0.1 Basic homotopy

Let $M$, $N$ be finite dimensional manifolds. We choose a point $x_0 \in M$ and a point $y_0 \in N$; they are called the basepoints of $M$ and $N$. We denote $\Gamma(M, N)$ the space of all smooth functions $f : M \to N$. (By smooth we mean continuous and $r$-times differentiable, with $0 \leq r \leq \infty$). We denote $\Gamma_*(M, N)$ the subspace of $\Gamma(M, N)$ consisting of functions that preserve basepoints, i.e. $f(x_0) = y_0$.

We say that two maps $f, g \in \Gamma(M, N)$ are homotopic (and write $f \simeq g$) if there exists a continuous map $F : M \times I \to N$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$. Intuitively, $F$ gives a one parameter family of maps, depending continuously on $t$, that interpolates between $f$ and $g$. Sometimes it is convenient to put into evidence the dependence on the parameter, and write $f_t = F(\cdot, t)$; then $f_0 = f$, $f_1 = g$. In the case when $M$, $N$ have basepoints and $f, g \in \Gamma_*(M, N)$, one requires $F(x_0, t) = y_0$ for all $t$. If $f_1 \simeq f_2$ are maps from $N$ to $P$ and $g_1 \simeq g_2$ are maps from $M$ to $N$, then $f_1 \circ g_1 \simeq f_2 \circ g_2$.

The relation of being homotopic is an equivalence relation. The quotient of $\Gamma(M, N)$ by this relation, i.e. the set of homotopy classes of maps from $M$ to $N$, is denoted $[M, N]$. Similarly one defines $[M, N]_*$, the set of homotopy classes of basepoint-preserving maps.

The set of homotopy classes thus defined do not depend on $r$, the class of differentiability of the maps. In fact, from the mathematical point of view, it is most natural to assume that $M$ and $N$ are only topological spaces and that the maps are only continuous ($r = 0$).

Two spaces $M$ and $N$ are said to have the same homotopy type if there are maps $f : M \to N$ and $g : N \to M$ such that $f \circ g \simeq Id_M$ and $g \circ f \simeq Id_N$. It is easy to see that if $M$ and $N$ have the same homotopy type, then $[P, M] = [P, N]$ and $[M, Q] = [N, Q]$ for all spaces $P$, $Q$. A space $N$ is said to be contractible if it is homotopy equivalent to a point or in other words if the identity map is homotopic to the constant map. Stated more explicitly, this means that there is a continuous map $F : I \times N \to N$ such that $F(0, y) = y$ and $F(1, y) = y_0$. For example, all vectorspaces are contractible. It is enough to take the origin as basepoint and consider $F(t, y) = ty$. If $N$ is contractible, then $[M, N]_*$ is the trivial set consisting of a single element. To see this it is sufficient to note that for any map $f : M \to N$, $Id_N \circ f = f$ is homotopic to $y_0 \circ f = y_0$. So from the point of view of homotopy a contractible space is equivalent to a single point.

In the case when $M$ is a sphere $S^M = \{x \in \mathbb{R}^{m+1} \mid x_1^2 + \cdots + x_{m+1}^2 = 1\}$ with
$m \geq 1$, the sets of homotopy classes can be given a group structure. This case is so important that it deserves a special name: the space $\pi_m(N) = [S^m, N]$ is called $m$-th homotopy group of $N$.

We first show how the group structure is defined in the case $m = 1$ ($\pi_1(N)$ is also called the fundamental group of $N$). We think of $S^1$ as an open interval $I = [0, 1]$ with the endpoints identified; the basepoint of $S^1$ corresponds to 0 (or 1). A basepoint preserving map $f : S^1 \to N$ is just a loop starting and ending at $y_0$. Given two loops $f_1, f_2$ we can define a third loop $f_1 \cdot f_2$ by “going first around $f_1$, then $f_2$ at double speed”:

$$f_1 \cdot f_2(t) = \begin{cases} f_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ f_2(2t - 1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}.$$ 

If we denote $[f] \in \pi_1(N)$ the homotopy class of the loop $f$, then $[f_1][f_2] = [f_1 \cdot f_2]$ defines a group multiplication in $\pi_1(N)$.

In the case $m \geq 2$, we think of $S^m$ as the $m$-cube $I^m$ with all points of the boundary identified. Note that if we call $t_1, \ldots, t_m$ the coordinates in $I^m$, the boundary $\partial I^m$ of the cube consists of all points for which at least one of the coordinates is equal to 0 or 1. A map $f : I^m \to N$ such that for all $x \in \partial I^m$, $f(x) = y_0$ can be regarded as a map $f : S^m \to N$, and thus defines a homotopy class in $\pi_m(N)$. We define $f_1 \cdot f_2$ by

$$f_1 \cdot f_2(t_1, \ldots, t_m) = \begin{cases} f_1(2t_1, t_2, \ldots, t_m) & \text{for } 0 \leq t_1 \leq \frac{1}{2} \\ f_2(2t_1 - 1, t_2, \ldots, t_m) & \text{for } \frac{1}{2} \leq t_1 \leq 1 \end{cases}.$$ 

The group structure in $\pi_m(N)$ is then defined as in the case $m = 1$.

The groups $\pi_m(N)$ for $m \geq 2$ are always abelian, whereas $\pi_1(N)$ need not be abelian. The following sequence of drawings is the proof of this statement in the case $m = 2$. The first square represents the homotopy between $f_1$ (yellow) and $f_2$ (red), as given in (A.2). Black areas (including the contours of the rectangles) are points where the value of the function is $y_0$. By a continuous sequence of deformations one arrives at interchanging the order of $f_1$ and $f_2$ in the homotopy. It is also immediately clear why this cannot be done for $m = 1$.

The definition of $\pi_m(N)$ given above works also in the case $m = 0$. The sphere $S^0$ consists of the two points +1 and −1. One of them, for example +1, can be taken as basepoint. A basepoint-preserving map $f : S^0 \to N$ maps +1 to $y_0$ and −1 to some point $y$ of $N$. Thus there is a bijective correspondence
between $\Gamma_* (M, N)$ and $N$. Two maps $f$ and $f'$ are homotopic if $y = f(-1)$ and $y' = f'(-1)$ belong to the same arcwise connected component of $N$. Thus $\pi_0 (N) = [S^0, N]* = \{\text{arcwise connected components of } N\}$. This set does not have a group structure in general.

Summarizing, the homotopy groups give some information about the topology of a manifold. $\pi_0 (N) \neq 0$ if $N$ has more than one connected component, $\pi_1 (N) \neq 0$ if $N$ is multiply connected, $\pi_m (N) \neq 0$ if $N$ contains non-contractible $m$-spheres. One can prove that if $M$ is a smooth manifold then the homotopy groups characterize its homotopy type.

If $f : N \to Q$ is a smooth map, there are natural homomorphisms $\pi_k (f) : \pi_k (N) \to \pi_k (Q)$ for all $k$, defined as follows: $\pi_k (f)$ maps the homotopy class of a map $g : S^k \to N$ to the homotopy class of the map $f \circ g : S^k \to Q$. One can easily check that these are homomorphisms.

We will mostly need the homotopy groups of the spheres, $\pi_m (S^n)$. These can be conveniently summarized in a table:

<table>
<thead>
<tr>
<th></th>
<th>$n = 1$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
<th>$n = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 1$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$m = 2$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$m = 3$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
</tr>
<tr>
<td>$m = 4$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
</tr>
</tbody>
</table>

All the elements on the diagonal are $\mathbb{Z}$ by a theorem of Hopf. All the elements above the diagonal are zero. The elements below the diagonal are generally nonzero and are the tricky ones. The only one that we shall need is $\pi_2 (S^3)$, which was also worked out by Hopf. It generator is the homotopy class of the Hopf map (the projection of the Hopf bundle).

Let $M$ and $N$ have the same dimension $n$. We denote $\omega = \frac{1}{n!} \omega_{i_1 \cdots i_n} dy^{i_1} \wedge \cdots \wedge dy^{i_n}$ a volume-form on $N$. For example, if $N$ is endowed with a riemannian metric $h = h_{\alpha \beta} dy^\alpha \otimes dy^\beta$ it is natural to consider the riemannian
volume form $\omega = \sqrt{\det h} \, dy^1 \wedge \cdots \wedge dy^n$. Given a map $\varphi : M \to N$ we define the winding number of $\varphi$

$$W(\varphi) = \frac{\int_M \varphi^* \omega}{\int_N \omega} = \frac{1}{\Vol(N)} \int_M d^n x \varepsilon^\mu_1 \cdots \varepsilon^\mu_n \partial_{\mu_1} \varphi \cdots \partial_{\mu_n} \varphi^n \omega_1 \cdots n .$$

The geometrical meaning of this quantity can be understood as follows. Recall that a point $x \in M$ is a regular point for the map $\varphi$ if the tangent map $T\varphi|_x$ is surjective (i.e., in coordinates, if $\det(\partial_{\mu} \varphi^\alpha)(x) \neq 0$). A point $y \in \text{Im}\varphi \subset N$ is regular value for $\varphi$ if all the points in its pre-image $\varphi^{-1}(y)$ are regular points. It can be proven that if $y$ is any regular value of $\varphi$, then

$$W(\varphi) = \left( \# \text{ of points in } \varphi^{-1}(y) \text{ with } \det(\partial_{\mu} \varphi^\alpha) > 0 \right) - \left( \# \text{ of points in } \varphi^{-1}(y) \text{ with } \det(\partial_{\mu} \varphi^\alpha) < 0 \right) . \quad (1)$$

In particular the winding number is the integer topological invariant that characterizes the maps from $S^n$ to $S^n$. 