0.1 Homotopy

0.1.1 Basic definitions

Let $M$, $N$ be finite dimensional manifolds. We choose a point $x_0 \in M$ and a point $y_0 \in N$; they are called the basepoints of $M$ and $N$. We denote $\Gamma(M, N)$ the space of all smooth functions $f : M \to N$. (By smooth we mean continuous and $r$-times differentiable, with $0 \leq r \leq \infty$). We denote $\Gamma^*(M, N)$ the subspace of $\Gamma(M, N)$ consisting of functions that preserve basepoints, i.e. $f(x_0) = y_0$.

We say that two maps $f, g \in \Gamma(M, N)$ are homotopic (and write $f \simeq g$) if there exists a continuous map $F : M \times I \to N$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$. Intuitively, $F$ gives a one parameter family of maps, depending continuously on $t$, that interpolates between $f$ and $g$. Sometimes it is convenient to put into evidence the dependence on the parameter, and write $f_t = F(\cdot, t)$; then $f_0 = f$, $f_1 = g$. In the case when $M$, $N$ have basepoints and $f, g \in \Gamma^*(M, N)$, one requires $F(x_0, t) = y_0$ for all $t$. If $f_1 \simeq f_2$ are maps from $N$ to $P$ and $g_1 \simeq g_2$ are map from $M$ to $N$, then $f_1 \circ g_1 \simeq f_2 \circ g_2$.

The relation of being homotopic is an equivalence relation. The quotient of $\Gamma(M, N)$ by this relation, i.e. the set of homotopy classes of maps from $M$ to $N$, is denoted $[M, N]$. Similarly one defines $[M, N]^*$, the set of homotopy classes of basepoint-preserving maps.

The set of homotopy classes thus defined do not depend on $r$, the class of differentiability of the maps. In fact, from the mathematical point of view, it is most natural to assume that $M$ and $N$ are only topological spaces and that the maps are only continuous ($r=0$).

Two spaces $M$ and $N$ are said to have the same homotopy type if there are maps $f : M \to N$ and $g : N \to M$ such that $f \circ g \simeq Id_M$ and $g \circ f \simeq Id_N$. It is easy to see that if $M$ and $N$ have the same homotopy type, then $[P, M] = [P, N]$ and $[M, Q] = [N, Q]$ for all spaces $P, Q$. A space $N$ is said to be contractible if it is homotopy equivalent to a point or in other words if the identity map is homotopic to the constant map. Stated more explicitly, this means that there is a continuous map $F : I \times N \to N$ such that $F(0, y) = y$ and $F(1, y) = y_0$. For example, all vectorspaces are contractible. It is enough to take the origin as basepoint and consider $F(t, y) = ty$. If $N$ is contractible, then $[M, N]^*$ is the trivial set consisting of a single element. To see this it is sufficient to note that for any map $f : M \to N$, $Id_N \circ f = f$ is homotopic to $y_0 \circ f = y_0$. So from the point of view of homotopy a contractible space is
equivalent to a single point.

A map \( p : P \rightarrow M \) is said to be a fibration if it has the homotopy lifting property, which means that given a homotopy \( f_t : Q \rightarrow M \) and a lift of \( f_0 \), namely a map \( \tilde{f}_0 : Q \rightarrow P \) such that \( \mu \circ \tilde{f}_0 = f_0 \), then there exists a homotopy \( \tilde{f}_t \) such that \( \mu \circ \tilde{f}_t = f_t \). In particular, there exist a lift of \( f_1 \) and it is homotopic to the lift of \( f_0 \). Important special cases of fibrations are fiber bundles. We shall return to this later.

In the case when \( M \) is a sphere \( S^m = \{ x \in \mathbb{R}^{m+1} \mid x_1^2 + \cdots + x_{m+1}^2 = 1 \} \) with \( m \geq 1 \), the sets of homotopy classes can be given a group structure. This case is so important that it deserves a special name: the space \( \pi_m(N) = [S^m, N]_* \) is called \( m \)-th homotopy group of \( N \).

We first show how the group structure is defined in the case \( m = 1 \) (\( \pi_1(N) \) is also called the fundamental group of \( N \)). We think of \( S^1 \) as an open interval \( I = [0, 1] \) with the endpoints identified; the basepoint of \( S^1 \) corresponds to 0 (or 1). A basepoint preserving map \( f : S^1 \rightarrow N \) is just a loop starting and ending at \( y_0 \). Given two loops \( f_1, f_2 \) we can define a third loop \( f_1 \cdot f_2 \) by “going first around \( f_1 \), then \( f_2 \) at double speed”:

\[
f_1 \cdot f_2(t) = \begin{cases} f_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ f_2(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}.
\]

If we denote \([f] \in \pi_1(N)\) the homotopy class of the loop \( f \), then \([f_1][f_2] = [f_1 \cdot f_2]\) defines a group multiplication in \( \pi_1(N) \).

In the case \( m \geq 2 \), we think of \( S^m \) as the \( m \)-cube \( I^m \) with all points of the boundary identified. Note that if we call \( t_1, \ldots, t_m \) the coordinates in \( I^m \), the boundary \( \partial I^m \) of the cube consists of all points for which at least one of the coordinates is equal to 0 or 1. A map \( f : I^m \rightarrow N \) such that for all \( x \in \partial I^m \), \( f(x) = y_0 \) can be regarded as a map \( f : S^m \rightarrow N \), and thus defines a homotopy class in \( \pi_m(N) \). We define \( f_1 \cdot f_2 \) by

\[
f_1 \cdot f_2(t_1, \ldots, t_m) = \begin{cases} f_1(2t_1, t_2, \ldots, t_m) & \text{for } 0 \leq t_1 \leq \frac{1}{2} \\ f_2(2t_1 - 1, t_2, \ldots, t_m) & \text{for } \frac{1}{2} \leq t_1 \leq 1 \end{cases}.
\]

The group structure in \( \pi_m(N) \) is then defined as in the case \( m = 1 \).

The groups \( \pi_m(N) \) for \( m \geq 2 \) are always abelian, whereas \( \pi_1(N) \) need not be abelian. The following sequence of drawings is the proof of this statement in the case \( m = 2 \). The first square represents the homotopy between \( f_1 \) (yellow) and \( f_2 \) (red), as given in (A.2). Black areas (including the contours
of the rectangles) are points where the value of the function is $y_0$. By a continuous sequence of deformations one arrives at interchanging the order of $f_1$ and $f_2$ in the homotopy. It is also immediately clear why this cannot be done for $m=1$.

The definition of $\pi_m(N)$ given above works also in the case $m=0$. The sphere $S^0$ consists of the two points $+1$ and $-1$. One of them, for example $+1$, can be taken as basepoint. A basepoint-preserving map $f : S^0 \to N$ maps $+1$ to $y_0$ and $-1$ to some point $y$ of $N$. Thus there is a bijective correspondence between $\Gamma_*(M,N)$ and $N$. Two maps $f$ and $f'$ are homotopic if $y=f(-1)$ and $y'=f'(-1)$ belong to the same arcwise connected component of $N$. Thus $\pi_0(N) = [S^0, N]_* = \{\text{arcwise connected components of } N\}$. This set does not have a group structure in general.

Summarizing, the homotopy groups give some information about the topology of a manifold. $\pi_0(N) \neq 0$ if $N$ has more than one connected component, $\pi_1(N) \neq 0$ if $N$ is multiply connected, $\pi_m(N) \neq 0$ if $N$ contains non-contractible $m$-spheres. One can prove that if $M$ is a smooth manifold then the homotopy groups characterize its homotopy type.

If $f : N \to Q$ is a smooth map, there are natural homomorphisms $\pi_k(f) : \pi_k(N) \to \pi_k(Q)$ for all $k$, defined as follows: $\pi_k(f)$ maps the homotopy class of a map $g : S^k \to N$ to the homotopy class of the map $f \circ g : S^k \to Q$. One can easily check that these are homomorphisms.

### 0.1.2 Some useful results

**The winding number**

Let $M$ and $N$ be compact, connected manifolds without boundary, both of dimension $n$. We denote $\omega = \frac{1}{n!} \omega_{i_1 \ldots i_n} dy^{i_1} \wedge \cdots \wedge dy^{i_n}$ a volume-form on $N$. For example, if $N$ is endowed with a riemannian metric $h = h_{\alpha \beta} dy^\alpha \otimes dy^\beta$ it is natural to consider the riemannian volume form $\omega = \sqrt{\det h} dy^1 \wedge \cdots \wedge dy^n$. 
Given a map $\varphi : M \to N$ we define the winding number of $\varphi$

$$W(\varphi) = \frac{\int_M \varphi^* \omega}{\int_N \omega} = \frac{1}{\text{Vol}(N)} \int_M d^n x \varepsilon^{\mu_1 \cdots \mu_n} \partial_{\mu_1} \varphi^1 \cdots \partial_{\mu_n} \varphi^n \omega_{1 \cdots n}.$$  

The geometrical meaning of this quantity can be understood as follows. Recall that a point $x \in M$ is a regular point for the map $\varphi$ if the tangent map $T\varphi |_x$ is surjective (i.e., in coordinates, if $\det(\partial_{\mu} \varphi^\alpha)(x) \neq 0$). A point $y \in \text{Im} \varphi \subset N$ is regular value for $\varphi$ if all the points in its pre-image $\varphi^{-1}(y)$ are regular points. It can be proven that if $y$ is any regular value of $\varphi$, then

$$W(\varphi) = \left(\# \text{ of points in } \varphi^{-1}(y) \text{ with } \det(\partial_{\mu} \varphi^\alpha) > 0\right) - \left(\# \text{ of points in } \varphi^{-1}(y) \text{ with } \det(\partial_{\mu} \varphi^\alpha) < 0\right).$$  

(1)

In particular the winding number is the integer topological invariant that characterizes the maps from $S^n$ to $S^n$.

**Homotopy groups of spheres**

We will often need the homotopy groups of the spheres, $\pi_m(S^n)$. These are summarized in the following table.

<table>
<thead>
<tr>
<th>$S^1$</th>
<th>$S^2$</th>
<th>$S^3$</th>
<th>$S^4$</th>
<th>$S^5$</th>
<th>$S^6$</th>
<th>$S^7$</th>
<th>$S^8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_1$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_2$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_3$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_4$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_5$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_6$</td>
<td>0</td>
<td>$\mathbb{Z}_{12}$</td>
<td>$\mathbb{Z}_{12}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_7$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z} \times \mathbb{Z}_{12}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$\pi_8$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_{24}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
</tr>
</tbody>
</table>

All the elements above the diagonal are zero. All the elements on the diagonal are given by the theorem of Hopf discussed in the preceding section,
and the integer classifying the maps is the winding number. The part below the diagonal is more complicated, but there are regularities. For example, the second and third column are the same from $\pi_3$ onwards. This is due to the properties of the Hopf map, as we shall see later. From the third column onwards, all the elements on the second and third lower diagonal are $\mathbb{Z}_2$; from the fifth column onwards, all the elements on the fourth lower diagonal are $\mathbb{Z}_{24}$.

**Homotopy groups of the classical groups**

The next table gives the homotopy groups of the classical (unitary, orthogonal, symplectic) groups, for sufficiently large $N$ (the condition is indicated in the second row). These homotopy groups are periodic modulo 2 (for the unitary groups) and modulo 8 (for the orthogonal and symplectic groups). This is known as Bott periodicity. The homotopy groups of the symplectic groups are the same as those of the orthogonal groups, but shifted by four.

<table>
<thead>
<tr>
<th>$\pi_n(.)$</th>
<th>$U(N)$</th>
<th>$O(N)$</th>
<th>$Sp(N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_0$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_1$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_3$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$\pi_4$</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\pi_5$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>$\pi_6$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_7$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$\pi_8$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>0</td>
</tr>
</tbody>
</table>

| $\pi_0(G)$ is the set of connected components. $\pi_0(O(3))$ is related to parity. |
| $\pi_1(O(3)) = \pi_1(SO(3))$ is related to spin in dimensions $d \geq 3$. |
| $\pi_3(Sp(1)) = \pi_3(SU(2))$ is related to instantons. |
| $\pi_3(U(N)) = \pi_3(SU(N))$ is related to the existence of Skyrmions. |
• $\pi_5(U(3)) = \pi_5(SU(3))$ is related to the spin of Skyrmions.

• $\pi_4(SU(2)) = \pi_4(Sp(1))$ is related to a global anomaly.

0.1.3 The homotopy exact sequence

An exact sequence of groups is a sequence of group homomorphisms $h_i : G_i \to G_{i+1}$ such that $\text{im } h_i = \ker h_{i+1}$. When an exact sequence is known to exist, then knowledge of the properties of some groups or homomorphisms can be used to infer properties of other groups or homomorphisms.

A principal bundle is a space $P$ on which a group $H$ acts freely (i.e. without fixed points) from the right. The quotient $M = P/H$ is called the “base space”, $P$ is called the “total space” and there is a natural projection $\mu : P \to M$ that maps each $p \in P$ to its equivalence class $\lbrack p \rbrack = p \mod H \in M$. As usual in homotopy theory, $P$ and $M$ are equipped with basepoints $p_0$ and $\lbrack p_0 \rbrack$. The orbit through $p_0$ can be identified with the group $H$ by identifying $p_0$ with the identity of $H$. This gives an injective map $\iota : H \to P$. The maps $\iota$ and $\mu$ are such that $\mu \circ \iota$ is the constant map $H \to M$ with image $\lbrack p_0 \rbrack$.

The standard example of a principal bundle, and the one that we shall be mostly interested in, is a Lie group $G$, with $\iota : H \to G$ a Lie subgroup and $M = G/H$ the space of right cosets.

It can be shown that a fiber bundle is a fibration, so the map $\mu$ has the homotopy lifting property. This property is crucial to prove the long exact sequence of homotopy groups that is associated to a principal bundle.

Given a map $f : M \to N$ there is an induced homomorphism of homotopy groups $f_* : \pi_n(M) \to \pi_n(N)$, defined as follows. Given a map $h : S^n \to M$, and denoting $[h] \in \pi_n(M)$ its homotopy class, $f_*([h]) = [g \circ h] \in \pi_n(N)$.

Now consider a principal bundle and the homotopy groups of $H$, $P$ and $M$. We have homomorphisms

$$\pi_n(H) \xrightarrow{\iota_*} \pi_n(P) \xrightarrow{\mu_*} \pi_n(M)$$

Since $\mu \circ \iota = [p_0]$, $\text{im } \iota_* \subset \ker \mu_*$. Conversely, if $f : S^n \to P$ is such that $\mu \circ f$ is homotopic to a constant (i.e. $[f] \in \ker \mu_*$), by the homotopy lifting property there exists a map $f'$, homotopic to $f$, such that $\mu \circ f' = [p_0]$. Thus $\text{im } \iota_* \supset \ker \mu_*$. Altogether we have found that $\text{im } \iota_* = \ker \mu_*$, therefore the sequence (2) is exact at $\pi_n(P)$.
Now we can tie together the short sequences (2) for different $n$ into a long exact sequence, by defining homomorphisms $\partial: \pi_n(M) \to \pi_{n-1}(H)$ and showing that $\text{im } \mu_* = \ker \partial$ and $\text{im } \partial = \ker \iota_*$. 

$$\ldots \to \pi_{n+1}(M) \xrightarrow{\partial} \pi_n(H) \xrightarrow{\iota_*} \pi_n(P) \xrightarrow{\mu_*} \pi_n(M) \xrightarrow{\partial} \pi_{n-1}(H) \to \ldots$$

$$\ldots \xrightarrow{\partial} \pi_0(H) \xrightarrow{\iota_*} \pi_0(P) \xrightarrow{\mu_*} \pi_0(M) \quad (3)$$

The last three sets do not have a group structure, but the sequence is still exact if we define the kernel of a based map to consist of those elements of the domain, that are mapped to the basepoint of the target.

We will now define the map $\partial$. Let $D^n = \{x \in \mathbb{R}^n | x_1^2 \ldots x_n^2 \leq 1\}$ be the closed unit ball and $\delta_n$ the inclusion of the unit sphere $S^n$ as the boundary of $B^{n+1}$. Furthermore, let $\gamma_n : D^n \to S^n$ be the map that identifies all points of the boundary as the basepoint of $S^n$.

Pick a map $f : S^n \to M$. Since $D^n$ is contractible, $f \circ \gamma_n : D^n \to M$ is homotopic to a constant. The constant map $D^n \to M$ has a lift, which is the constant map $D^n \to P$. By the homotopy lifting property, also $f \circ \gamma_n$ has a lift $\lambda : D^n \to P$. This is shown in the following commutative diagram:

Now consider the map $\lambda \circ \delta_{n-1} : S^{n-1} \to P$. Since $\mu \circ \lambda \circ \delta_{n-1}$ is the constant map, there must exist a map $\psi : S^{n-1} \to H$ such that $\lambda \circ \delta_{n-1} = \iota \circ \psi$. We define $\partial([f]) = [\psi]$.

### 0.2 Geometry of $SU(2)$ and the Hopf map

Now consider the group $SU(2)$. An element $U \in SU(2)$ is a complex matrix

$$U = \begin{bmatrix} x_4 + ix_3 & x_2 + ix_1 \\ -x_2 + ix_1 & x_4 - ix_3 \end{bmatrix}$$

with

$$\det U = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 . \quad (4)$$

So $SU(2)$ is the unit sphere in $\mathbb{R}^4$. The element with coordinates $(0,0,0,1)$ corresponds to the unit matrix $I$ and can be thought of as the north pole of
The sphere. Then, the south pole is the element with coordinates \((0, 0, 0, -1)\), corresponding to the matrix \(-I\).

The Lie algebra of \(SU(2)\) consists of the anti-hermitian matrices. It can be identified geometrically with the plane \(x_4 = 1\) in \(\mathbb{R}^4\). It is customary to take as basis in the Lie algebra the matrices

\[ t_i = -\frac{i}{2} \sigma_i \]

where \(\sigma_i\) are the Pauli matrices. This normalization of the generators is chosen so that they satisfy the algebra

\[ [t_i, t_j] = \epsilon_{ijk} t_k . \]

The Lie algebra of \(SU(2)\) is therefore isomorphic to the Lie algebra of \(SO(3)\). A concrete isomorphism is given by mapping \(t_i\) to \(T_i\), where

\[ (T_i)_{jk} = -\epsilon_{ijk} . \]

This also defines an isomorphism of a neighbourhood of the identity in \(SU(2)\) to a neighbourhood of the identity in \(SO(3)\), and we can use this fact to introduce the Euler angles as coordinates on \(SU(2)\). Define:

\[ U(\Theta, \Phi, \Psi) = \exp(\Phi t_3) \exp(\Theta t_2) \exp(\Psi t_3) \]

\[ = \begin{bmatrix}
\cos \frac{\Theta}{2} \exp \left[ -\frac{i}{2} (\Phi + \Psi) \right] & -\sin \frac{\Theta}{2} \exp \left[ -\frac{i}{2} (\Phi - \Psi) \right] \\
\sin \frac{\Theta}{2} \exp \left[ \frac{i}{2} (\Phi - \Psi) \right] & \cos \frac{\Theta}{2} \exp \left[ \frac{i}{2} (\Phi + \Psi) \right]
\end{bmatrix} \]

Every matrix \(U \in SU(2)\) can be written in this way, provided that \(0 < \Phi \leq 2\pi\), \(0 \leq \Theta \leq \pi\), \(0 < \Psi \leq 4\pi\). There is a homomorphism from \(SU(2)\) to \(SO(3)\) that maps \(U(\Theta, \Phi, \Psi)\) to

\[ R(\Theta, \Phi, \Psi) = \exp(\Phi T_3) \exp(\Theta T_2) \exp(\Psi T_3) \]

\[ = \begin{bmatrix}
\cos \Theta \cos \Phi \cos \Psi - \sin \Phi \sin \Psi & -\cos \Theta \cos \Phi \sin \Psi - \sin \Phi \cos \Psi & \sin \Theta \cos \Psi \\
\cos \Theta \sin \Phi \cos \Psi + \cos \Phi \sin \Psi & -\cos \Theta \sin \Phi \sin \Psi + \cos \Phi \cos \Psi & \sin \Theta \sin \Phi \\
-\sin \Theta \cos \Psi & \sin \Theta \sin \Psi & \cos \Theta
\end{bmatrix} \]

It is a double covering because \(R(\Theta, \Phi, \Psi + 2\pi) = R(\Theta, \Phi, \Psi)\) but

\[ U(\Theta, \Phi, \Psi + 2\pi) = -U(\Theta, \Phi, \Psi) . \]

As a consequence the range of \(\Psi\) as a coordinate in \(SU(2)\) is twice the range of \(\Psi\) as a coordinate in \(SO(3)\). Note also that \(R(\Theta, \Phi, \Psi)\) is the adjoint representation of \(U(\Theta, \Phi, \Psi)\).
0.2. GEOMETRY OF SU(2) AND THE HOPF MAP

As on any Lie group, one can define the Lie-algebra-valued Maurer-Cartan forms

\[ L = U^{-1}dU, \quad R = dUU^{-1}. \]  

(6)

The form \( L \) is invariant under the action of left multiplication \( U \mapsto gU \) and \( R \) is invariant under the right multiplication \( U \mapsto Ug \). They can be decomposed on the basis of the Lie algebra

\[ L = L^a_t a, \quad R = R^a_t a, \]  

(7)

where \( L^a, R^a \), with \( a = 1, 2, 3 \) are ordinary differential forms on \( SU(2) \). Given any coordinate system \( \{y^\alpha\} \), they can be decomposed on a natural basis

\[ L^a = L^a_\alpha dy^\alpha, \quad R^a = R^a_\alpha dy^\alpha. \]  

(8)

The components of the Maurer-Cartan forms in Euler coordinates can be calculated directly by inserting (5) in (6) and decomposing

\[ U^{-1}dU = L^a_\alpha dy^\alpha t_a, \quad dUU^{-1} = R^a_\alpha dy^\alpha t_a \]

In this way one finds

\[
\begin{align*}
L^1 &= \sin \Psi d\Theta - \sin \Theta \cos \Psi d\Phi, \\
L^2 &= \cos \Psi d\Theta + \sin \Theta \sin \Psi d\Phi, \\
L^3 &= d\Psi + \cos \Theta d\Phi, \\
R^1 &= -\sin \Phi d\Theta + \sin \Theta \cos \Phi d\Psi, \\
R^2 &= \cos \Phi d\Theta + \sin \Theta \sin \Phi d\Psi, \\
R^3 &= d\Phi + \cos \Theta d\Psi.
\end{align*}
\]  

(9)

One can then explicitly verify the Maurer-Cartan equations:

\[
\begin{align*}
dL^a + \frac{1}{2} f_{bc}^a L^b \wedge L^c &= 0; \\
dR^a - \frac{1}{2} f_{bc}^a R^b \wedge R^c &= 0.
\end{align*}
\]

The left-invariant forms \( L^a \) are linearly independent and therefore form a global field of bases for one-forms. Then, there is a dual field of bases for vectors \( L_a \):

\[ L_a = L^\alpha_a \partial_\alpha, \]  

(10)

where the matrix \( L^\alpha_a \) is the inverse of the matrix \( L^a_\alpha \). The vectors \( L_a \) are left-invariant. Similarly one defines a basis of right-invariant vectorfields

\[ R_a = R^\alpha_a \partial_\alpha. \]  

(11)
In Euler coordinates

\[ L_1 = \sin \Psi \frac{\partial}{\partial \Theta} - \frac{1}{\sin \Theta} \cos \Psi \frac{\partial}{\partial \Phi} + \cot \Theta \cos \Psi \frac{\partial}{\partial \Psi} , \]
\[ L_2 = \cos \Psi \frac{\partial}{\partial \Theta} + \frac{1}{\sin \Theta} \sin \Psi \frac{\partial}{\partial \Phi} - \cot \Theta \sin \Psi \frac{\partial}{\partial \Psi} , \]
\[ L_3 = \frac{\partial}{\partial \Psi} , \]
\[ R_1 = -\sin \Phi \frac{\partial}{\partial \Theta} - \cot \Theta \cos \Phi \frac{\partial}{\partial \Phi} + \frac{1}{\sin \Theta} \cos \Phi \frac{\partial}{\partial \Psi} , \]
\[ R_2 = \cos \Phi \frac{\partial}{\partial \Theta} - \cot \Theta \sin \Phi \frac{\partial}{\partial \Phi} + \frac{1}{\sin \Theta} \sin \Phi \frac{\partial}{\partial \Psi} , \]
\[ R_3 = \frac{\partial}{\partial \Phi} . \]  \hspace{1cm} (12)

These vector fields are the infinitesimal generators of the action of the group on itself. More precisely, the vector fields \( L_a \) generate the right multiplication and \( R_a \) generate the left multiplication.

A direct calculation gives the Lie brackets

\[ [L_a, L_b] = \epsilon_{abc} L_c \]  \hspace{1cm} (13)
\[ [R_a, R_b] = -\epsilon_{abc} R_c \]  \hspace{1cm} (14)
\[ [R_a, L_b] = 0 . \]  \hspace{1cm} (15)

Since \([X, Y] = L_X Y\) (the Lie derivative) the last bracket expresses the fact that the vector fields \( L_a \) are left-invariant and the \( R_a \) are right-invariant.

In general, the adjoint representation matrices are given by:

\[ Ad(g)^a_b = R(g)^a_\alpha L(g)^\alpha_b . \]

One can indeed check that

\[ Ad(U(\Theta, \Phi, \Psi)) = R(\Theta, \Phi, \Psi) . \]

Given an inner product \( \gamma_{ab} \) in the Lie algebra, we can construct a left- and a right- invariant metric on \( SU(2) \)

\[ h^{(L)}_{\alpha \beta} = L^a_\alpha L^b_\beta \gamma_{ab} \]

and

\[ h^{(R)}_{\alpha \beta} = R^a_\alpha R^b_\beta \gamma_{ab} . \]
However, if the inner product is $Ad$-invariant, both metrics agree and are bi-invariant. Thus for example the inner product

$$\gamma(v, w) = C \text{tr}(vw)$$

is $Ad$-invariant, because

$$\text{tr}(g^{-1}vg^{-1}wg) = \text{tr}(vw).$$

The components of this inner product are

$$\gamma_{ab} = C \text{tr}(ta tb) = -\frac{C}{2} \delta_{ab}.$$

It is convenient to choose $C = -2$, so that $\gamma_{ab} = \delta_{ab}$, in which case the $L_a$ and $R_a$ are orthonormal bases (“triads”) for the bi-invariant metric $h_{\alpha\beta}$. Then, in Euler coordinates, the bi-invariant metric is

$$ds^2 = (L^1)^2 + (L^2)^2 + (L^3)^2$$

$$= d\Theta^2 + d\Phi^2 + d\Psi^2 + 2 \cos \Theta d\Phi d\Psi$$

$$= h_{\alpha\beta} dy^\alpha dy^\beta. \quad (16)$$

The corresponding volume element is

$$\omega = L^1 \wedge L^2 \wedge L^3$$

$$= \sin \Theta d\Theta \wedge d\Phi \wedge d\Psi$$

$$= \sqrt{\det h} dy^1 \wedge dy^2 \wedge dy^3. \quad (17)$$

With this volume form, the volume of $SU(2)$ is $\int \omega = 16\pi^2$, which is eight times the volume of the unit three-sphere. Indeed the invariant metric $h$ is the metric of a three-sphere of radius 2 (the metric induced from the Euclidean metric in $\mathbb{R}^4$ by the embedding (4) is $\frac{1}{2} h_{\alpha\beta}$). The curvature scalar is $R = 3/2$.

0.3 Conventions

Signature: $- + \ldots +$. 
0.3.1 Yang-Mills theory

Let $\psi$ be a matter field in a representation $\rho$ of the gauge group $G$. The Lie algebra of $G$ is $\mathfrak{g}$. A basis of generators of $\mathfrak{g}$ in the representation $\rho$ is $T_a$. They satisfy the algebra

$$[T_a, T_b] = f_{ab}^c T_c .$$

(18)

The matrices $T_a$ are assumed to be antihermitian: $T_a^\dagger = -T_a$. Then, the structure constants are real. The generators are normalized so that

$$\text{tr}T_a T_b = -\frac{1}{2} \delta_{ab} .$$

For example for $SU(2)$, $T_a = -\frac{i}{2} \sigma_a$ whereas for $SU(3)$, $T_a = -\frac{i}{2} \lambda_a$, where $\lambda_a$ are the Gell-Mann matrices.

The structure constants of the Lie algebra of $SU(2)$ are $f_{abc} = \varepsilon_{abc}$ (in the adjoint representation the generators are $(T_a)_{bc} = -\varepsilon_{abc}$). The gauge field is $A_\mu = A_\mu^a T_a$.

*Unscaled version.* In perturbation theory it is convenient to have the coupling constant in the definition of the covariant derivative and curvature. The covariant derivative of the matter field is

$$D_\mu \psi = \partial_\mu \psi + e A_\mu^a T_a \psi .$$

(19)

(indices of the matter representation are not made explicit here) and the nonabelian field strength is

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e f_{bc}^a A_\mu^b A_\nu^c ,$$

(20)

The Yang–Mills action is

$$S_{YM} = -\frac{1}{4} \int d^n x F_{\mu\nu}^a F_\mu^{a\nu} .$$

(21)

The action is invariant under the local gauge transformations

$$\psi' = g^{-1} \psi ,$$

(22)

$$A'_\mu = g^{-1} A_\mu g + \frac{1}{e} g^{-1} \partial_\mu g$$

(23)

which imply

$$D_\mu \psi' = g^{-1} D_\mu \psi ,$$

(24)

$$F'_{\mu\nu} = g^{-1} F_{\mu\nu} g .$$

(25)
0.3. CONVENTIONS

Rescaled version. In many cases, and in particular to discuss geometrical properties, it is more convenient to rescale the field $A$ by a factor $1/e$. Then, the covariant derivative of the matter field is

$$D_\mu \psi = \partial_\mu \psi + A^a_\mu T_a \psi .$$  \hfill (26)

and the nonabelian field strength is

$$F^{a}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu ,$$  \hfill (27)

In this case the Yang–Mills action reads

$$S_{YM} = -\frac{1}{4e^2} \int d^nx F^{a}_{\mu\nu} F^{a\mu\nu} ,$$  \hfill (28)

The nonabelian gauge transformations then read

$$A_\mu \rightarrow g^{-1} A_\mu g + g^{-1} \partial_\mu g ,$$  \hfill (29)

Abelian case. The Lie algebra of $U(1)$ consists of the purely imaginary numbers and one can take as a basis element $T = -i$. In this case $A_\mu$ always represents the real and unscaled field, not the Lie algebra-valued field $-i A_\mu$. Thus the abelian covariant derivative is

$$D_\mu \psi = \partial_\mu \psi - ie A_\mu \psi .$$  \hfill (30)

The gauge transformations of $-i A_\mu$ are still given by (29), thus multiplying by $i$ we get

$$A'_\mu = A_\mu + \frac{i}{e} g^{-1} \partial_\mu g = A_\mu - \frac{i}{e} \partial_\mu \alpha ,$$  \hfill (31)

$$\psi' = g^{-1} \psi = e^{-i\alpha(x)} \psi .$$  \hfill (32)