# Preface

This book is an introduction to statistical field theory, an important subject of theoretical physics that has undergone formidable progress in recent years. Most of the attractiveness of this field comes from its profound interdisciplinary nature and its mathematical elegance; it sets outstanding challenges in several scientific areas, such as statistical mechanics, quantum field theory, and mathematical physics.

Statistical field theory deals, in short, with the behavior of classical or quantum systems consisting of an enormous number of degrees of freedom. Those systems have different phases, and the rich spectrum of the phenomena they give rise to introduces several questions: What is their ground state in each phase? What is the nature of the phase transitions? What is the spectrum of the excitations? Can we compute the correlation functions of their order parameters? Can we estimate their finite size effects? An ideal guide to the fascinating area of phase transitions is provided by a remarkable model, the Ising model.

There are several reasons to choose the Ising model as a pathfinder in the field of critical phenomena. The first one is its simplicity – an essential quality to illustrate the key physical features of the phase transitions, without masking their derivation with worthless technical details. In the Ising model, the degrees of freedom are simple boolean variables $\sigma_{\vec{i}}$, whose values are $\sigma_{\vec{i}} = \pm 1$, defined on the sites $\vec{i}$ of a $d$-dimensional lattice. For these essential features, the Ising model has always played an important role in statistical physics, both at the pedagogical and methodological levels.

However, this is not the only reason of our choice. The simplicity of the Ising model is, in fact, quite deceptive. Despite its apparent innocent look, the Ising model has shown an extraordinary ability to describe several physical situations and a remarkable theoretical richness. For instance, the detailed analysis of its properties involves several branches of mathematics, quite distinguished for their elegance: here we mention only combinatoric analysis, functions of complex variables, elliptic functions, the theory of nonlinear differential and integral equations, the theory of the Fredholm determinant and, finally, the subject of infinite dimensional algebras. Although this is only a partial list, it is sufficient to prove that the Ising model is an ideal playground for several areas of pure and applied mathematics.

Equally rich is its range of physical aspects. Therefore, its study offers the possibility to acquire a rather general comprehension of phase transitions. It is time to say a few words about them: phase transitions are remarkable collective phenomena, characterized by sharp and discontinous changes of the physical properties of a statistical system. Such discontinuities typically occur at particular values of the external parameters (temperature or pressure, for instance); close to these critical values, there is a divergence of the mean values of many thermodynamical quantities, accompanied by anomalous fluctuations and power law behavior of correlation functions. From an experimental point of view, phase transitions have an extremely rich phenomenology, ranging from the superfluidity of certain materials to the superconductivity of others, from the mesomorphic transformations of liquid crystals to the magnetic properties of iron. Liquid helium $He^4$, for instance, shows exceptional superfluid properties at temperatures lower than $T_c = 2.19\,K$, while several alloys show phase transitions equally remarkable, with an abrupt vanishing of the electrical resistance for very low values of the temperature.

The aim of the theory of phase transitions is to reach a general understanding of all the phenomena mentioned above on the basis of a few physical principles. Such a theoretical synthesis is made possible by a fundamental aspect of critical phenomena: their universality. This is a crucial property that depends on two basic features: the internal symmetry of the order parameters and the dimensionality of the lattice. In short, this means that despite the differences that two systems may have at their microscopic level, as long as they share the two features mentioned above, their critical behaviors are surprisingly identical.[1] It is for these universal aspects that the theory of phase transitions is one of the pillars of statistical mechanics and, simultaneously, of theoretical physics. As a matter of fact, it embraces concepts and ideas that have proved to be the building blocks of the modern understanding of the fundamental interactions in Nature. Their universal behavior, for instance, has its natural demonstration within the general ideas of the renormalization group, while the existence itself of a phase transition can be interpreted as a spontaneously symmetry breaking of the hamiltonian of the system. As is well known, both are common concepts in another important area of theoretical physics: quantum field theory (QFT), i.e. the theory that deals with the fundamental interactions of the smallest constituents of the matter, the elementary particles.

The relationship between two theories that describe such different phenomena may appear, at first sight, quite surprising. However, as we will see, it will become more comprehensible if one takes into account two aspects: the first one is that both theories deal with systems of infinite degrees of freedom; the second is that, close to the phase transitions, the excitations of the systems have the same dispersion relations as the elementary particles.[2] Due to the essential identity of the two theories, one should not be surprised to discover that the two-dimensional Ising model, at temperature $T$ slightly away from $T_c$ and in the absence of an external magnetic field, is equivalent to a fermionic neutral particle (a Majorana fermion) that satisfies a Dirac equation. Similarly, at $T = T_c$ but in the presence of an external magnetic field $B$, the two-dimensional Ising model may be regarded as a quantum field theory with eight scalar particles of different masses.

The use of quantum field theory – i.e. those formalisms and methods that led to brilliant results in the study of the fundamental interactions of photons, electrons, and all other elementary particles – has produced remarkable progress both in the understanding of phase transitions and in the computation of their universal quantities. As will be explained in this book, our study will significantly benefit from such a possibility: since phase transitions are phenomena that involve the long distance scales of

[1] This becomes evident by choosing an appropriate combination of the thermodynamical variables of the two systems.

[2] The explicit identification between the two theories can be proved by adopting for both the path integral formalism.

the systems – the infrared scales – the adoption of the continuum formalism of field theory is not only extremely advantageous from a mathematical point of view but also perfectly justified from a physical point of view. By adopting the QFT approach, the discrete structure of the original statistical models shows itself only through an ultraviolet microscopic scale, related to the lattice spacing. However, it is worth pointing out that this scale is absolutely necessary to regularize the ultraviolet divergencies of quantum field theory and to implement its renormalization.

The main advantage of QFT is that it embodies a strong set of constraints coming from the compatibility of quantum mechanics with special relativity. This turns into general relations, such as the completeness of the multiparticle states or the unitarity of their scattering processes. Thanks to these general properties, QFT makes it possible to understand, in a very simple and direct way, the underlying aspects of phase transitions that may appear mysterious, or at least not evident, in the discrete formulation of the corresponding statistical model.

There is one subject that has particularly improved thanks to this continuum formulation: this is the set of two-dimensional statistical models, for which one can achieve a classification of the fixed points and a detailed characterization of their classes of universality. Let us briefly discuss the nature of the two-dimensional quantum field theories.

Right at the critical points, the QFTs are massless. Such theories are invariant under the conformal group, i.e. the set of geometrical transformations that implement a scaling of the length of the vectors while preserving their relative angle. But, in two dimensions conformal transformations coincide with mappings by analytic functions of a complex variable, characterized by an infinite-dimensional algebra known as a Virasoro algebra. This enables us to identify first the operator content of the models (in terms of the irreducible representations of the Virasoro algebra) and then to determine the exact expressions of the correlators (by solving certain linear differential equations). In recent years, thanks to the methods of conformal field theory, physicists have reached the exact solutions of a huge number of interacting quantum theories, with the determination of all their physical quantities, such as anomalous dimensions, critical exponents, structure constants of the operator product expansions, correlation functions, partition functions, etc.

Away from criticality, quantum field theories are, instead, generally massive. Their analysis can often be carried out only by perturbative approaches. However, there are some favorable cases that give rise to integrable models of great physical relevance. The integrable models are characterized by the existence of an infinite number of conserved charges. In such fortunate circumstances, the exact solution of the off-critical models can be achieved by means of $S$-matrix theory. This approach makes it possible to compute the exact spectrum of the excitations and the matrix elements of the operators on the set of these asymptotic states. Both these data can thus be employed to compute the correlation functions by spectral series. These expressions enjoy remarkable convergence properties that turn out to be particularly useful for the control of their behaviors both at large and short distances. Finally, in the integrable cases, it is also possible to study the exact thermodynamical properties and the finite size effects of the quantum field theories. Exact predictions for many universal quantities can also be obtained. For the two-dimensional Ising model, for instance, there are two distinct integrable theories, one corresponding to its thermal perturbation (i.e. $T \neq T_c$, $B = 0$), the other to the magnetic deformation ($B \neq 0$, $T = T_c$). In the last case, a universal quantity is given, for instance, by the ratio of the masses of the lowest excitations, expressed by the famous golden ratio $m_2/m_1 = 2\cos(\pi/5) = (\sqrt{5}+1)/2$.

In addition to their notable properties, the exact solution provided by the integrable theories is an important step towards the general study of the scaling region close to the critical points. In fact, they permit an efficient perturbative scheme to study non-integrable effects, in particular to follow how the mass spectrum changes by varying the coupling constants. Thanks to this approach, new progress has been made in understanding several statistical models, in particular the class of universality of the Ising model by varying the temperature $T$ and the magnetic field $B$. Non-integrable field theories present an extremely interesting set of new physical phenomena, such as confinement of topological excitations, decay processes of the heavier particles, the presence of resonances in scattering processes, or false vacuum decay, etc. The analytic control of such phenomena is one of the most interesting results of quantum field theory in the realm of statistical physics.

This book is a long and detailed journey through several fields of physics and mathematics. It is based on an elaboration of the lecture notes for a PhD course, given by the author at the International School for Advances Studies (Trieste). During this elaboration process, particular attention has been paid to achieving a coherent and complete picture of all surveyed topics. The effort done to emphasize the deep relations among several areas of physics and mathematics reflects the profound belief of the author in the substantial unity of scientific knowledge.

This book is designed for students in physics or mathematics (at the graduate level or in the last year of their undergraduate courses). For this reason, its style is greatly pedagogical; it assumes only some basis of mathematics, statistical physics, and quantum mechanics. Nevertheless, we count on the intellectual curiosity of the reader.