expression of the grand-canonical partition function of a fictitious one-dimensional gas

$$\mathcal{Z}(mr) = \sum_{n=0}^{\infty} z^n\, Z_n. \tag{20.8.52}$$

To set up the vocabulary of this analogy, let's identify the coordinates of the gas particles with the rapidities θ_i, with the Boltzmann weight relative to the interactive potential of the gas with the modulus squared of the form factors

$$e^{-V(\theta_1,\ldots,\theta_n)} \equiv |\langle 0\,|\mathcal{O}(0)|\theta_1,\ldots,\theta_n\rangle|^2. \tag{20.8.53}$$

Finally, let's identify the fugacity of the gas with

$$z(\theta) = \frac{1}{2\pi}\,e^{-mr\cosh\theta}. \tag{20.8.54}$$

We have defined in this way the *Feynman gas* that was analyzed at the end of Chapter 2. The only difference with respect to the standard case is the coordinate dependence of the fugacity of this gas. Although the coordinates of the particles of this gas span the infinite real axis, the effective volume of the system is however determined by the region in which the fugacity (20.8.54) is significantly different from zero, as shown in Fig. 20.9. Note that $z(\theta)$ is a function that rapidly goes to zero outside a finite interval and, in the limit $mr \to 0$, presents a plateau of height $z_c = 1/(2\pi)$ and width

$$L \simeq 2\,\log\frac{1}{mr}.$$

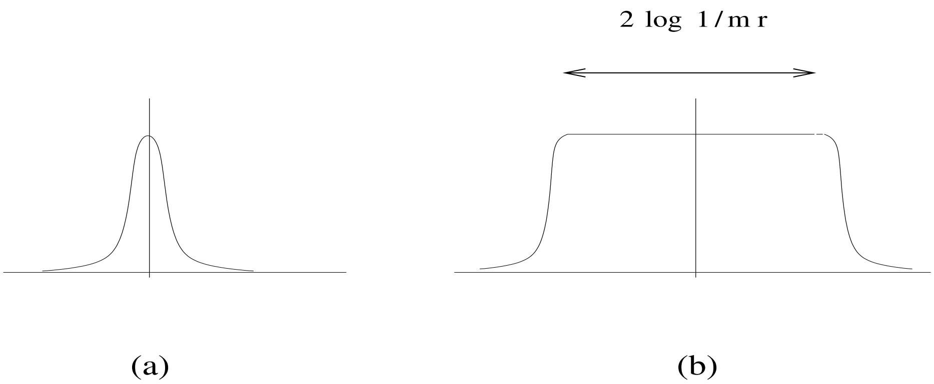


Fig. 20.9 *Plot of the fugacity as a function of θ: (a) for finite values of (mr); (b) in the limit $(mr) \to 0$.*

The equation of state of a one-dimensional gas is given by

$$\mathcal{Z} = e^{p(z)L},$$

where $p(z)$ is the pressure as a function of the fugacity. Following this analogy, for the two-point correlation function in the limit $(mr) \to 0$, we have

$$G(r) = \mathcal{Z} = e^{p(z_c)\,L} \simeq e^{2p(z_c)\log 1/(mr)} = \left(\frac{1}{mr}\right)^{2p(z_c)}, \qquad (20.8.55)$$

i.e. a power law behavior! Moreover, comparing with the short-distance behavior of the correlator given in eqn (20.8.51), there is an interesting result: the conformal weight can be expressed in terms of the pressure of this fictitious one-dimensional gas, evaluated at the plateau value of the fugacity

$$2\,\Delta = p(1/2\pi). \qquad (20.8.56)$$

Besides the thermodynamics of the Feynman gas, the conformal weight of the operators can also be extracted by applying the sum rule given by the Δ-theorem (see Chapter 15)

$$\Delta = -\frac{1}{2\langle\mathcal{O}\rangle}\int_0^\infty dr\, r\,\langle\Theta(r)\mathcal{O}(0)\rangle. \qquad (20.8.57)$$

To compute this quantity, it is necessary to know the form factors of the operator $\mathcal{O}(x)$ and the trace of the stress–energy tensor $\Theta(x)$.

c-theorem sum rule. Additional control of the ultraviolet limit of the theory is provided by the sum-rule of the c-theorem: it gives the central charge of conformal field theory associated to the ultraviolet limit of the massive theory through the integral

$$c = \frac{3}{2}\int_0^\infty dr\, r^3\,\langle\Theta(r)\Theta(0)\,\rangle_c.$$

Using the spectral representation of this correlator we have

$$c = \sum_{n=1}^{\infty} c_n, \tag{20.8.58}$$

where the n-particle contribution is

$$\begin{aligned} c_n &= \frac{12}{n!} \int_0^{\infty} \frac{d\mu}{\mu^3} \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \\ &\times \delta\left(\sum_{i=1}^{n} \sinh\theta_i\right) \delta\left(\sum_{i=1}^{n} \cosh\theta_i - \mu\right) |\langle 0|\Theta(0)|\theta_1, \ldots, \theta_n\rangle|^2. \end{aligned} \tag{20.8.59}$$

Usually this series presents very fast behavior. This permits us to obtain rather accurate estimations of the central charge c, with an explicit check of the entire formalism of the S-matrix and form factors. It is easy to understand the reason for this fast convergence by studying the integrand, shown in Fig. 20.10: the term r^3 kills the singularity of the correlator at short distance (therefore the integrand vanishes at the origin), while it weights the correlator more at large distances. But this is just the region where a few terms of the spectral series are very efficient in approximating the correlation function with high accuracy.

Asymptotic behavior. Finally, let's discuss the upper bound on the asymptotic behavior of the form factors dictated by the ultraviolet behavior of the correlator (20.8.51). To establish this bound, let's start by noting that in a massive theory we have

$$M_p \equiv \int d^2x\, |x|^p \,\langle\, \mathcal{O}(x)\mathcal{O}(0)\,\rangle_c \qquad < +\infty \qquad \text{if} \qquad p > 4\Delta_{\mathcal{O}} - 2. \tag{20.8.60}$$

Employing now the spectral representation of the correlator (20.5.32) and integrating over p, μ, and x, we get

$$M_p \sim \sum_{n=1}^{\infty} \int_{\theta_1 > \ldots > \theta_n} d\theta_1 \ldots d\theta_n \frac{|F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n)|^2}{(\sum_{k=1}^{n} m_k \cosh\theta_k)^{p+1}}\, \delta\left(\sum_{k=1}^{n} m_k \sinh\theta_k\right). \tag{20.8.61}$$

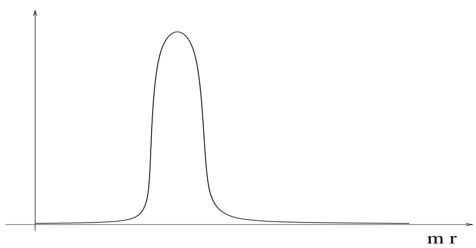


Fig. 20.10 *Plot of the integrand $r^3\langle\Theta(r)\Theta(0)\rangle$ in the C-theorem sum rule.*

Equation (20.8.60) can now be used to find an upper limit on the real quantity y_Φ, defined by

$$\lim_{|\theta_i| \to \infty} F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n) \sim e^{y_\Phi |\theta_i|}. \tag{20.8.62}$$

In fact, taking the limit $\theta_i \to +\infty$ in the integrand of (20.8.61), the delta-function forces some other rapidities to move at $-\infty$ as $-\theta_i$. Because the matrix element $F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n)$ depends on the differences of the rapidities, it contributes to the integrand with the factor $e^{2y_\Phi |\theta_i|}$ in the limit $|\theta_i| \to \infty$. Hence, eqn (20.8.60) leads to the condition

$$y_{\mathcal{O}} \leq \Delta_{\mathcal{O}}. \tag{20.8.63}$$

This equation provides information on the partial degree of the polynomial $Q_n^{\mathcal{O}}$. Note, however, that this conclusion may not apply for non-unitary theories because not all terms of the expansion on the intermediate states are necessarily positive in this case.

20.9 The Ising Model at $T \neq T_c$

In this section we present the form factors and the correlation functions of the relevant operators $\epsilon(x)$, $\sigma(x)$, and $\mu(x)$ of the two-dimensional Ising model when the temperature T is away from its critical value. From the duality of the model, we can discuss equivalently the case $T > T_c$ or $T < T_c$. Suppose the system is in the high-temperature phase where the scattering theory of the off-critical model involves only one particle with an S-matrix $S = -1$. There are no bound states. The particle A can be considered as being created by the magnetization operator $\sigma(x)$, so that it is odd under the Z_2 symmetry of the Ising model, with its mass given by $m = |T - T_c|$.

Let's now employ the form factor equations to find the matrix elements of the various operators on the multiparticle states. The first step is the determination of the function $F_{min}(\theta)$ which satisfies

$$\begin{aligned} F_{\min}(\theta) &= -F_{\min}(-\theta) \\ F_{\min}(i\pi - \theta) &= F_{\min}(i\pi + \theta). \end{aligned} \tag{20.9.64}$$

The minimal solution is

$$F_{\min}(\theta) = \sinh \frac{\theta}{2}. \tag{20.9.65}$$

20.9.1 The Energy Operator

Let's initially discuss the form factors of the energy operator $\epsilon(x)$ or, equivalently, those of the trace of the stress–energy tensor, since the two operators are related by

$$\Theta(x) = 2\pi m\, \epsilon(x). \tag{20.9.66}$$

This is an even operator under the Z_2 symmetry and therefore it has matrix elements only on states with an even number of particles, F_{2n}^{Θ}. The recursive equations of the kinematical poles are particularly simple

$$-i \lim_{\tilde{\theta} \to \theta} (\tilde{\theta} - \theta) F_{2n+2}^{\Theta}(\tilde{\theta} + i\pi, \theta, \theta_1, \theta_2, \ldots, \theta_{2n}) = \left(1 - (-1)^{2n}\right) F_{2n}^{\Theta}(\theta_1, \ldots, \theta_{2n}) = 0. \tag{20.9.67}$$

Taking into account the normalization of the trace operator $F_2^{\Theta}(i\pi) = 2\pi m^2$, the simplest solution of all these equations is

$$F_{2n}^{\Theta}(\theta_1, \ldots, \theta_{2n}) = \begin{cases} -2\pi i\, m^2 \sinh \frac{\theta_1-\theta_2}{2} & , n = 2 \\ 0 & , \text{otherwise.} \end{cases} \tag{20.9.68}$$

In the light of the discussion in Section 20.4, note that the identification of the operator Θ with this specific sequence of form factors is equivalent to putting equal to zero all coefficients of the kernel solutions $F_{2n}^{(i)}$ at all the higher levels.

We have an explicit check that (20.9.68) is the correct sequence of the form factors of the trace operator which comes from its two-point correlation function and from the c-theorem. For the correlator we get

$$\begin{aligned} G^{\Theta}(r) = \langle \Theta(r)\Theta(0)\rangle &= \frac{1}{2} \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} |F_2^{\Theta}(\theta_{12})|^2 \, e^{-mr(\cosh\theta_1 + \cosh\theta_1)} \\ &= \frac{m^4}{2} \int d\theta_1\, d\theta_2 \, \sinh^2 \frac{\theta_1 - \theta_2}{2} \, e^{-mr(\cosh\theta_1 + \cosh\theta_2)} \\ &= \frac{m^4}{4} \int d\theta_1\, d\theta_2 \, [\cosh(\theta_1 - \theta_2) - 1] \; e^{-mr(\cosh\theta_1 - \cosh\theta_2)} \\ &= m^4 \left(\left[\int d\theta \cosh\theta \, e^{-mr\cosh\theta} \right]^2 - \left[\int d\theta \, e^{-mr\cosh\theta} \right]^2 \right) \\ &= m^4 \left(K_1^2(mr) - K_0^2(mr) \right) \end{aligned} \tag{20.9.69}$$

where, in the last line, we used the integral representation of the modified Bessel functions

$$K_\nu(z) = \int_0^\infty dt \, \cosh \nu t \, e^{-z\cosh t}.$$

Hence, we have

$$G^{\Theta}(r) = \langle \Theta(r)\Theta(0)\rangle = m^4 \left[K_1^2(mr) - K_0^2(mr) \right]. \tag{20.9.70}$$

whose plot is in Fig. 20.11. This function has the correct ultraviolet behavior associated to the energy operator

$$G^{\Theta}(r) \to \frac{m^2}{|x|^2}, \qquad |x| \to 0. \tag{20.9.71}$$

Substituting the expression above in the c-theorem, we get the correct value of the central charge of the Ising model

$$c = \frac{3}{2} \int_0^\infty dr \, r^3 \langle \Theta(r)\Theta(0)\rangle = \frac{1}{2}. \tag{20.9.72}$$

20.9.2 Magnetization Operators

In the high-temperature phase, the order parameter $\sigma(x)$ is odd under the Z_2 symmetry while the disorder operator $\mu(x)$ is even. Hence, $\sigma(x)$ has matrix elements on states

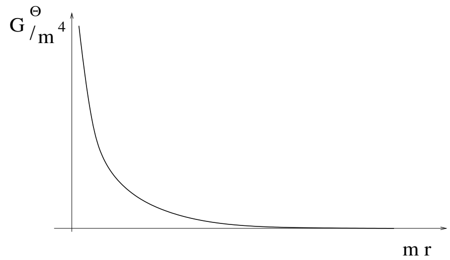


Fig. 20.11 *Plot of the two-point correlation function of the trace of the stress–energy tensor for the thermal Ising model.*

with an odd number of particles, F^{σ}_{2n+1}, whereas $\mu(x)$ is on an even number, F^{μ}_{2n}. In writing down the residue equations relative to the kinematical poles, we have to take into account that the operator μ has a semilocal index equal to 1/2 with respect to the operator $\sigma(x)$ that creates the asymptotic states. Denoting by F_n the form factors of these operators (for n even they refer to $\mu(x)$ while for n odd to $\sigma(x)$), we have the recursive equation

$$-i \lim_{\tilde{\theta}\to\theta} (\tilde{\theta} - \theta) F_{n+2}(\tilde{\theta} + i\pi, \theta, \theta_1, \theta_2, \ldots, \theta_{2n}) = 2\, F_n(\theta_1, \ldots, \theta_{2n}). \tag{20.9.73}$$

As for any form factor equation, these equations admit an infinite number of solutions that can be obtained by adding all possible kernel solutions at each level. The minimal solution is the one chosen to identify the form factors of the order and disorder operators

$$F_n(\theta_1, \ldots, \theta_n) \;=\; H_n \prod_{i<j}^{n} \tanh \frac{\theta_i - \theta_j}{2}. \tag{20.9.74}$$

The normalization coefficients satisfy the recursive equation

$$H_{n+2} \;=\; i\, H_n.$$

The solutions with n even are therefore fixed by choosing $F_0 = H_0$, namely with a non-zero value of the vacuum expectation of the disorder operator

$$F_0 \;=\; \langle 0|\mu(0)|0\rangle \;=\; \langle \mu \rangle, \tag{20.9.75}$$

while those with n odd are determined by the real constant F_1 relative to the one-particle matrix element of $\sigma(x)$

$$F_1 \;=\; \langle 0|\sigma(0)|A\rangle. \tag{20.9.76}$$

Adopting the conformal normalization of both operators

$$\langle \sigma(x)\sigma(0)\rangle \;=\; \langle \mu(x)\mu(0)\rangle \simeq \frac{1}{|x|^{1/4}}, \qquad |x| \to 0 \tag{20.9.77}$$

it is possible to show that $F_0 = F_1$ and the vacuum expectation value F_0 can be computed using eqn (20.7.47)

$$F_0 = F_1 = 2^{1/3}\, e^{-1/4}\, A^3\, m^{1/4}, \tag{20.9.78}$$

where $A = 1.282427..$ is called the Glasher constant. Vice versa, if we choose $F_0 = F_1 = 1$ (as we do hereafter), for the ultraviolet behavior of the correlation functions we have

$$\langle\sigma(x)\sigma(0)\rangle = \langle\mu(x)\mu(0)\rangle \simeq \frac{2^{-1/3}e^{1/4}\,A^{-3}}{|x|^{1/4}} = \frac{0.5423804\ldots}{|x|^{1/4}}, \qquad |x| \to 0. \tag{20.9.79}$$

There are several ways to check the correct identification of the form factors of the order/disorder operators. A direct way is to employ the Δ-theorem. In fact, using the matrix elements of $\mu(x)$ and $\Theta(x)$, we can compute their correlator, following the same procedure as in eqn (20.9.69)

$$\begin{aligned}\langle\Theta(r)\mu(0)\rangle &= \frac{1}{2}\int \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi} F^{\Theta}(\theta_{12})\,\bar{F}^{\mu}(\theta_{12})\,e^{-mr(\cosh\theta_1+\cosh\theta_2)}\\ &= -m^2\,\langle\mu\rangle\left[\frac{e^{-2mr}}{2mr} + Ei(-2mr)\right]\end{aligned} \tag{20.9.80}$$

where

$$Ei(-x) = -\int_x^{\infty}\frac{dt}{t}\,e^{-t}.$$

Substituting this correlator in the formula of the Δ-theorem, one obtains the correct value of the conformal weight of the disorder operator

$$\Delta = -\frac{1}{2\langle\mu\rangle}\int_0^{\infty} dr\, r\langle\Theta(r)\mu(0)\rangle = \frac{1}{4\pi}\int_0^{\infty} d\theta\,\frac{\sinh^2\theta}{\cosh^3\theta} = \frac{1}{16}. \tag{20.9.81}$$

Another way to determine the conformal weight of the magnetization operators consists of solving the thermodynamics of the Feynman gas associated to the form factors. Using the nearest-neighbor approximation discussed in Chapter 2, the pressure of this gas satisfies the integral equation (Problem 2)

$$z_c^{-1} = 2\pi = \int_0^{\infty} dx\,\tanh^2\frac{x}{2}\,e^{-px}, \tag{20.9.82}$$

whose numerical solution is

$$p \simeq 0.12529\ldots \tag{20.9.83}$$

Comparing with the exact value

$$p = 2\Delta = \frac{1}{8} = 0.125, \tag{20.9.84}$$

we see that the relative precision is less than one part in a thousand! This result confirms the validity of the form factor solution for the magnetization operators and, furthemore, it explicitly shows the convergence property of the spectral series.

20.9.3 The Painlevé Equation

The two-point correlation functions of the magnetization operators are given by

$$\begin{aligned}\langle \mu(r)\mu(0)\rangle &= \textstyle\sum_{n=0}^{\infty} g_{2n}(r)\\ \langle \sigma(r)\sigma(0)\rangle &= \textstyle\sum_{n=0}^{\infty} g_{2n+1}(r)\end{aligned}$$

where

$$g_n(r) = \frac{1}{n!}\int\left[\prod_{k=1}^{n}\frac{d\theta_k}{2\pi}\,e^{-mr\cosh\theta_k}\right]\prod_{i<j}\tanh^2\frac{\theta_{ij}}{2}.$$

These expressions can be further elaborated: imposing $u_i = e^{\theta_i}$ and using

$$\tanh^2\frac{\theta_i-\theta_j}{2} = \left(\frac{u_i-u_j}{u_i+u_j}\right)^2,$$

we get

$$\prod_{i<j}\tanh^2\frac{\theta_{ij}}{2} = \prod_{i<j}\left(\frac{u_i-u_j}{u_i+u_j}\right)^2 = \det W, \tag{20.9.85}$$

where the matrix elements of the operator W are

$$W_{ij} = \frac{2\sqrt{u_i\,u_j}}{u_i+u_j}.$$

Combining the two correlators

$$G^{(\pm)}(r) = \langle\mu(r)\mu(0)\rangle \pm \langle\sigma(r)\sigma(0)\rangle = \sum_{n=0}^{\infty}\lambda^n\,g_n(r) \tag{20.9.86}$$

(with $\lambda = \pm 1$) and using (20.9.85) we obtain

$$G^{(\pm)}(r) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int\left[\prod_{k=1}^{n}\frac{d\theta_k}{2\pi}\,e^{-mr\cosh\theta_k}\right]\det W. \tag{20.9.87}$$

The last expression is nothing else but the Fredholm determinant of an integral operator V, whose kernel is

$$\begin{aligned}V(\theta_i,\theta_j,r) &= \frac{E(\theta_i,r)\,E(\theta_j,r)}{u_i+u_j}\\ E(\theta_i,r) &= (2u_i\,e^{-mr\cosh\theta_i})^{1/2}.\end{aligned}$$

Hence

$$G^{(\pm)}(r) = \text{Det}\,(1+\lambda\,V). \tag{20.9.88}$$

The remarkable circumstance that the correlation functions are expressed in terms of the Fredholm determinant of an integral operator is crucial for studying their properties. The detailed discussion is beyond the scope of this book and here we simply present the main conclusions.

First of all, the expression given in eqn (20.9.88) permits us to solve *exactly* the thermodynamics of the Feynman gas associated to the form factors of the correlation function $G^{(+)}(r)$. The exact expression of the pressure of the Feynman gas is given by

$$
\begin{aligned}
p(z) &= \frac{1}{4}\int \frac{dp}{2\pi}\,\log\left[1+\left(\frac{2\pi z}{\sinh \pi p}\right)^2\right] \\
&= \frac{1}{4\pi}\arcsin(2\pi z) - \frac{1}{4\pi^2}\arcsin^2(2\pi z).
\end{aligned}
$$

Substituting in this formula the plateau value of the fugacity, $z = z_c = 1/(2\pi)$, one obtains the exact value of the conformal weight of the magnetization operators, $p = 2\Delta = 1/8$.

Secondly, using the Fredholm determinant (20.9.88), it is possible to show that the correlators can be concisely written as

$$
\begin{pmatrix} \langle \mu(r)\mu(0)\rangle \\ \langle \sigma(r)\sigma(0)\rangle \end{pmatrix} = \begin{pmatrix} \cosh \frac{\Psi(s)}{2} \\ \sinh \frac{\Psi(s)}{2} \end{pmatrix} \exp\left[-\frac{1}{4}\int_s^\infty dt\, t\left[\left(\frac{d\Psi}{dt}\right)^2 - \sinh^2\Psi\right]\right] \qquad (20.9.89)
$$

($s = mr$), where $\Psi(s)$ is a function solution of the differential equation

$$
\frac{d^2\Psi}{ds^2} + \frac{1}{s}\frac{d\Psi}{ds} = 2\sinh(2\Psi), \qquad (20.9.90)
$$

with boundary conditions

$$
\begin{aligned}
\Psi(s) &\simeq -\log s + \text{costant}, \quad s \to 0 \\
\Psi(2) &\simeq 2/\pi\, K_0(2s), \quad s \to \infty.
\end{aligned} \qquad (20.9.91)
$$

With the substitution $\eta = e^{-\Psi}$, the differential equation becomes the celebrated Painlevé differential equation of the third kind

$$
\frac{\eta^{''}}{\eta} = \left(\frac{\eta^{'}}{\eta}\right)^2 - \frac{1}{s}\left(\frac{\eta^{'}}{\eta}\right) + \eta^2 - \frac{1}{\eta^2}. \qquad (20.9.92)
$$

This equation was originally obtained by T.T. Wu, B. McCoy, C. Tracy and E. Barouch by studying the scaling limit of the lattice Ising model. It has also been derived by M. Jimbo, T. Miwa, and K. Ueno by using the monodromy theory of differential equations.

20.10 Form Factors of the Sinh–Gordon Model

In this section we study the form factors of an integrable lagrangian theory, the one defined by the Sinh–Gordon model. The action is

$$
\mathcal{S} = \int d^2x\left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{g^2}\cosh g\phi(x)\right], \qquad (20.10.93)
$$

and it possesses the Z_2 symmetry $\phi \to -\phi$. The exact S-matrix relative to the particle created by the field $\phi(x)$ is given by

$$S(\theta, B) \;=\; \frac{\tanh \frac{1}{2}(\theta - i\frac{\pi B}{2})}{\tanh \frac{1}{2}(\theta + i\frac{\pi B}{2})}, \tag{20.10.94}$$

where B is a function of the coupling constant g:

$$B(g) \;=\; \frac{2g^2}{8\pi + g^2}. \tag{20.10.95}$$

The theory does not have bound states, therefore the form factors satisfy the recursive equations coming from the kinematic poles only. As we already discussed in Chapter 18, the S-matrix is invariant under the transformation

$$B \to 2 - B \tag{20.10.96}$$

namely, under the weak–strong duality

$$g \to \frac{8\pi}{g}. \tag{20.10.97}$$

The Z_2 symmetry implies that the even (odd) operators have form factors different from zero only on asymptotic states with an even (odd) number of particles. The simplest odd field is just $\phi(x)$, with the normalization given by

$$F_1^{\phi}(\theta) \;=\; \langle 0 \mid \phi(0) \mid \theta \rangle_{\rm in} \;=\; \frac{1}{\sqrt{2}}. \tag{20.10.98}$$

One of the most important fields is the stress–energy tensor

$$T_{\mu\nu}(x) \;=\; 2\pi \left(:\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}(x)\;:\right) \tag{20.10.99}$$

where : : denotes the normal order of the composite operators. Its trace $T_\mu^\mu(x) = \Theta(x)$ is normalized as

$$F_2^{\Theta}(\theta_{12} = i\pi) \;=\; {}_{\rm out}\langle \theta_1 \mid \Theta(0) \mid \theta_2 \rangle_{\rm in} \;=\; 2\pi m^2, \tag{20.10.100}$$

while F_1^{Θ} is a free parameter. In the following we will only discuss the case $F_1^{\Theta} = 0$: this is equivalent to regarding the Sinh–Gordon model as a deformation of the conformal field theory with central charge $c = 1$ (see Chapter 16 and Problem 1 at the end of the chapter).

20.10.1 Minimal Form Factor

The first step to the solution of the form factor equation consists of finding the minimal two-particle form factor. Expressing the S-matrix as

$$S(\theta) \;=\; \exp\left[8 \int_0^{\infty} \frac{dx}{x} \sinh\left(\frac{xB}{4}\right) \sinh\left(\frac{x}{2}(1 - \frac{B}{2})\right) \sinh \frac{x}{2} \sinh\left(\frac{x\theta}{i\pi}\right)\right].$$

we have

$$F_{\min}(\theta, B) = \mathcal{N} \exp\left[8\int_0^\infty \frac{dx}{x}\frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}(1-\frac{B}{2})\right)\sinh\frac{x}{2}}{\sinh x}\sin^2\left(\frac{x\hat{\theta}}{2\pi}\right)\right] \tag{20.10.101}$$

($\hat{\theta} \equiv i\pi - \theta$), with the normalization given by

$$\mathcal{N} = \exp\left[-4\int_0^\infty \frac{dx}{x}\frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}(1-\frac{B}{2})\right)\sinh\frac{x}{2}}{\sinh^2 x}\right].$$

The analytic structure of this function can be studied using its representation in terms of an infinite product of Γ functions (see Problem 3)

$$F_{\min}(\theta, B) = \prod_{k=0}^{\infty}\left|\frac{\Gamma\left(k+\frac{3}{2}+\frac{i\hat{\theta}}{2\pi}\right)\Gamma\left(k+\frac{1}{2}+\frac{B}{4}+\frac{i\hat{\theta}}{2\pi}\right)\Gamma\left(k+1-\frac{B}{4}+\frac{i\hat{\theta}}{2\pi}\right)}{\Gamma\left(k+\frac{1}{2}+\frac{i\hat{\theta}}{2\pi}\right)\Gamma\left(k+\frac{3}{2}-\frac{B}{4}+\frac{i\hat{\theta}}{2\pi}\right)\Gamma\left(k+1+\frac{B}{4}+\frac{i\hat{\theta}}{2\pi}\right)}\right|^2. \tag{20.10.102}$$

$F_{\min}(\theta, B)$ has a simple zero at $\theta = 0$ since $S(0) = -1$ and its asymptotic behavior is

$$\lim_{\theta\to\infty} F_{\min}(\theta, B) = 1.$$

It satisfies the functional equation

$$F_{\min}(i\pi+\theta, B)F_{\min}(\theta, B) = \frac{\sinh\theta}{\sinh\theta + \sinh\frac{i\pi B}{2}} \tag{20.10.103}$$

which can be proved by employing its representation (20.10.102). For the numerical evalutation of this function it is useful to use the mixed representation given by

$$F_{\min}(\theta, B) = \mathcal{N}\prod_{k=0}^{N-1}\left[\frac{\left(1+\left(\frac{\hat{\theta}/2\pi}{k+\frac{1}{2}}\right)^2\right)\left(1+\left(\frac{\hat{\theta}/2\pi}{k+\frac{3}{2}-\frac{B}{4}}\right)^2\right)\left(1+\left(\frac{\hat{\theta}/2\pi}{k+1+\frac{B}{4}}\right)^2\right)}{\left(1+\left(\frac{\hat{\theta}/2\pi}{k+\frac{3}{2}}\right)^2\right)\left(1+\left(\frac{\hat{\theta}/2\pi}{k+\frac{1}{2}+\frac{B}{4}}\right)^2\right)\left(1+\left(\frac{\hat{\theta}/2\pi}{k+1-\frac{B}{4}}\right)^2\right)}\right]^{k+1}$$
$$\times \exp\left[8\int_0^\infty \frac{dx}{x}\frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}(1-\frac{B}{2})\right)\sinh\frac{x}{2}}{\sinh^2 x}(N+1-N\,e^{-2x})\,e^{-2Nx}\sin^2\left(\frac{x\hat{\theta}}{2\pi}\right)\right].$$

The convergence of the integral in this formula can be improved by increasing the value of N.

20.10.2 Recursive Equations

The Sinh–Gordon model does not have bound states. Hence the only recursive equations come from the kinematical poles relative to the three-particle clusters. Using the identity

$$(p_1+p_2+p_3)^2 - m^2 = 8m^2\cosh\frac{1}{2}\theta_{12}\cosh\frac{1}{2}\theta_{13}\cosh\frac{1}{2}\theta_{23},$$

all possible poles in these channels are taken into account using the parameterization

$$F_n(\theta_1,\ldots,\theta_n) \,=\, H_n\, Q_n(x_1,\ldots,x_n) \prod_{i<j} \frac{F_{\min}(\theta_{ij})}{x_i + x_j} \tag{20.10.104}$$

where $x_i = e^{\theta_i}$ and H_n are normalization factors. The expression above has simple poles each time the difference of two rapidities θ_{ij} is equal to $i\pi$. The functions $Q_n(x_1,\ldots,x_n)$ are symmetric polynomials in x_i. For the form factors of the scalar operators, the total degree of these polynomials must be equal to that of the denominator, given by $n(n-1)/2$. The partial degree of Q_n depends instead on the asymptotic behavior of the operator $\mathcal{O}$. With the parameterization above, the recursive equations can be expressed as recursive equations for the polynomials Q_n

$$(-)^n\, Q_{n+2}(-x,x,x_1,\ldots,x_n) \,=\, x\mathcal{C}_n(x,x_1,x_2,\ldots,x_n)\, Q_n(x_1,x_2,\ldots,x_n) \tag{20.10.105}$$

where we have introduced the function

$$\mathcal{C}_n \,=\, \frac{-i}{4\sin(\pi B/2)} \left(\prod_{i=1}^{n} \left[(x+\omega x_i)(x-\omega^{-1}x_i) \right] - \prod_{i=1}^{n} \left[(x-\omega x_i)(x+\omega^{-1}x_i) \right] \right)$$

with $\omega = \exp(i\pi B/2)$. The normalization constants H_n in (20.10.104) satisfy the recursive equations

$$H_{2n+1} = H_1 \mu^{2n}, \quad H_{2n} = H_2 \mu^{2n-2},$$

with

$$\mu \equiv \left(\frac{4\sin(\pi B/2)}{F_{min}(i\pi, B)} \right)^{\frac{1}{2}}$$

where H_1 and H_2 are the initial conditions, fixed by the operator. Using the generating function of the elementary symmetric polynomials, the function $\mathcal{C}_n$ can be written as

$$\mathcal{C}_n(x,x_1,\ldots,x_n) \,=\, \sum_{k=1}^{n} \sum_{m=1,odd}^{k} [m]\, x^{2(n-k)+m} \sigma_k^{(n)} \sigma_{k-m}^{(n)} (-1)^{k+1}_{.} \tag{20.10.106}$$

where we have introduced the symbol $[n]$ defined by

$$[n] \,\equiv\, \frac{\sin(n\frac{B}{2})}{\sin\frac{B}{2}}.$$

Note that the elementary symmetric polynomials satsify the recursive equation

$$\sigma_k^{(n+2)}(-x,x,x_1,\ldots,x_n) \,=\, \sigma_k^{(n)}(x_1,x_2,\ldots,x_n) - x^2 \sigma_{k-2}^{(n)}(x_1,x_2,\ldots,x_n). \tag{20.10.107}$$

20.10.3 General Properties of the Q_n Solutions

The form factors of the derivative operators present a factorized form: for instance, for the operator $\partial\bar{\partial}\phi$ we have $Q_n = \sigma_{n-1}\sigma_1\tilde{Q}_n$. For this reason, it is convenient to focus attention on the so-called *irreducible operators*, whose form factors cannot be factorized, and use them as building blocks for the form factors of all other operators. The polynomials Q_n of the irreducible operators satisfy a series of interesting results coming from the recursive equations (20.10.105). Let's initially show that the partial degree of Q_n satisfies the inequality

$$\deg(Q_n) \leq n - 1. \tag{20.10.108}$$

It is easy to see that this result holds for Q_1 and Q_2. To show that it also holds for the higher polynomials, let us consider the two cases (a) $Q_n \neq 0$ and (b) $Q_n = 0$ separately.

- In case (a) the proof is by induction. Assume $\deg(Q_n) \leq n - 1$. Since $\mathcal{C}_n$ is bilinear in $\sigma^{(n)}$ (see eqn 20.10.106), the partial degree of $Q_{n+2}(-x, x, x_1, \ldots x_n)$ in the variables $x_1, \ldots x_n$ is less than or equal to $n + 1$. But the partial degree of $Q_{n+2}(x_1, x_2, \ldots x_{n+2})$ is equal to the partial degree of $Q_{n+2}(-x, x, x_1 \ldots, x_n)$, hence the partial degree of Q_{n+2} must be less than or equal to $n + 1$.
- In case (b), the space of the solutions is given by the kernel of the operator $\mathcal{C}$, namely
$$Q_{n+2}(-x, x, \ldots, x_{n+2}) = 0.$$
In the space of the polynomials $\mathcal{P}$ of total degree $(n + 2)(n + 1)/2$, there is only one solution of this equation, given by
$$Q_{n+2} = \prod_{i<j}^{n+2}(x_i + x_j). \tag{20.10.109}$$
This polynomial has partial degree $n + 1$ and coincides with the polynomial of the denominator of eqn. (20.10.104).

We have thus shown that the partial degree of Q_n must be less than or equal to $(n - 1)$ for any irreducible scalar operator. The first consequence is that the form factors of these operators cannot diverge when $\theta_i \to \infty$. The second consequence is the presence of an additional parameter at each step of the iterative procedure. This comes from a simple argument: the dimension of the space of the polynomials Q_n is given by the dimension of the space of the polynomials Q_{n-2} *plus* the dimension of the kernel. Since the kernel is one dimensional, the dimension of the space of the solutions increases exactly by one at each iterative step. With the initial conditions $\dim(Q_1) = \dim(Q_2) = 1$, we finally get

$$\dim(Q_{2n-1}) = \dim(Q_{2n}) = n. \tag{20.10.110}$$

Hence the most general form factor of an irreducible scalar operator belongs to a linear space that can be spanned by a basis Q_n^k:

$$Q_{2n}(A_1^{(2n)}, \ldots, A_n^{(2n)}) = \sum_{p=1}^{n} A_p^{(2n)} Q_{2n}^p \tag{20.10.111}$$
$$Q_{2n-1}(A_1^{(2n-1)}, \ldots, A_n^{(2n-1)}) = \sum_{p=1}^{n} A_p^{(2n-1)} Q_{2n-1}^p .$$

Each polynomial above defines a matrix element of an operator of the Sinh–Gordon model. Note that the dimension of this linear space grows exactly as the number of powers ϕ^k $(k < n)$ of the elementary field. This means that the matrix elements of the composite operators ϕ^k can be obtained as linear combinations of the above functions.

20.10.4 The Elementary Solutions

A remarkable class of solutions of the recursive equations (20.10.105) is given by[5]

$$Q_n(k) \;=\; ||M_{ij}(k)||, \tag{20.10.112}$$

where $M_{ij}(k)$ is the $(n-1) \times (n-1)$ matrix

$$M_{ij}(k) \;=\; \sigma_{2i-j}\, [i - j + k]. \tag{20.10.113}$$

and $||M||$ denotes the determinant of the matrix M. These polynomials are called *elementary solutions*: they depend on an arbitrary integer k and satisfy

$$Q_n(k) \;=\; (-1)^{n+1} Q_n(-k). \tag{20.10.114}$$

Although all $Q_n(k)$ are solutions of (20.10.105), not all of them are linearly independent. The simplest reason is that the dimension of the space of the solutions at the level $N = 2n$ (or $N = 2n - 1$) is at most n. Among the first representatives we have

$$Q_3(k) \;=\; \left|\left| \begin{array}{cc} [k]\sigma_1 & [k+1]\sigma_3 \\ [\text{k-1}] & [k]\sigma_2 \end{array} \right|\right| .$$

Using the trigonometric identity $[n]^2 - [n-1][n+1] = 1$, it is easy to see that this expression satisfies eqn. (20.10.105) (with $A_0^1 = 1$) for any integer k. These solutions allow us to express at once all the form factors of the elementary field $\phi(x)$ and the trace $\Theta(x)$ of the stress–energy tensor. In fact, it is possible to prove that the matrix elements of $\phi(x)$ are given by $Q_n(0)$. Note that the form factors relative to an even number of particles are automatically zero, in agreement with the Z_2 symmetry of the model. Those with an odd number of asymptotic particles vanish when $\theta_i \to \infty$, in agreement with the perturbative evaluation of these matrix elements given by the Feynman diagrams. The form factors of $\Theta(x)$ are instead given by the even polynomials $Q_{2n}(1)$, which go to a finite limit when $\theta_i \to \infty$, once again in agreement with their

[5] For simplicity we have suppressed the dependence of $Q_n(k)$ on the variables x_i.

Table 20.1: Approximate values of the central charge of the Sinh–Gordon model obtained by using only the two-particle form factor of $\Theta(x)$ in the c-theorem.

B	$\frac{g^2}{4\pi}$	$\Delta\, c^{(2)}$
$\frac{1}{10}$	$\frac{2}{19}$	0.9989538
$\frac{3}{10}$	$\frac{6}{17}$	0.9931954
$\frac{2}{5}$	$\frac{1}{2}$	0.9897087
$\frac{1}{2}$	$\frac{2}{3}$	0.9863354
$\frac{2}{3}$	1	0.9815944
$\frac{7}{10}$	$\frac{14}{13}$	0.9808312
$\frac{4}{5}$	$\frac{4}{3}$	0.9789824
1	2	0.9774634

perturbative computation. A further confirmation of the validity of this identification can be obtained by using the c-theorem. Employing just the two-particle form factor, we have the following approximated value of the ultraviolet central charge

$$c^{(2)} = \frac{3}{2F_{min}^2(i\pi)} \int_0^\infty \frac{d\theta}{\cosh^4\theta}\, |F_{min}(2\theta)|^2. \qquad (20.10.115)$$

The numerical values for different values of the coupling constant $g^2/4\pi$ are collected in Table 20.1. From this table one can see that the sum rule is saturared by the two-particle form factor even for large values of the coupling constant: this proves once again the fast convergent behavior of the spectral series.

It is interesting to understand which are the operators $\Psi_k(x)$ associated to the elementary solutions $Q_n(k)$ ($k \neq 0$). For the sequence of form factors related to $Q_n(k)$, let's choose the normalization as follows

$$H_1^k = \mu[k], \quad H_2^k = \mu^2\,[k]. \qquad (20.10.116)$$

The present conjecture is that the operators Ψ_k correspond to the vertex operators $e^{kg\phi}$. A non-trivial check of this conjecture is provided by the computation of the conformal weights $\Delta_k(g)$ that emerge in their ultraviolet limits. These quantities can be computed by analyzing the limit $x \to 0$ of the correlation function

$$\begin{aligned} G_{k,m}(x) &= \langle \Psi_k(x)\, \Psi_m(0)\rangle \\ &= \sum_{n=0}^{\infty} \int \frac{d\beta_1 \dots d\beta_n}{n!(2\pi)^n} F_n^{\Psi_k}(\beta_1 \dots \beta_n) F_n^{\Psi_m}(\beta_n \dots \beta_1) \exp\left(-mr \sum_{i=1}^{n} \cosh\beta_i\right). \end{aligned}$$

At first order in g, we have $\Delta_k(g) = -k^2 g^2/8\pi$ which coincides with the conformal weight of the vertex operators $e^{k\,g\,\phi(x)}$, computed using the gaussian conformal theory.

20.11 The Ising Model in a Magnetic Field

The Ising model in a magnetic field has quite a rich S-matrix: it has eight massive exitations and 36 elastic scattering amplitudes, some of them with higher order poles.

In addition to the functional and recursive equations, the form factors of this theory also satisfy other recursive equations related to the higher poles of the S-matrix. The relative formulas can be found in the papers by G. Delfino, G. Mussardo, and P. Simonetti quoted at the end of the chapter. Here we only report the main results about the form factors of the energy operator $\epsilon(x)$ and of the magnetization operator $\sigma(x)$. In this theory, the latter operator is proportional to the trace

$$\Theta(x) = 2\pi h(2 - 2\Delta_\sigma)\,\sigma(x). \tag{20.11.117}$$

Relying on the fast convergence of the spectral series, for the correlation functions of these operators we can focus our attention on the one and two-particle form factors. To begin with, let's fix some notation. For the S-matrix of the particles A_a and A_b we have

$$S_{ab}(\theta) = \prod_{\alpha \in \mathcal{A}_{ab}} (f_\alpha(\theta))^{p_\alpha} \tag{20.11.118}$$

where

$$f_\alpha(\theta) \equiv \frac{\tanh\frac{1}{2}(\theta + i\pi\alpha)}{\tanh\frac{1}{2}(\theta - i\pi\alpha)}. \tag{20.11.119}$$

The set of the numbers $\mathcal{A}_{ab}$ and their multiplicity p_α can be found in Table 18.3 of Chapter 18. It is convenient to parameterize the two-particle form factors of this theory as

$$F_{ab}^{\mathcal{O}}(\theta) = \frac{Q_{ab}^{\Phi}(\theta)}{D_{ab}(\theta)} F_{ab}^{min}(\theta), \tag{20.11.120}$$

where $D_{ab}(\theta)$ and $Q_{ab}^{\mathcal{O}}(\theta)$ are polynomials in $\cosh\theta$: the latter is fixed by the singularities of the S-matrix, the former depends on the operator $\mathcal{O}(x)$. The minimal form factors can be written as

$$F_{ab}^{min}(\theta) = \left(-i\sinh\frac{\theta}{2}\right)^{\delta_{ab}} \prod_{\alpha\in\mathcal{A}_{ab}} (G_\alpha(\theta))^{p_\alpha}, \tag{20.11.121}$$

where

$$G_\alpha(\theta) = \exp\left\{2\int_0^\infty \frac{dt}{t}\frac{\cosh\left(\alpha - \frac{1}{2}\right)t}{\cosh\frac{t}{2}\sinh t}\sin^2\frac{(i\pi - \theta)t}{2\pi}\right\}. \tag{20.11.122}$$

For large values of the rapidity, we have

$$G_\alpha(\theta) \sim \exp(|\theta|/2), |\theta| \to \infty, \tag{20.11.123}$$

independently of the index α.

From the analysis of the singularities of the form factors, one can arrive at the following expression for the denominator

$$D_{ab}(\theta) = \prod_{\alpha\in\mathcal{A}_{ab}} (\mathcal{P}_\alpha(\theta))^{i_\alpha} (\mathcal{P}_{1-\alpha}(\theta))^{j_\alpha}, \tag{20.11.124}$$

where

$$
\begin{array}{llll} i_\alpha = n+1, & j_\alpha = n, & \text{se} & p_\alpha = 2n+1; \\ i_\alpha = n, & j_\alpha = n, & \text{se} & p_\alpha = 2n, \end{array} \tag{20.11.125}
$$

having introduced the notation

$$
\mathcal{P}_\alpha(\theta) \equiv \frac{\cos \pi\alpha - \cosh\theta}{2\cos^2 \frac{\pi\alpha}{2}}. \tag{20.11.126}
$$

Both quantities $F_{ab}^{min}(\theta)$ and $D_{ab}(\theta)$ are normalized to be equal to 1 when $\theta = i\pi$.

The polynomials of the numerator can be expressed as

$$
Q_{ab}^{\mathcal{O}}(\theta) = \sum_{k=0}^{N_{ab}^{\mathcal{O}}} c_{ab,\mathcal{O}}^{(k)} \cosh^k \theta. \tag{20.11.127}
$$

The condition $\left[F_{ab}^{\mathcal{O}}(\theta)\right]^* = F_{ab}^{\mathcal{O}}(-\theta)$ follows from the monodromy condition satisfied by the form factors and from the property $S_{ab}^*(\theta) = S_{ab}(-\theta)$. This means that the coefficients $c_{ab,\mathcal{O}}^{(k)}$ are real numbers and their values identify the different operators.

The degrees of the polynomials are fixed by the conformal weight of the operators and, for both $\sigma(x)$ and $\epsilon(x)$, we have in particular $N_{11}^{\Phi} \leq 1$. Therefore the initial conditions of the recursive equation for the form factors of the two relevant operators consists of *two* free parameters, i.e. the coefficients $c_{11,\mathcal{O}}^{(0)}$ and $c_{11,\mathcal{O}}^{(1)}$. Furthemore, it can be checked that the number of free parameters does not increase implementation the bootstrap equations. Consider, for instance the condition $N_{12}^{\mathcal{O}} \leq 2$, which seems to imply three new coefficients $c_{12,\mathcal{O}}^{(k)}$ $(k = 1, 2, 3)$ for $F_{12}^{\mathcal{O}}(\theta)$. However, the amplitudes $S_{11}(\theta)$ and $S_{12}(\theta)$ have three common bound states. This circumstance gives rise to three equations

$$
\frac{1}{\Gamma_{11}^c} \text{Res}_{\theta = iu_{11}^c} F_{11}^{\Phi}(\theta) = \frac{1}{\Gamma_{12}^c} \text{Res}_{\theta = iu_{12}^c} F_{12}^{\Phi}(\theta), \qquad c = 1, 2, 3
$$

that permit us to fix the three coefficients $c_{12,\mathcal{O}}^{(k)}$ in terms of the two coefficients $c_{11,\mathcal{O}}^{(k)}$.

Table 20.2: Central charge given by the partial sum of the form factors entering the c-theorem. $c_{ab..}$ denotes the contribution of the state $A_a A_b ..$. The exact result is $c = 1/2$.

c_1	0.472038282
c_2	0.019231268
c_3	0.002557246
c_{11}	0.003919717
c_4	0.000700348
c_{12}	0.000974265
c_5	0.000054754
c_{13}	0.000154186
c_{partial}	0.499630066

Table 20.3: Conformal weights $\Delta_{\mathcal{O}}$ given by the partial sum of the form factors of the correlation functions entering the Δ-theorem. $\Delta_{ab..}$ denotes the contribution of the state $A_a A_b..$. The exact values are $\Delta_\sigma = 1/16 = 0.0625$ and $\Delta_\varepsilon = 1/2$.

	σ	ϵ
Δ_1	0.0507107	0.2932796
Δ_2	0.0054088	0.0546562
Δ_3	0.0010868	0.0138858
Δ_{11}	0.0025274	0.0425125
Δ_4	0.0004351	0.0069134
Δ_{12}	0.0010446	0.0245129
Δ_5	0.0000514	0.0010340
Δ_{13}	0.0002283	0.0065067
Δ_{partial}	0.0614934	0.4433015

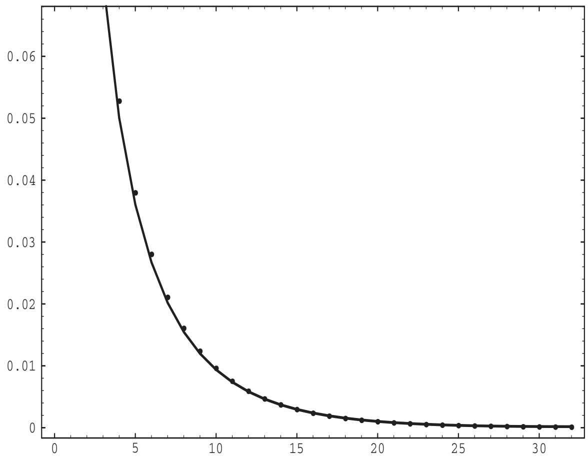


Fig. 20.12 *Plot of the correlation function $\langle\sigma(r)\sigma(0)\rangle$ for the Ising model in a magnetic field. The continuous line is the determination obtained with the first eight form factors, while the dots are the numerical determination of the correlators obtained by a Monte Carlo simulation.*

There is additional information about the numerator Q_{ab} of the operator $\Theta(x)$. In fact, from the conservation law $\partial_\mu T^{\mu\nu} = 0$ it follows that the polynomials Q_{ab}^{Θ} contain the factor

$$\left(\cosh\theta + \frac{m_a^2 + m_b^2}{2m_a m_b}\right)^{1-\delta_{ab}} . \tag{20.11.128}$$

The determination of the coefficients $c_{ab}^{(k)}$ and the one-particle form factors of the two operators $\sigma \sim \Theta$ and ϵ has been done in the papers cited at the end of the chapter and their values can be found there.

Employing these lowest form factors one can compute the correlation functions and perform some non-trivial checks by applying the sum rules of the c-theorem and Δ-theorem. The relative results are given in Tables 20.2 and 20.3. A successful check of the correlation function $\langle \sigma(r)\sigma(0) \rangle$ has also been done versus the numerical determination of this function, as shown in Fig. 20.12.

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Problems

1. Form factors of a free theory

Consider the theory of a free bosonic field $\phi(x)$ associated to a particle A of mass m.

a Compute the form factors of $\phi(x)$ and prove that $\langle 0|\phi(0)|A\rangle = 1/\sqrt{2}$. Show that the euclidean correlation function is given by

$$\langle \phi(x)\phi(0)\rangle \,=\, \frac{1}{\pi}\, K_0(mr).$$

b Show that the arbitrariness of the one-particle form factor of the trace of the stress–energy tensor

$$F_1^{\Theta} = \langle 0|\Theta(0)|A\rangle \equiv -\sqrt{2\pi}\, m^2\, Q$$

corresponds to the possibility of redefining the stress–energy tensor by adding a total divergence

$$\Theta(x) = 2\pi \left(m^2\phi^2 + \frac{Q}{\sqrt{\pi}} \Box\phi \right).$$

c Use the c-theorem and the form factors of $\Theta(x)$ to show that the central charge in the ultraviolet region is given by

$$c = 1 + 12Q^2.$$

2. Feynman gas

a Derive the equation of state of the Feynman gas associated to the form factors of the magnetization operators in the nearest neighbor approximation. Prove that the pressure $p(z)$ satisfies the integral equation (20.9.82).

b Justify the accuracy of the approximation of the conformal weights computing the average number of particles per unit length by means of the formula

$$\frac{\langle N\rangle}{L} = z\,\frac{\partial p}{\partial z}$$

and checking the very dilute nature of the gas.

3. Infinite products

Using the integral

$$\int \frac{dt}{t} e^{-\beta t} \sin^2 \frac{\alpha t}{2} = \frac{1}{4} \log \frac{\alpha^2 + \beta^2}{\beta^2},$$

and the identity satisfied by the Γ functions

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\gamma)\Gamma(\beta-\gamma)} = \prod_{k=0}^{\infty} \left[\left(1 + \frac{\gamma}{\alpha+k}\right) \left(1 - \frac{\gamma}{\beta+k}\right) \right],$$

to derive the expression for $F_{\min}(\theta)$ of the Sinh–Gordon model.

4. Cluster properties

Consider the form factors of a scattering theory based on the functions

$$f_x(\theta) = \frac{\tanh \frac{1}{2}(\theta + i\pi x)}{\tanh \frac{1}{2}(\theta - i\pi x)}$$

that have the property $\lim_{\theta\to\infty} f_x(\theta) = 1$.

a Using the Watson equation satisfied by the form factors $F_n^{\mathcal{O}_a}(\theta_1, \ldots, \theta_n)$ of an operator $\mathcal{O}_a$, prove that taking the limit

$$\lim_{\Lambda \to \infty} F_n^{\mathcal{O}_a}(\beta_1 + \Delta, \ldots, \beta_m + \Delta, \beta_{m+1}, \ldots, \beta_n) = F_m^{\mathcal{O}_b}(\beta_1, \ldots, \beta_m) F_{n-m}^{\mathcal{O}_c}(\beta_{m+1}, \ldots, \beta_n)$$

the form factor factorizes in terms of two functions both satisfying the Watson equations. Hence they can be considered the form factors of the operators $\mathcal{O}_b$ and $\mathcal{O}_c$. This expresses the cluster property of the form factors.

b Prove that the form factors of the elementary solutions of the Sinh–Gordon model are self-clustering quantities.

5. Correlation functions of the Ising model

Use the fermionic representation of the energy operator of the Ising model, $\epsilon = i\bar{\psi}\,\psi$, and the mode expansion of the fermionic field in terms of the creation and annihilation operators, to compute the matrix elements of $\epsilon(x)$ and its two-point correlation function.

6. Form factors of the Yang–Lee model

Using the form factors of the Sinh–Gordon model, obtain the form factors of the Yang–Lee model by using the analytic continuation $B \to -\frac{2}{3}$.