1 Antiferromagnetism

Consider the Ising model with \( J < 0 \). The nearest neighbor interaction favors oppositely oriented spins, so the ground state has antiferromagnetic order. Divide the lattice into sublattices \( A \), \( B \), such that the nearest neighbors of every site of one sublattice belong to the other sublattice. Define the magnetization and the staggered magnetization

\[
M = \langle \sum_A S_i + \sum_B S_i \rangle, \quad S = \langle \sum_A S_i - \sum_B S_i \rangle.
\]

The order parameter is \( S \). Using the mean field approximation show that there is a phase transition and compute the critical temperature. The magnetization is always proportional to the external magnetic field, so the material behaves as a paramagnet.

2 More on kinks

2.1 Energy bound

In a general scalar theory in 1+1 dimensions with action \( (1.1) \), let \( \dot{\phi} \) and \( \phi' \) be the derivatives with respect to time and space, respectively. Start from the observation that

\[
0 \leq \int dx \left( \frac{1}{\sqrt{2}} \phi' + \sqrt{V} \right)^2.
\]

and derive a bound on the static energy

\[
E_S \geq \left| \int_{\phi_-}^{\phi_+} d\phi \sqrt{2V} \right|.
\]

Rederive the equipartition of energy.
2.2 Interaction between kinks

A kink has a linear dimension $\sim 1/(\sqrt{\lambda} f)$, but in some approximation can be viewed as a particle of mass $M = E_S \sim \sqrt{\lambda} f^3$. Two widely separated static kinks can be treated as two particles exerting a mutual force. The force can be calculated as follows. Assume there is an antikink at $x = -a$ and a kink at $x = a$, with $a \gg 1/(\sqrt{\lambda} f)$. The configuration can be described as

$$\phi(x) = \phi_1(x) + \phi_2(x) + 1$$

(3)

where $\phi_1(x)$ is the field of the static antikink and $\phi_2(x)$ is the field of the static kink. Without using this ansatz, let $-a \ll b \ll a$ and consider the momentum of the field in the half space left of $b$:

$$P = -\int_{-\infty}^{b} dx \phi \phi' .$$

Derive a general formula for the force $F = \frac{dP}{dt}$. Since the integrand is a total derivative, the force can be expressed in terms of the field and its derivatives at $-\infty$ and $b$. Now use the ansatz. For $x \leq b$, the quantity $\phi_2 + 1$ is exponentially small, so when (3) is used, one can treat it as a small perturbation and keep only linear terms. Some terms cancel using the fact that the integrand in (1) is zero for the kink solution. Also use the equation of motion, to obtain

$$F = \phi_1' \phi_2' + (\phi_2 - 1) \phi_1''|_\infty^b .$$

Use the explicit form of $\phi_{1,2}$ to calculate the force (use the approximation $\tanh(x) \approx 1 - 2e^{-2x}$ for $x \gg 1$, and the analogous one for $x \ll 1$).

3 Formulae for solitons

Verify equations (1.82-83) for the skyrmion and equations (1.125-127), (1.130-132) for the monopole.

4 The conformal group in $d \geq 3$

Given a manifold with metric $g$, a diffeomorphism $\phi$ of $M$ is said to be a conformal isometry if

$$\phi^* g = \Omega^2 g$$

(4)
for some function $\Omega$. Clearly every isometry is a conformal isometry with $\Omega = 1$. An infinitesimal conformal isometry is a vector field $v$ such that

$$\mathcal{L}_v g = 2\omega(v) g .$$

(5)

This is called the conformal Killing equation. In components

$$(\mathcal{L}_v g)_{\mu\nu} = v^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu v^\rho + g_{\nu\rho} \partial_\mu v^\rho$$

(6)

can also be written in the form

$$(\mathcal{L}_v g)_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu ,$$

(7)

where $\nabla$ is the covariant derivative constructed with the metric $g$. Taking the trace of (5) we find that $\omega(v) = (1/d)\nabla_\mu v^\mu$. Then we can write the conformal Killing equation as

$$\nabla_\mu v_\nu + \nabla_\nu v_\mu = \frac{2}{d} \nabla_\rho v^\rho g_{\mu\nu} .$$

(8)

A solutions of this equation is called a conformal Killing vector.

Show that the following vector fields are conformal Killing vectors for Minkowski space:

$$P_\mu = \partial_\mu$$

(9)

$$L_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$$

(10)

$$D = x^\mu \partial_\mu , \text{ with } \omega(D) = 1$$

(11)

$$K_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu , \text{ with } \omega(K_\mu) = 2x_\mu$$

(12)

In particular $P_\mu$ and $L_{\mu\nu}$ are genuine Killing vectors (i.e. they have zero divergence) and generate Poincaré transformations. $D$ generates dilatations and $K_\mu$ generate so-called special conformal transformations. Show that these vector fields form a closed algebra.

Consider $\mathbb{R}^{d+2}$ with coordinates $z^a$ and a flat metric $\eta_{ab}$ of signature $- + \ldots + -$. The invariance group of this metric is $SO(2, d)$ and has generators $M_{ab} = z_a \partial_b - z_b \partial_a$. (As before, $z_a = \eta_{ab} z^b$). The algebra of these generators is

$$[M_{ab}, M_{cd}] = -\eta_{ac} M_{bd} + \eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{bd} M_{ac} .$$

(13)

Map the conformal Killing vectors to the generators $M_{ab}$ to show that the algebras are isomorphic.