1 The conformal group in $d \geq 3$

Given a manifold with metric $g$, a diffeomorphism $\phi$ of $M$ is said to be a \textit{conformal isometry} if

$$\phi^* g = \Omega^2 g$$

for some function $\Omega$. Clearly every isometry is a conformal isometry with $\Omega = 1$, but there may be conformal isometries that are not isometries. An infinitesimal conformal isometry is a vector field $v$ such that

$$\mathcal{L}_v g = 2\omega(v)g.$$  

This is called the conformal Killing equation (if $\omega = 0$ it is the “normal” Killing equation \footnote{educational reading: https://en.wikipedia.org/wiki/Wilhelm_Killing}). In components

$$(\mathcal{L}_v g)_{\mu\nu} = v^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu v^\rho + g_{\nu\rho} \partial_\mu v^\rho$$

can also be written in the form

$$(\mathcal{L}_v g)_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu,$$

where $\nabla$ is the covariant derivative constructed with the metric $g$. Taking the trace of (2) we find that $\omega(v) = (1/d)\nabla_\mu v^\mu$. Then we can write the conformal Killing equation as

$$\nabla_\mu v_\nu + \nabla_\nu v_\mu = \frac{2}{d} \nabla_\rho v^\rho g_{\mu\nu}.$$  

A solutions of this equation is called a \textit{conformal Killing vector}.

We want to determine all the conformal Killing vectors of $d$-dimensional Minkowski space (metric $\eta$ with signature $- + \ldots +$). Equation (5) then simplifies to

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{d} \partial_\rho v^\rho \eta_{\mu\nu}.$$  

A flat space can be identified both with its tangent and cotangent space, so a point with coordinates $x^\mu$ can also be seen as a vector and we denote $x_\mu = \eta_{\mu\nu} x^\nu$ the corresponding one-form. Deriving twice the conformal Killing equation and tracing, deduce that

$$\Box (\partial_\mu v^\mu) = 0$$
Thus we can write
\[ v_\mu = a_\mu + b_{\mu\rho} x^\rho + c_{\mu\rho\sigma} x^\rho x^\sigma, \]
where \( a, b, c \) are constant. Reinserting in (6) find that \( a \) and the antisymmetric part of \( b \) are the generators of the Poincaré group. On the other hand the symmetric part of \( b \) must be proportional to the metric. Deduce that the corresponding solution for \( v \) is the generator of dilatations: \( D = x^\mu \partial_\mu \).

For the terms proportional to \( c \), it is convenient to first manipulate the conformal Killing equation by taking a derivative, say \( \partial_\rho \), of both sides and then cyclically permute \( \mu, \nu, \rho \). Subtract the first equation from the other two and get the relation
\[ 2 \partial_\mu \partial_\nu v_\rho = 2 d (\eta_{\rho\mu} \partial_\nu + \eta_{\rho\nu} \partial_\mu - \eta_{\mu\nu} \partial_\rho) \partial_\lambda v_\lambda. \]

Now inserting the \( c \)-term in this equation find that
\[ c_{\rho\mu\nu} = \eta_{\rho\mu} b_\nu + \eta_{\rho\nu} b_\mu - \eta_{\mu\nu} b_\rho, \]
where \( b_\mu = -\frac{1}{d} c_{\lambda\mu} \). The corresponding generators \( K_\mu \), parametrized by \( b_\mu \), are called special conformal transformations.

So we have the following list of conformal Killing vectors
\[ P_\mu = \partial_\mu, \] (7)
\[ L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu \] (8)
\[ D = x^\mu \partial_\mu, \text{ with } \omega(D) = 1 \] (9)
\[ K_\mu = 2 x_\mu x^\nu \partial_\nu - x^2 \partial_\mu, \text{ with } \omega(K_\mu) = 2 x_\mu \] (10)
of which \( P_\mu \) and \( L_{\mu\nu} \) are genuine Killing vectors (\( \omega = 0 \)). Show that these vectorfields form a closed algebra.

Consider \( \mathbb{R}^{d+2} \) with coordinates \( z^a \) and a flat metric \( \eta_{ab} \) of signature \(-+\ldots+-\). The invariance group of this metric is \( SO(2, d) \) and has generators \( M_{ab} = z_a \partial_b - z_b \partial_a \). (As before, \( z_a = \eta_{ab} z^b \)). The algebra of these generators is
\[ [M_{ab}, M_{cd}] = -\eta_{ac} M_{bd} + \eta_{ad} M_{bc} + \eta_{be} M_{ad} - \eta_{bd} M_{ac}. \] (11)

Map the conformal Killing vectors to the generators \( M_{ab} \) to show that the algebras are isomorphic.