Chapter 1

Anomalies

WARNING:
IN THIS CHAPTER THERE ARE STILL INCONSISTENCIES IN THE NOTATION

It is sometimes impossible to quantize a system preserving all its classical symmetries. One then says that there is an anomaly. There are various types of anomalies, both from a mathematical and from a physical point of view. One can distinguish between anomalies for discrete groups of transformations, for finite dimensional continuous groups (Lie groups) and for infinite dimensional groups. Another distinction of a more physical nature is whether the invariance that cannot be preserved is a genuine symmetry of the system (meaning that it consists of transformations that can be physically observed) or a gauge invariance (in which case the transformed object is physically indistinguishable from the original one).

In the case of a (finite- or infinite-dimensional) continuous group, the anomaly manifests itself in the failure of a conservation law. The physical implications of the anomaly are then very different depending on the nature of the classical conservation law. In the case of a genuine continuous sym-
metrical (typically a symmetry with constant transformation parameters) the current of interest is the Noether current. The anomaly appears as a nonzero divergence of this current and does not have harmful consequences. The prototype is the Adler-Bell-Jackiw (ABJ) anomaly in the axial current. This case will be discussed in section XXX. On the other hand in the case of a current coupled to gauge fields (when the transformation parameters are functions on spacetime), failure of current conservation jeopardizes the consistency of the theory. We will generally refer to such anomalies as “gauge anomalies”. In these cases the anomaly is a pathology of the theory and requiring the absence of anomalies becomes a powerful tool for selecting physically viable theories.

One of the most important examples of anomalies affects scale transformations. The running of couplings breaks scale invariance even in theories that are classically scale invariant. This anomaly manifests itself in a nonvanishing trace of the energy-momentum tensor and is therefore usually called the trace anomaly. We shall not consider the trace anomaly, nor other anomalies that affect space-time transformations. We shall restrict ourselves to anomalies for internal transformations, which are closely related to the topics developed in the preceding chapters.

From the mathematical point of view the existence of anomalies is related to a very rich vein of algebraic and geometrical results. In particular, global (axial) anomalies are intimately related to the index theorem for the Dirac operator and the existence of gauge anomalies can be proven using a generalization of the index theorem involving two-parameter families of Dirac operators. The whole subject can also be recast in cohomological language. Altogether, there are few other fields where the progress of physics and mathematics have been so close.

### 1.1 The axial anomaly

Here we consider the ABJ anomaly, which was historically one of the earliest examples of anomaly. It appears in the case of a single complex massless
1.1. THE AXIAL ANOMALY

Dirac field coupled to electromagnetism. One finds that it is impossible to satisfy simultaneously the conservation of the vector and of the axial symmetry. Therefore this theory is anomalous. Depending on the regularization we choose, we can decide which symmetry is actually realized in the quantum theory: since the vector symmetry is in some sense more important than the axial symmetry, one usually prefers to give up the latter. Once this is understood, one then says that the axial symmetry is anomalous. In the next section we shall generalize the results to the case of a multiplet of fermions fields, carrying a representation of some global symmetry group.

We begin by setting up some notation. The action for a fermion in \( n \) spacetime dimensions, coupled to an external electromagnetic potential \( A_\mu \) is

\[
S_F(\psi, \bar{\psi}, A) = - \int d^n x \bar{\psi} \left( \gamma^\mu D_\mu + m \right) \psi . \tag{1.1}
\]

Our conventions for the gamma matrices are given in Appendix XXX. In particular, we recall the chirality operator \( \gamma^A \) (often called \( \gamma^5 \)), that anti-commutes with the gamma matrices and squares to one.

The action (1.1) is invariant under the (global) vector transformations

\[
\psi' = e^{-i\alpha} \psi ; \quad \bar{\psi}' = \bar{\psi} e^{i\alpha} \tag{1.2}
\]

with associated vector current

\[
j^\mu_V = \bar{\psi} \gamma^\mu \psi . \tag{1.3}
\]

One is also interested in the axial transformations

\[
\psi' = e^{i\beta \gamma^A} \psi ; \quad \bar{\psi}' = \bar{\psi} e^{i\beta \gamma^A} . \tag{1.4}
\]

Under an infinitesimal axial transformation the variation of the action (1.1) is

\[
\delta S = -2i\beta m \int d^n x \bar{\psi} \gamma^A \psi , \tag{1.5}
\]

showing that (1.4) is a symmetry of the Dirac action only if the mass vanishes. In this case the corresponding Noether current is

\[
j^\mu_A = \bar{\psi} \gamma^A \gamma^\mu \psi . \tag{1.6}
\]

In general, the divergence of this current is

\[
\partial_\mu j^\mu_A = -2m \bar{\psi} \gamma^A \psi , \tag{1.7}
\]

so the axial current is conserved only if the mass is zero: in the following we will consider the massless situation without losing any generality.
1.1.1 Point splitting

Let us now quantize the theory and ask whether \( \partial_\mu \langle j^\mu_A \rangle = 0 \) in the massless case (it is understood that \( j^\mu_A \) now denotes the quantum operator corresponding to the axial current (1.6) and the brackets its vacuum expectation value in the \( A_\mu \) background). The formal manipulations leading to the result (1.7) cannot be trusted because the operator \( j^\mu_A \) is the product of two fields at the same spacetime point and is therefore singular: in other words the naive definition of composite operator in quantum field theory leads to divergent result. One has to resort to some kind of regularization. Physically, the most transparent regularization for problems of this type is point splitting: the axial current operator is defined to be the \( \epsilon \to 0 \) limit of the following expression:

\[
\begin{align*}
    j^\mu_A(x, \epsilon) &= \bar{\psi}(x + \frac{\epsilon}{2}) \gamma^A \gamma^\mu \exp \left( i e a \int_{x - \frac{\epsilon}{2}}^{x + \frac{\epsilon}{2}} A \right) \psi(x - \frac{\epsilon}{2}).
\end{align*}
\]

(1.8)

The regulator \( \epsilon \) is a vector representing an infinitesimal displacement in spacetime; in order not to break Lorentz invariance it will be necessary, in taking the limit \( \epsilon \to 0 \), to average over all directions.

The exponential contains an arbitrary parameter \( a \). Under the local gauge transformation

\[
\begin{align*}
    \psi'(x) &= e^{-ia(x)}\psi(x) \quad ; \quad \bar{\psi}'(x) = e^{ia(x)}\bar{\psi}(x) \quad ; \quad A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \alpha
\end{align*}
\]

it becomes

\[
\exp \left( i e a \int_{x - \frac{\epsilon}{2}}^{x + \frac{\epsilon}{2}} A' \right) = \exp \left( -ia \alpha \left( x + \frac{\epsilon}{2} \right) \right) \exp \left( i e a \int_{x - \frac{\epsilon}{2}}^{x + \frac{\epsilon}{2}} A \right) \exp \left( i a \alpha \left( x - \frac{\epsilon}{2} \right) \right).
\]

For \( a = 1 \) the two outer exponentials cancel the transformation of the fermions and the regulated current (1.8) is gauge invariant. To compute the divergence of the current in the quantum theory, the prescription is to take first the divergence and then the limit.

Using the equations of motion

\[
\begin{align*}
    \gamma^\mu \partial_\mu \psi &= i e \gamma^\mu A_\mu \psi - m \psi, \\
    \partial_\mu \bar{\psi} \gamma^\mu &= -i e \bar{\psi} \gamma^\mu A_\mu + m \bar{\psi},
\end{align*}
\]

(1.9)
one finds that
\[ \partial_\mu j_\mu^A(x, \epsilon) = ie j_\mu^A(x, \epsilon) \left[ A_\mu \left( x - \frac{\epsilon}{2} \right) - A_\mu \left( x + \frac{\epsilon}{2} \right) + a \partial_\mu \int_{x + \frac{\epsilon}{2}} \right] , \] (1.10)

plus the classical term given in (1.7), that we shall disregard from now on.

For small \( \epsilon \) the square bracket can be expanded as
\[ \epsilon^\alpha (\partial_\alpha A_\mu - a \partial_\mu A_\alpha) + O(\epsilon^2) = \epsilon^\alpha (F_{\alpha\mu} + (1 - a) \partial_\mu A_\alpha) + O(\epsilon^2) . \]

Note that the classical result is recovered if one takes the limit \( \epsilon \to 0 \) naively. We have already said that this incorrect, being the limit in \( j_\mu^A(x, \epsilon) \) singular. To see this concretely, let us take the vacuum expectation value of both sides of (1.10). We find
\[ \langle \partial_\mu j_\mu^A(x, \epsilon) \rangle = -ie \langle j_\mu^A(x, \epsilon) \rangle \epsilon^\alpha (F_{\alpha\mu} + (1 - a) \partial_\mu A_\alpha) + O(\epsilon^2) . \] (1.11)

### 1.1.2 Calculation of the anomaly

We will now show that the second term does not vanish in the limit, because the coefficient of \( \epsilon \) is divergent. The v.e.v. on the r.h.s. can be rewritten, for \( \epsilon^0 > 0 \),
\[ \langle j_\mu^A \rangle = -Tr \gamma^A \gamma^\mu \left( T \psi \left( x - \frac{\epsilon}{2} \right) \bar{\psi} \left( x + \frac{\epsilon}{2} \right) \right) e^{ieA_\mu} \]
\[ = -Tr \gamma^A \gamma^\mu G \left( x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2} \right) e^{ieA_\mu} , \] (1.12)

where the trace is over Dirac indices, \( T \) denotes time ordering and \( G(x, y) \) denotes the Dirac propagator in an external electromagnetic field, defined by
\[ \gamma^\mu (\partial_\mu - ieA_\mu(x)) G(x, y) = \delta(x - y) . \] (1.13)

Note that due to the presence of the external field, \( G \) is not simply a function of the difference \( x - y \). Let \( S(x - y) \) denote the free Dirac propagator, which is defined by equation (1.13) with \( A_\mu \) set equal to zero. Multiplying (1.13) by \( S \) on the left (in the sense of kernel composition) and using
\[ -\frac{\partial}{\partial y^\mu} S(x - y) \gamma^\mu = \delta(x - y) \]
one finds the equation

\[ G(x, y) = S(x - y) + ie \int dz S(x - z) \gamma^\mu A_\mu(z) G(z, y). \] (1.14)

This equation can be solved by iteration:

\[ G \left( x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2} \right) = S(-\epsilon) + ie \int d^2 y S \left( x - \frac{\epsilon}{2} - y \right) \gamma^\mu A_\mu(y) S \left( y - x - \frac{\epsilon}{2} \right) \] (1.15)

\[-e^2 \int dy dz S \left( x - \frac{\epsilon}{2} - y \right) \gamma^\mu A_\mu(y) S(y - z) \gamma^\nu A_\nu(z) S \left( z - x - \frac{\epsilon}{2} \right) + \ldots\]

This is represented graphically in Fig. XXX, where the single lines represent free propagators, and the crosses represent insertions of the external field. At this point the analysis begins to depend upon the dimension of spacetime. The free propagator can be written in Fourier space

\[ S(x) = \int \frac{d^2 p}{(2\pi)^2} e^{-ip \cdot x} \frac{\gamma^\rho p^\rho}{p^2} = \gamma^\rho S_\rho(x) \]

and inserting in (1.15) we see that the first term diverges for \( \epsilon \to 0 \) like \( \epsilon^{-(n-1)} \), the second like \( \epsilon^{-(n-2)} \) and so on, until the \( n \)-th term, which diverges logarithmically. All subsequent terms are convergent.

The expression (1.15) contains an odd number of gamma matrices. When inserted in (1.12) we have a trace of \( \gamma^A \) times an even number of gamma matrices. The first nonzero term occurs when the number of gamma matrices is equal to the spacetime dimensions. This leading term is proportional to the totally antisymmetric Levi-Civita symbol. It is always linearly divergent and inserted in (1.12) it gives a finite contribution. The subsequent logarithmically divergent and finite terms of \( G \) are irrelevant in the limit \( \epsilon \to 0 \).

Let us discuss first the case \( n = 2 \). We have to take into account only the first term of the expansion (1.15) so the right hand side of (1.11) becomes:

\[ ie \text{Tr}[\gamma^A \gamma^\mu \gamma^\nu] S_\nu(-\epsilon) \epsilon^\alpha F_{\alpha \mu} + O(\epsilon). \]

The trace gives

\[ \text{Tr} \left[ \gamma^A \gamma^\mu \gamma^\nu \right] = 2i \epsilon^{\mu\nu} \]
while for the fermionic propagator we obtain:

\[ S_\nu(-\epsilon) = -\frac{1}{2\pi} \frac{\epsilon_\nu}{\epsilon^2}. \]  

(1.16)

Taking the limit in \( \epsilon \) and averaging over all the directions gives

\[ \lim_{\epsilon \to 0} \frac{\epsilon_\alpha \epsilon_\nu}{\epsilon^2} = \frac{1}{2} \eta_{\alpha\nu}. \]  

(1.17)

The factor 1/2 comes from imposing that both sides of the equation have the same trace. In this way we arrive at the following expression for the anomaly:

\[ \langle \partial_\mu j_\mu^A \rangle = \frac{e}{2\pi} \varepsilon^{\mu\nu} F_{\mu\nu}. \]  

(1.18)

We note that this is twice the integrand of the topological invariant \( c_1 \) defined in ???.

A slightly more complicated calculation leads to the following result in four dimensions:

\[ \partial_\mu j_\mu^A = \frac{e^2}{16\pi^2} \varepsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda}. \]  

(1.19)

Some remarks are now in order: first of all we notice that the anomaly appears to be a finite object and this is not a coincidence. Looking at the previous computations we learn that a classical zero with a quantum infinity give rise to a finite term in the conservation laws: this is always the way in which anomalies arise in quantum field theory. Moreover if we restore the Planck constant \( h \) we find that the coefficient of the anomaly depends linearly on it, manifesting the pure quantum mechanical nature of the phenomenon.

At this point one may ask what is the destiny of the vector current (1.3) and of its conservation law: it is easy to understand that no anomaly arises using this regularization procedure. The trace over Dirac matrices produces a tensor that, after averaging over the \( \epsilon \) directions, saturates two symmetric indices with the strength field generated by the exponential string: a simple computation in \( d = 2, 4 \) may convince the reader.

### 1.1.3 Other axial anomalies

In the preceding calculation we have considered a fermion coupled to the electromagnetic field. Let us now generalize the previous results to the case of a multiplet of fermions \( \psi^A \), carrying a representation of a global symmetry
group $G$ (in realistic applications, $G$ is $SU(N)$, and is called the flavor group). The matrices representing the generators in the Lie algebra of $G$ will be denoted $T_a$. They are assumed to be antihermitian and to satisfy

$$T_a^\dagger = -T_a, \quad [T_a, T_b] = f_{abc} T_c, \quad f^*_{abc} = f_{abc}, \quad \text{tr} T_a T_b = -\frac{1}{2} \delta_{ab}. \quad (1.20)$$

For example for $SU(2)$, $T_a = -\frac{i}{2} \sigma_a$ whereas for $SU(3)$, $T_a = -\frac{i}{2} \lambda_a$, where $\lambda_a$ are the Gell-Mann matrices. We will not write explicitly the indices of the fermions, neither the spinor indices nor the indices pertaining to the representation of $G$. Thus $\psi$ will now denote a column vector on which the group acts by left multiplication and the Dirac conjugate $\bar{\psi}$ is a row vector on which the group acts by right multiplication. The action can be written again as in (1.1), with our new interpretation of symbols. The field

$$A_\mu = A^a_\mu T_a \quad (1.21)$$

is an external (non-dynamical) non-abelian gauge field that couples to the fermions. $^2$ This action has a global symmetry $U(1)_V \times G_V$, where $U(1)_V$ is defined by (1.2), with all components of $\psi$ transforming by the same phase, and $G_V$ is defined by

$$\psi' = g^{-1} \psi; \quad \bar{\psi}' = \bar{\psi} g, \quad (1.22)$$

where $g = e^{-ia^a T_a}$. The Noether current associated to $U(1)_V$ is (1.3), and the Noether current associated to $G_V$ is

$$j^V_\mu = \bar{\psi} T_a \gamma^\mu \psi. \quad (1.23)$$

It transforms according to the adjoint representation. In the case $m = 0$ (1.1) is also invariant under a group $U(1)_A \times G_A$, where $U(1)_A$ is defined by (1.4), and $G_A$ is given by the transformations

$$\psi' = e^{ia^a T_a \gamma^A} \psi; \quad \bar{\psi}' = \bar{\psi} e^{ia^a T_a \gamma^A}. \quad (1.24)$$

The Noether current corresponding to $U(1)_A$ is (1.6), and the current associated to $G_A$ is

$$j^A_\mu = \bar{\psi} T_a \gamma^A \gamma^\mu \psi. \quad (1.25)$$

$^2$In the Abelian case, according to XXX, there is a single anti-Hermitian generator $T_1 = i/\sqrt{2}$ and we should write $A_\mu = A^1_\mu i/\sqrt{2}$. We will avoid this awkward notation, by treating this case separately.
1.2. *THE INDEX THEOREM*

As in the abelian case, the vector and the axial current cannot be simultaneously conserved. The anomaly can be computed using the method described above, the only difference consists in replacing the exponential in (1.8) by a path ordered exponential, and one finds

\[ \partial_\mu j_\mu^A = \frac{ie}{2\pi} \epsilon^{\mu\nu} \text{tr} F_{\mu\nu} \quad \text{for} \quad n = 2 \quad (1.26) \]

\[ \partial_\mu j_\mu^A = \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma} \quad \text{for} \quad n = 4 \quad (1.27) \]

\( F_{\mu\nu} \) now being the non-abelian field strength. Note that the r.h.s. of (1.26) is zero for the group \( SU(N) \), and the r.h.s. of (1.27) is twice the topological invariant \( c_2 \). The argument given above can be generalized straightforwardly also to the calculation of the (covariant) divergence of the non-singlet current (1.25). The result is

\[ (D_\mu j_\mu^A)_a = \frac{e}{2\pi} \epsilon_{\mu\nu} \text{tr} T_a F_{\mu\nu} , \quad \text{for} \quad n = 2 \quad (1.28) \]

\[ (D_\mu j_\mu^A)_a = \frac{ie^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr} T_a F_{\mu\nu} F_{\rho\sigma} , \quad \text{for} \quad n = 4 \quad (1.29) \]

We note that the factors of \( i \) in the formulas (1.26) and (1.29) are needed for these traces to be real, since they contain an odd number of antihermitian matrices. We end the section stressing again that the axial symmetry we have discussed is a *global* symmetry: no local invariance has been allowed in the classical action for the axial transformation. The anomalies we have studied are therefore global anomalies.

### 1.2 The index theorem

The careful reader will have noticed that the anomaly of the \( G_A \)-current is twice the integrand of the topological invariants \( c_1 \) and \( c_2 \) introduced in sections XXX. This is no coincidence, and in fact the axial anomaly can be shown to affect the phenomenon of theta vacua in gauge theories in the presence of fermion fields. We begin by observing that since the fermionic configuration space is linear, the general topological arguments for the existence of theta sectors given in sections 2.4 and 2.5 continue to hold (the same remark had been made in section 2.8 concerning the coupling of the gauge theory to scalars). However, *massless* fermions have a dramatic influence on
the dynamics of such theories: it turns out that the v.e.v.s of gauge invariant
operators become independent of $\theta$. The proof of this statement relies on a
profound mathematical result, known as the Atiyah-Singer index theorem,
that encodes the topological meaning of the axial anomaly.

In order to state this theorem in a simple form, we pass to the Euclidean
signature and assume that spacetime is even dimensional, compact and with-
out boundary. As usual, this can be achieved by imposing suitable boundary
conditions on all fields: for instance we can take $S^n$ as the compact space-time.

In this case the Dirac operator:

$$D = \gamma^\mu (i\partial_\mu + A_\mu)$$

is self-adjoint and its spectrum is discrete in the space $V$ of fermionic fields
such that $\int d\mu \bar{\psi}\psi$ is finite, $d\mu$ being the natural measure of $S^n$. Since $(\gamma^A)^2 = 1$, we can split

$$V = V_+ \oplus V_-$$

where $V_\pm$ are eigenspaces of $\gamma^A$ with eigenvalues $\pm1$ respectively. Now let
\{\psi_n\} be a complete set orthonormal eigenfunctions of $D$:

$$D\psi_n = \lambda_n \psi_n \quad ; \quad \int d\mu \bar{\psi}_m \psi_n = \delta_{mn} .$$

Since $\gamma^A$ anticommutes with $\gamma^\mu$, if $\psi$ is in $V_+$, $D\psi$ is in $V_-$, and vice-versa. So, if $\lambda_n \neq 0$, $\psi_n$ cannot be an eigenfunction of $\gamma^A$. However, the eigenfunctions
with zero eigenvalue (the zero modes) can be chosen to belong either to $V_+$ or
$V_-$. We will call $n_+$ and $n_-$ the numbers of linearly independent zero modes
of $D$ with positive and negative chirality respectively. The index theorem
states that

$$n_+ - n_- = c_n , \quad (1.30)$$

where $c_n$ is the topological invariant defined by (2.4.3) and (2.5.1) for $n = 2$
and 4 respectively. The proof of this theorem is beyond the scope of these
notes. Here we shall instead give a heuristic derivation that highlights its
connection to the axial anomaly.

1.2.1 Derivation from the anomaly

Let us start with a massive fermion interacting with an external gauge field
via the vector current (1.23) (or (1.3) if $G = U(1)$) in four dimension. Here
the mass plays the role of a regulator and will be sent to zero in the end. From the anomaly, the divergence of the axial current is

$$\partial_\mu \langle j^\mu_A \rangle = -2m \langle \bar{\psi} \gamma^A \psi \rangle + \frac{i}{8\pi} \text{Tr} F^\mu_\nu F^{\mu\nu}$$

(The factor $i$ appears because we are now in Euclidean signature.) We integrate both sides and take the expectation value in the fermionic vacuum. The l.h.s. gives

$$\int d^4x \partial_\mu j^\mu_A = \int d\Sigma \mu j^\mu_A = 0$$

because the fermion field is massive (in this case the current has a smooth behaviour in the limit or if were in the compact situation no singularity occurs). From the r.h.s. we obtain

$$2m \int d^4x \langle \bar{\psi} \gamma^A \psi \rangle = -2ic_2.$$ 

Now we want to evaluate the v.e.v. on the left:

$$\langle \bar{\psi} \gamma^A \psi \rangle = \left( \int d\psi d\bar{\psi} e^{-S_F} (\int d^4x \bar{\psi} \gamma^A \psi) \right) \left( \int d\psi d\bar{\psi} e^{-S_F} \right)^{-1}.$$ (1.31)

The eigenfunctions of $D$ defined in (1) are also eigenfunctions of $D - im$ with eigenvalues $\lambda_n - im$. Thus we can decompose

$$\psi(x) = \sum_n a_n \psi_n(x) ; \quad \bar{\psi}(x) = \sum_n \bar{a}_n \bar{\psi}_n(x),$$

$$S_F(\psi, \bar{\psi}, A) = \sum_n \bar{a}_n a_n (\lambda_n - im) ; \quad (d\psi d\bar{\psi}) = \prod_n da_n d\bar{a}_n$$

The functional integrals in (1.31) can be performed using Berezin’s rules for the integration over fermion fields:

$$\int da_n a_m = \delta_{nm} ; \quad \int d\bar{a}_n \bar{a}_m = \delta_{nm}$$

The numerator of (1.31) is

$$\prod_n \int da_n d\bar{a}_n (1 - \bar{a}_n a_n (\lambda_n - im)) \left[ \sum_{rs} \bar{a}_r a_s \int d^4y \bar{\psi}_r(y) \gamma^A \psi_s(y) \right]$$

$$= \sum_r \int d^4y \bar{\psi}_r(y) \gamma^A \psi_r(y) \prod_{s \neq r} (\lambda_s - im),$$ (1.32)
while the denominator is
\[
\int (d\psi d\bar{\psi}) e^{-S_F} = \prod_n \int d\bar{a}_n da_n (1 - (\lambda_n - im)\bar{a}_n a_n)
\]
\[
= \prod_n (\lambda_n - im) = \det (D - im) .
\] (1.33)

In the above formulas the formal determinant of the Dirac operator appears. It can be given a meaning by choosing a specific regularization procedure. In any case, for the calculation we are interested in we do not need its details, as we shall see. Therefore,
\[
\int d^4 y \langle \bar{\psi}(y) \gamma^A \psi(y) \rangle = \sum_r \int d^4 y \bar{\psi}_r(y) \gamma^A \psi_r(y) \frac{\lambda_r - im}{\lambda_r - im}
\]
Since $D$ anticommutes with $\gamma^A$, if $\psi_n$ is an eigenfunction with eigenvalue $\lambda_n$, $\gamma^A \psi_n$ is an eigenfunction with eigenvalue $-\lambda_n$. Therefore, using the orthogonality (1.2) we find that if $\lambda_s \neq 0$, $\int d^4 y \bar{\psi}_s(y) \gamma^A \psi_s(y) = 0$. On the other hand, if $\psi_n$ is a zero mode, also $\gamma^A \psi_n$ is a zero mode, and we have chosen the zero modes to have definite chirality. Therefore, if $\lambda_n = 0$, $\int d^4 y \bar{\psi}_s(y) \gamma^A \psi_s(y) = \pm 1$, depending on the chirality. So we find that

\[
\int d^n y \langle \bar{\psi}(y) \gamma^A \psi(y) \rangle = -\frac{1}{im} (n_+ - n_-)
\]
and inserting back in (1.2.1) we obtain the index theorem (1.30). We have shown in this way that with certain assumptions about the boundary conditions, the index theorem follows from the existence of the axial anomaly. Conversely we can understand the appearance of the axial anomaly not as an accident of the quantum field theory but as a consequence of the dependence of the spectrum of the Dirac operator on the topology of the gauge field.

1.2.2 Consequences for the theta sectors

In the previous discussion the gauge field was treated as a fixed background. Let us now see the implications of these results for the quantization of the full theory, with dynamical gauge field. We are specifically interested in the tunnelling amplitude through the noncontractible path in $Q$, since this amplitude was responsible for the $\theta$–dependence of the vacuum energy, as
we have seen in Sections 2.7-9. This amplitude is given by the Euclidean functional integral

\[
\int_{c_{n/2} \neq 0} (dA \bar{\psi} d\psi) e^{-S(\psi, \bar{\psi}, A)} = \int_{c_{n/2} \neq 0} (dA) e^{-S_{\text{YM}}(A)} \det(D),
\]

where we have used (1.33). The integral is restricted to those gauge field configurations which have nonvanishing topological invariant. Now the index theorem (1.30) implies that for these fields there must be at least one zero mode, and therefore the determinant of the Dirac operator is identically zero, for all \( A \). This implies that the tunnelling amplitude is zero, and therefore the theta vacua are all degenerate, in sharp contrast to what happened without massless fermions. Note that in deriving this result we did not have to make use of the WKB approximation.

There is also a formal argument that directly relates the existence of the anomaly to the degeneracy of the theta vacua. Consider a gauge- and chiral-invariant operator \( O \). The v.e.v. of \( O \) in the vacuum specified by a value \( \theta \) can be computed as

\[
\langle O \rangle = \delta \log Z_\theta(J) / \delta J,
\]

where

\[
Z_\theta(J) = \int (dA \bar{\psi} d\psi) e^{-S_{\text{YM}}(A) + i \theta c_2(A) - S_F(\psi, \bar{\psi}, A)} + \int d^4x J O.
\]

To simplify the notation we do not write explicitly the gauge fixing and ghost terms, since they are irrelevant for what follows. A priori, the v.e.v. of \( O \) seems to depend upon \( \theta \). Let us now examine how \( Z_\theta \) behaves under the chiral transformations (1.4). Since the action is invariant under chiral transformations, if the measure was also invariant, the whole functional integral would be invariant. But this is incompatible with the statement that there is an anomaly, so the measure cannot be invariant. Without discussing this in detail, we can infer how the measure has to transform, from our knowledge of the anomaly. Under an infinitesimal chiral transformation \( \delta \psi = \delta \alpha \gamma^A \bar{\psi} \) the transformation of the measure can be written as \( (d\bar{\psi} d\psi) = (d\bar{\psi}' d\psi') e^{\delta S} \). The variation of the action can be inferred from Noether’s theorem to be

\[
\delta S = \int d^4x \delta \mathcal{L} = \int d^4x \partial_\mu j^\mu_\psi = 2i \delta \alpha c_2(A).
\]
Therefore the effect of a chiral transformation on the fermion fields is equivalent to a shift of $\theta$ by $2\delta\alpha$:

$$Z_\theta(J) = \int (dA d\bar{\psi}' d\psi') e^{-S_{YM}(A)+i(\theta+2\alpha)c_2(A)-S_F(\psi',\bar{\psi}',A)+\int d^4x J^\mu} = Z_{\theta+2\alpha}(J).$$

In the last step we have replaced $\psi'$ and $\bar{\psi}'$ by $\psi$ and $\bar{\psi}$, since these are integration variables. The conclusion is that the value of $\theta$ is irrelevant: the expectation value of every gauge and chiral invariant observable is independent of $\theta$.

We emphasize once again that this does not mean that there are no theta sectors anymore. The topological arguments remain valid. One can also argue that the theta sectors have to be still distinct in order that the cluster property be satisfied. All that has happened is that the theta sectors are now completely degenerate.

### 1.3 Gauge anomalies

Next we consider anomalies in a current that couples to gauge fields. We will call these “gauge anomalies”. Let us consider quite generally a fermionic current $J_a^\mu$ coupled to a gauge field $A^a_\mu$ via an interaction term $L_I = J_a^\mu A^a_\mu$.

To begin with, we do not specify whether $J$ is a vector, axial or other current. All we assume is that the classical action $S$ is gauge invariant. The classical current can be defined as

$$J_a^\mu = \frac{\delta S}{\delta A^a_\mu}.$$

Functional integration over the fermions yields a contribution to the action for the gauge fields:

$$W[A] = -i \ln \int (d\psi d\bar{\psi}) e^{iS_F[\psi,\bar{\psi},A]}.$$

We will loosely refer to $W$ as the effective action. The expectation value of the current in the fermionic vacuum is given by

$$\langle J_a^\mu \rangle = \frac{\delta W}{\delta A^a_\mu}. \quad (1.34)$$
1.3. GAUGE ANOMALIES

For an infinitesimal gauge transformation parameter \( \epsilon = \epsilon^a T^a \), define the operator

\[
\delta_\epsilon = \int d^2x \, D_\mu \epsilon^a(x) \frac{\delta}{\delta A^a_\mu(x)}.
\]

It can be thought of as a vector tangent to the gauge orbit through \( A \) in the space of all gauge fields. The derivative of \( W \) in the direction of this vector is

\[
\delta_\epsilon W[A] = \int d^2x \, D_\mu \epsilon^a(x) \frac{\delta W}{\delta A^a_\mu(x)} = \int d^2x \, \epsilon^a(x) \langle J^a_\mu \rangle = - \int d^2x \, \epsilon^a(x) \langle D_\mu J^a_\mu(x) \rangle.
\] (1.35)

Since \( \epsilon \) is arbitrary, we see that gauge invariance of the effective action \( W \) is equivalent to the covariant conservation of the current. This is a very important property in the full quantum gauge theory: in perturbation theory, it ensures unitarity and renormalizability.

In section 5.1 we discussed the case when \( J^\mu = j^\mu_V \). We proved that there exists a quantization scheme that preserves the conservation of this current, violating the conservation of the axial current. Since the axial current was not coupled to gauge fields, no problem arose in that case.

The situation is different if the coupling is not purely vectorial. The most general situation is to have the vector and axial currents coupled to two different gauge fields

\[
\mathcal{L}_I = j^\mu_V A^a_\mu + j^\mu_A A^a_\mu.
\]

In order to ensure the invariance of the action under the vector and axial gauge transformations (1.22) and (1.24) the gauge fields have to transform as follows:

\[
\begin{align*}
\delta_{V\epsilon} A^a_\mu &= D_\mu \epsilon ; \\
\delta_{A\epsilon} A^a_\mu &= [A^a_\mu, \epsilon] ;
\end{align*}
\] (1.36)

\[
\begin{align*}
\delta_{V\epsilon} A^a_\mu &= [A^a_\mu, \epsilon] ; \\
\delta_{A\epsilon} A^a_\mu &= D_\mu \epsilon .
\end{align*}
\] (1.37)

These transformations obey the following algebra:

\[
\begin{align*}
[\delta_{V\epsilon_1}, \delta_{V\epsilon_2}] &= \delta_{V[\epsilon_1, \epsilon_2]} ; \\
[\delta_{V\epsilon_1}, \delta_{A\epsilon_2}] &= \delta_{A[\epsilon_1, \epsilon_2]} ; \\
[\delta_{A\epsilon_1}, \delta_{A\epsilon_2}] &= \delta_{V[\epsilon_1, \epsilon_2]} ;
\end{align*}
\] (1.38) (1.39) (1.40)
CHAPTER 1. ANOMALIES

The vector and axial transformations are deeply entangled. It is convenient to define left and right currents

\[ j_{L\mu}^a = \frac{j_V^\mu - j_A^\mu}{2} = \bar{\psi} T_a \gamma^\mu \left( \frac{1 - \gamma^5}{2} \right) \psi \]  

(1.41)

\[ j_{R\mu}^a = \frac{j_V^\mu + j_A^\mu}{2} = \bar{\psi} T_a \gamma^\mu \left( \frac{1 + \gamma^5}{2} \right) \psi \]  

(1.42)

and left and right gauge fields

\[ A_{L\mu}^a = A_V^a - A_A^a \]  

(1.43)

\[ A_{R\mu}^a = A_V^a + A_A^a \]  

(1.44)

In term of these new variables the interaction reads

\[ \mathcal{L}_I = j_{L\mu}^a A_{L\mu}^a + j_{R\mu}^a A_{R\mu}^a , \]

and defining \( \delta_L = \delta_V - \delta_A \) and \( \delta_R = \delta_V + \delta_A \) the algebra becomes

\[ [\delta_{Le_1}, \delta_{Le_2}] = \delta_{L[e_1,e_2]} ; \]

(1.45)

\[ [\delta_{Le_1}, \delta_{Re_2}] = 0 ; \]

(1.46)

\[ [\delta_{Re_1}, \delta_{Re_2}] = \delta_{R[e_1,e_2]} . \]

(1.47)

In terms of these variables the left and right gauge transformations are completely decoupled. The left and right gauge fields transform in the usual way under the left and right gauge transformations and are coupled to the left and right currents respectively.

In discussing the possible anomalies of this theory it is therefore more convenient to use the left–right decomposition than the vector–axial decomposition. Since the left and right sectors of the theory are classically decoupled, it will be enough to study only one of them. From now on we will assume that only the left handed component of the fermion is coupled to a gauge field (henceforth denoted \( A \)); this is equivalent to setting \( A_R = 0 \).

We are now going to consider anomalies for the local gauge transformations in this chirally coupled theory. Some new features, related to dynamical character of the anomalous current, come into play. We start with a chirally modified non-abelian version of (1.1):

\[ S_F^L(\psi, \bar{\psi}, A) = \int d^{2n}x \bar{\psi} (i \gamma^\mu D^L_\mu) \psi , \]  

(1.48)
1.3. GAUGE ANOMALIES

where the new operator is defined as:

\[ D^L_\mu = \partial_\mu - ie \left( \frac{1 - \gamma^A}{2} \right) A_\mu. \]  

(1.49)

This action has a local symmetry \( G_L \),

\[
\psi'_L = g^{-1} \psi_L ; \quad \bar{\psi}'_L = \bar{\psi}_L g, \tag{1.50}
\]

\[
\psi'_R = \psi_R \quad \bar{\psi}'_R = \bar{\psi}_R, \tag{1.51}
\]

\[
A'_\mu = \frac{i}{e} g \partial_\mu g^{-1} + g A_\mu g^{-1}. \tag{1.52}
\]

where \( \psi_L = \left( \frac{1 - \gamma^A}{2} \right) \psi, \psi_R = \left( \frac{1 - \gamma^A}{2} \right) \psi \) and \( g = e^{-i \alpha^a T_a} \). Interactions of this type actually occur in the Standard Model.

We omit the calculation of the gauge anomaly. The result is

\[
[D_\mu \langle j_L^{\mu\nu} \rangle]^a = \pm \frac{e}{4\pi} \varepsilon^{\mu\nu\rho\lambda} tr T_a \partial_\mu A_\nu \quad \text{for} \quad n = 2 ;
\]

\[
[D_\mu \langle j_L^{\mu\nu} \rangle]^a = \pm \frac{ie^2}{24\pi^2} \varepsilon^{\mu\nu\lambda\rho} tr T_a \partial_\mu \left( A_\nu \partial_\lambda A_\rho + \frac{1}{2} A_\nu A_\lambda A_\rho \right) \quad \text{for} \quad n = 4 ,\tag{1.53}
\]

where the overall sign depends on the chirality of the fermions.

The form (1.53) of the anomaly is by no means unique: it depends on the chosen regularization of the fermionic determinant. Another regularization could result in another form of the effective action differing by a local functional of the gauge field and its derivatives. The determinant is given by a sum of one loop graphs with any number of insertions of the external field \( A \). The first term contains one power of \( A \) and diverges like \( \Lambda^{n-1} \), where \( \Lambda \) is some ultraviolet cutoff; the second contains two powers of \( A \) and diverges like \( \Lambda^{n-2} \) and so on. The \( n \)-th term contains \( A^n \) and is logarithmically divergent. All subsequent terms are finite. Divergent terms give rise to ambiguities in the effective action. One is free to change the renormalization conditions so as to add to the effective action finite terms proportional to the coefficients of these divergences. Therefore, one is free to modify the effective action by adding a polynomial in \( A \) and its derivatives of order \( n \) (and containing terms of dimension \( n \)). If the expression (1.53) was itself the variation of such a polynomial, then by a different choice of renormalization one could obtain zero anomaly. It can be shown that this is not the case, so the anomaly is a genuine physical phenomenon.
Independently of this, one can redefine the current in such a way that its transformation property is covariant. One defines a new current:
\[ \hat{j}_\mu^a = j_\mu^a + X_\mu^a \]
where
\[ X_\mu^a = \begin{cases} 0 & \text{for } n = 2, \\ \pm \frac{i}{24\pi^2} \varepsilon^{\mu\nu\rho\sigma} \mathrm{tr} \left( A_\nu \partial_\rho A_\sigma + \partial_\nu A_\rho A_\sigma + \frac{3}{2} A_\nu A_\rho A_\sigma \right) & \text{for } n = 4 \end{cases} \]
This current has the property
\[ \delta_\varepsilon \langle \hat{j}_\mu^a (x) \rangle = - [\varepsilon, \langle \hat{j}_\mu^a \rangle]_a \]
thereby recovering the classical tensorial transformation. The polynomial \( X_\mu^a \) is called the Bardeen-Zumino counterterm and the covariant divergence of \( \langle \hat{j}_L^\mu \rangle \) is known as the covariant anomaly. In \( d = 2, 4 \) we get
\[ \langle D_\mu \hat{j}_L^\mu \rangle_a = \begin{cases} \pm \frac{e}{2\pi} \varepsilon^{\mu\nu} \mathrm{tr} T_a F_{\mu\nu} & \text{for } n = 2 \\ \pm i \frac{e^2}{16\pi^2} \varepsilon^{\mu\nu\lambda\rho} \mathrm{tr} T_a F_{\mu\nu} F_{\lambda\rho} & \text{for } n = 4 \end{cases} \]
We remark that this redefinition does not correspond to the addition of local terms in the effective action, therefore the physical meaning of this current is not linked directly to the local gauge invariance.

### 1.4 Gauge anomalies and cohomology

We have seen in the previous sections that the anomalies appear as effects of a regularization procedure; we have also remarked in sect XXX that the gauge anomaly is just the manifestation of the impossibility to construct a gauge-invariant functional of gauge fields, when integrating out chiral fermions. It is therefore clear that the underlying group structure must play an important role in determining the form of the anomaly itself.

At the level of infinitesimal gauge transformations, a simple integrability condition, known as the Wess-Zumino consistency condition, is strong enough to nearly completely determine the anomaly itself. At the level of finite transformations, the same condition leads to the definition of the Wess-Zumino functional. We shall discuss here the interrelation between these notions.
1.4. GAUGE ANOMALIES AND COHOMOLOGY

1.4.1 The WZ consistency condition

Define the anomaly $A$ as the anomalous divergence of the gauge current. Then from equation (1.35),

$$\delta_\epsilon W[A] = -A(\epsilon, A)$$  \hspace{1cm} (1.57)

The operators $\delta_\epsilon$ form a representation of the gauge algebra:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{[\epsilon_1, \epsilon_2]}.$$  \hspace{1cm} (1.58)

If we now apply the above operatorial relation to the vacuum functional $W[A]$ we get an equation for $A(\epsilon, A)$:

$$\delta_{\epsilon_1} A(\epsilon_2, A) - \delta_{\epsilon_2} A(\epsilon_1, A) = A([\epsilon_1, \epsilon_2], A).$$  \hspace{1cm} (1.59)

This is called the Wess-Zumino (WZ) consistency condition. If the anomaly is defined as gauge variation of the effective action $W$, as in (1.57), then it must satisfy the above constraint. Such anomalies are called consistent anomalies. If on the other hand one defines the anomaly as $\langle D_\mu J^\mu \rangle$, then the result of a calculation may or may not satisfy this condition, depending on the regularization procedure.

For example, multiplying (1.53) by an infinitesimal gauge parameter $\epsilon^a$ and integrating over spacetime we obtain the expressions $^3$

$$A(A, \epsilon) = \mp \frac{e}{4\pi} \int d^2 x \varepsilon^{\mu\nu} \text{tr} \partial_\mu \epsilon A_\nu \quad \text{for} \quad n = 2;$$

$$A(A, \epsilon) = \mp \frac{ie^2}{24\pi^2} \int d^4 x \varepsilon^{\mu\rho\lambda\nu} \text{tr} \partial_\mu \epsilon \left( A_\nu \partial_\lambda A_\rho + \frac{1}{2} A_\nu A_\lambda A_\rho \right)$$  \hspace{1cm} (1.60)

One can verify by explicit calculation that they satisfy the WZ consistency condition. On the other hand, if we do the same with the covariant anomalies (1.56)

$$A(A, \epsilon) = \frac{e}{2\pi} \int d^2 x \varepsilon^{\mu\nu} \text{tr} \epsilon F_{\mu\nu} \quad \text{for} \quad n = 2$$

$$A(A, \epsilon) = i \frac{e^2}{16\pi^2} \int d^4 x \varepsilon^{\mu\rho\lambda\nu} \text{tr} \epsilon F_{\mu\nu} F_{\lambda\rho} \quad \text{for} \quad n = 2.$$  \hspace{1cm} (1.61)

$^3$For the present purposes it proves convenient to perform an integration by parts so that one derivative acts on the gauge parameter. This is legitimate, since $\epsilon$ vanishes at infinity.
we find that they do not satisfy it. Therefore, these anomalies are not the variation of a functional $W(A)$, not even of a locally defined functional, as we shall now discuss.

The definitions in this section have a clear geometrical meaning. Let $C$ be the space of connections $A$, and $G$ the gauge group. For a fixed infinitesimal gauge transformation $\epsilon$, $\delta_\epsilon$ is a first order (functional) differential operator corresponding to the directional derivative along a vector field tangent to the orbits of the gauge group in the space of connections. We can think of it as a vertical vectorfield on $C$, i.e. a vectorfield that is in the kernel of the projection $C \to C/G$. Fix a reference gauge field $A$ and consider its gauge orbit $O_A$. It is diffeomorphic to the gauge group $G$. The anomaly $\mathcal{A}$ is a linear functional that maps vectorfields on $O_A$ to real numbers. Thus, we can think of it as a one-form on $O_A$. Equation (1.59) is the statement that $\mathcal{A}$ is a closed form. If $W$ was a globally well-defined functional on $O_A$, equation (1.57) would say that $\mathcal{A}$ is an exact form, and $\mathcal{A}$ would be in the trivial cohomology class in $H^1(O_A)$, or equivalently of $H^1(G)$. However, at this stage we do not really know $W$ well enough. Equation (1.57) must be interpreted as saying that $\mathcal{A}$ is locally exact, i.e. the differential a locally-defined functional $-W$.

In the next section we will consider the global properties of this functional and we shall see that it is not globally well-defined.

### 1.4.2 The WZ functional

In the preceding section we have considered the effect of infinitesimal gauge transformations on the fermionic determinant. Let us now consider the effect of finite gauge transformations. Define the Wess-Zumino functional $\Gamma_{WZ}(A,g)$ to be (minus) the change in the fermionic effective action under a gauge transformation:

$$W(A^g) - W(A) = -\Gamma_{WZ}(A,g) .$$

When $g$ differs infinitesimally from the identity, $\Gamma_{WZ}$ becomes the anomaly:

$$\Gamma_{WZ}(A, 1 + \epsilon) = \mathcal{A}(A, \epsilon) .$$

From the definition one finds that

$$\Gamma_{WZ}(A^{g_1}, g_2) - \Gamma_{WZ}(A, g_1 g_2) + \Gamma_{WZ}(A, g_1) = 0 .$$

\footnote{Unlike earlier sections, here we consider connections and gauge transformations on spacetime, not just space.}
This condition has a cohomological significance which is the analogue of the WZ consistency condition for finite transformations. A functional satisfying it is said to be a one-cocycle for the action of the gauge group with coefficients in the smooth functionals of $A_\mu$.

The way it was derived, $\Gamma_{WZ}$ depends on a connection $A$ and a gauge transformation $g$. However, $g$ is just a map from spacetime to the group $G$ and we can also think of it as a configuration for a chiral model. In this case we denote it as $U$ and we can think as $\Gamma_{WZ}(A,U)$ as a possible term in the action for a chiral model coupled to gauge fields. In this case it is more convenient to rewrite (1.4.2) in the form

$$\Gamma_{WZ}(A^g_\mu, U^g) - \Gamma_{WZ}(A, U) = -\Gamma_{WZ}(A, g) .$$

(1.63)

where $U^g = g^{-1}U$ can be thought of as the gauge transform of $U$ by $g$. Comparing with (A.2), this formula shows that the WZ functional has the same anomalous transformation property as the fermionic determinant. The important difference, that we shall now see, is that whereas $W$ is a non-local functional, $\Gamma_{WZ}$ is a local functional.

We can compute $\Gamma$ explicitly by integrating the anomaly. We begin by fixing a reference gauge field $A_{\mu}$. Then we can identify the orbit through $A_{\mu}$ with $G$ (mapping $g$ to $A^g_\mu$) and we can regard $W$ as a function on $G$. As above, we think of the anomaly $A$ as one-form on $G$. Since $A$ is closed, its integral along a curve does not change under continuous deformations of the curve, as long as the endpoints remain fixed. It can only change in a discontinuous way if we change the homotopy class of the curve. Let $g(r)$ be a one-parameter family of gauge transformations interpolating between $g$ and the identity:

$$\hat{g}(r) = e^{r a^a A_a} ; \quad \hat{g}(0) = e ; \quad \hat{g}(1) = g .$$

(1.64)

and let

$$\hat{A}_\mu(r) = \hat{g}^{-1} A_\mu \hat{g} + \hat{g}^{-1} \partial_\mu \hat{g}$$

(1.65)

be the gauge transform of $A_\mu$ at the point $r$ along the path. Then the WZ functional can be written

$$\Gamma_{WZ}[A, g] = \int_0^1 dr \ A(A^{g(r)}, \hat{g}^{-1} \partial \hat{g}) .$$

(1.66)
Let us perform the integral explicitly in $d = 2$. Using the anomaly (1.60) we have to compute

$$\frac{1}{4\pi} \int_0^1 dr \int d^2 x \varepsilon^{\mu \nu} \text{tr} \, \hat{g}^{-1} \partial_r \hat{g} \partial_\mu (\hat{g}^{-1} A_\nu \hat{g} + \hat{g}^{-1} \partial_\nu \hat{g})$$

$$\frac{1}{4\pi} \int_0^1 dr \int d^2 x \varepsilon^{\mu \nu} \partial_r \hat{g} \hat{g}^{-1} \left[ \partial_\mu A_\nu - \partial_\mu \hat{g} \hat{g}^{-1} A_\nu + A_\nu \partial_\mu \hat{g} \hat{g}^{-1} - \partial_\mu \hat{g} \hat{g}^{-1} \partial_\nu \hat{g} \hat{g}^{-1} \right].$$

Now we rewrite this in a covariant form in the three coordinates $r, x_1, x_2$, which parametrize a three-dimensional ball with boundary $S^2$ (it is assumed that the $r$-component of $A_\mu$ is zero):

$$\frac{1}{4\pi} \int d^3 x \varepsilon^{\lambda \mu \nu} \text{tr} \left[ \partial_\lambda \hat{g} \hat{g}^{-1} \partial_\mu A_\nu - \partial_\lambda \hat{g} \hat{g}^{-1} \partial_\mu \hat{g} \hat{g}^{-1} A_\nu - \frac{1}{3} \partial_\lambda \hat{g} \hat{g}^{-1} \partial_\mu \hat{g} \hat{g}^{-1} \partial_\nu \hat{g} \hat{g}^{-1} \right].$$

The first two terms are a total derivative, and can be rewritten as a two-dimensional integral, so we obtain

$$\Gamma_{WZ}(A, g) = - \frac{1}{4\pi} \int d^2 x \varepsilon^{\mu \nu} \text{tr} \left[ R_\mu A_\nu - \frac{1}{12\pi} \int d^3 x \varepsilon^{\lambda \mu \nu} \text{tr} \hat{R}_\lambda \hat{R}_\mu \hat{R}_\nu \right], \quad (1.67)$$

where $\hat{R}_\mu = \partial_\mu \hat{g} \hat{g}^{-1}$. We note that if $g = 1 + \epsilon$, $R_\mu = \partial_\mu \epsilon$, so the first term gives back $A(A, \epsilon)$, as expected. Also, we recognize that the second term is the correctly normalized WZW action $S_{WZW}$, defined in (??), depending on the extension of the field $g$ from the compactified two-dimensional space-time $S^2$ to the interior of a ball. Recall that, even though the integrand of $S_{WZW}(\hat{g})$ is the same as that of the winding number, $S_{WZW}(\hat{g})$ is not a topological invariant, since it depends on the boundary values of $\hat{g}$. The WZW action that appears in the WZ functional corresponds to the choice $n = 1$, or $c = 2\pi$, for the coefficient. Thus, the WZ functional can be viewed as a left-gauged extension of the WZW functional.

As an exercise, let us check that this functional satisfies the condition (1.63). We compute

$$- \frac{1}{4\pi} \int d^2 x \varepsilon^{\mu \nu} \text{tr} \left[ \partial_\mu (g^{-1} U)(g^{-1} U)^{-1} (g^{-1} A_\nu g + g^{-1} \partial_\nu g) \right] + S_{WZW}(g^{-1} U).$$

The first integral gives

$$\frac{1}{4\pi} \int d^2 x \varepsilon^{\mu \nu} \text{tr} \left[ R_\mu^g A_\nu + R_\mu^g R_\nu^g - R_\mu^U A_\nu - R_\nu^U R_\mu^g \right] \quad (1.68)$$
where we denote $R^g_{\mu} = \partial_{\mu} gg^{-1}$ and $R^U_{\mu} = \partial_{\mu}UU^{-1}$. The second term gives

$$-\frac{1}{12\pi} \int d^3 x \varepsilon^{\lambda\mu\nu} \text{tr} \left[ -\hat{R}_\lambda^g \hat{R}_\mu^g \hat{R}_\nu^g + 3 \hat{R}_\lambda^g \hat{R}_\mu^g \hat{R}_\nu^U - 3 \hat{R}_\lambda^g \hat{R}_\mu^U \hat{R}_\nu^U + \hat{R}_\lambda^U \hat{R}_\mu^U \hat{R}_\nu^U \right],$$

(1.69)

The first and last terms in this expression are equal to $-S_{WZW}(\hat{g})$ and $S_{WZW}(\hat{U})$ respectively. The two middle terms add up to a total derivative that exactly cancels the last term in (1.68). The second term in (1.68) vanishes identically. The remaining terms exactly reconstruct $-\Gamma_{WZ}(A,g) + \Gamma_{WZ}(A,U)$.

One can proceed in the same way in higher dimensions. Integrating the anomaly (1.60) one arrives at the following expression for the Wess-Zumino functional in four dimensions

$$\Gamma(A_\mu, g) = -i \frac{1}{48\pi^2} \int d^4 x \varepsilon^{\mu\nu\rho\sigma} \text{tr} \left[ (A_\mu \partial_\nu A_\rho + \partial_\mu A_\nu A_\rho + A_\mu A_\nu A_\rho) R_\sigma \right. - \frac{1}{2} A_\mu R_\nu A_\rho R_\sigma - A_\mu R_\nu R_\rho R_\sigma \bigg]$$

$$\left. - i \frac{1}{240\pi^2} \int_B d^5 x \varepsilon^{\lambda\mu\rho\sigma} \text{tr} R_\lambda R_\mu R_\nu R_\rho R_\sigma \right).$$

(1.70)

Once again we recognize that the last term is the WZW functional with the correctly normalized coefficient $c = 2\pi$. (FACTORS!!)

Finally, let us return to the question whether $W$ is a globally well-defined functional on the orbits of the gauge group. We will discuss this in the two-dimensional case, with gauge group $SU(2)$. Assuming that spacetime has been compactified to $S^2$, the gauge group is $\mathcal{G} = \Gamma_*(S^2, SU(2))$ and according to the lemma XXX, $\pi_1(\mathcal{G}) = \pi_3(S^2) = \mathbb{Z}$. Thus we need to worry whether $W$ is single-valued along a non-contractible path in the orbit. As observed above, the gauge variations of $W$ is the same as the gauge variation of the WZ functional. Therefore, up to an additive constant, these two functionals are the same, when restricted to a gauge orbit. We therefore ask whether $\Gamma_{WZ}$ is single-valued along a non-contractible path in the orbit. To answer this question one just has to integrate the anomaly along a closed loop, in which case in (1.64) we have to set $\hat{g}(1) = e$.

Now consider the four-dimensional case, with gauge group $SU(N)$, $N > 2$. Assuming that spacetime has been compactified to $S^4$, the gauge group is $\mathcal{G} = \Gamma_*(S^4, SU(N))$ and according to the lemma XXX, $\pi_1(\mathcal{G}) = \pi_5(S^4) = \mathbb{Z}$. The situation is the same as in the two-dimensional case.
1.5 The descent equations

The axial anomaly, the gauge anomaly and the Schwinger terms of gauge theories in different dimensions, are strictly related. In fact, we will see that one can obtain the solution of the WZ consistency condition, i.e. the consistent anomaly, including the correct normalization, by a series of purely geometrical operations, starting from the axial anomaly in a space of two more dimensions. We will define the cocycles \( \omega^k_r \), for \( k = 0, 1, 2 \), where \( r = 2n - k - 1 \) is the degree of \( \omega \) as a form (and hence also the dimension of the space over which \( \omega \) is to be integrated), and \( k \) is its degree as a form in the space of connections (more precisely, in an orbit of the gauge group in the space of connections). This means that, given \( k \) infinitesimal gauge transformation parameters \( \epsilon_1, \ldots, \epsilon_k \), and \( r \) vectorfields \( v_1, \ldots, v_r \), \( \omega^k_r(\epsilon_1, \ldots, \epsilon_k, v_1, \ldots, v_r) \) is a real number. The fact that these are cocycles means that they are closed, both as \( r \)-forms on space(time) and as \( k \)-forms on the orbit if the gauge group. The relations between all these forms constitute the so-called “descent equations”.

In order to minimize the index clutter it is convenient to use the algebra of differential forms. Hence we write \( A = A^a_{\mu} dx^\mu T_a \), \( F = \frac{1}{2} F^a_{\mu\nu} dx^\mu \wedge dx^\nu T_a \). The exterior derivative acting on a \( p \)-form \( \omega \) can be defined in a coordinate-independent way by specifying the result of acting with \( d\omega \) on \( p + 1 \) vector-fields:

\[
d\omega(v_1, \ldots, v_{p+1}) = \sum_{1<i<p+1} (-1)^{i+1} v_i (\omega(v_1, \ldots, \hat{v}_i, \ldots, v_{p+1})) + \sum_{1<i<j<p+1} (-1)^{i+j} \omega([v_i, v_j], v_1, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{p+1}) ,
\]

where a hat over a vector means that it is missing. If the components of \( \omega \) are defined by

\[
\omega = \frac{1}{p!} \omega_{\mu_1, \ldots, \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}
\]

we can also write

\[
d\omega = \frac{1}{p!} d\omega_{\mu_1, \ldots, \mu_p} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}
\]

where the differential acts only on the components. Thus we can write \( F = dA + A \wedge A \), and to further condense the notation also wedge products will not be written explicitly, so \( F = dA + A^2 \).
1.5. THE DESCENT EQUATIONS

One begins from the expression for the Chern character, a $2n$-form in $2n$ dimensions:

$$c_n = k_n \text{tr} F^n ,$$  \hspace{1cm} (1.74)

where $k_n$ is a normalization constant, depending on the group $G$, such that the integral of $c_n$ is an integer. It is gauge invariant

$$\delta_c c_n = 0$$  \hspace{1cm} (1.75)

and closed

$$dc_n = 0 .$$  \hspace{1cm} (1.76)

From Poincare’s lemma, one should be able to write $c_n$ at least locally as the exterior differential of an $2n-1$ form $\omega_0^{2n-1}$, called a Chern–Simons form

$$d\omega_0^{2n-1} = k_n \text{tr} F^n .$$ (1.77)

A general formula can be given in any dimension, but we limit ourselves to the cases $n = 2, 3, 4$, where we have

$$\omega_3^0(A) = k_2 \text{tr} \left( FA - \frac{1}{3} A^3 \right) ,$$  \hspace{1cm} (1.78)

$$\omega_5^0(A) = k_3 \text{tr} \left( F^2 A - \frac{1}{2} FA^3 + \frac{1}{10} A^5 \right) ,$$  \hspace{1cm} (1.79)

$$\omega_7^0(A) = k_4 \text{tr} \left( F^3 A - \frac{2}{5} F^2 A^3 - \frac{1}{5} F A F A^2 + \frac{1}{5} F A^5 - \frac{1}{35} A^7 \right) .$$ (1.80)

The gauge variations of the Chern–Simons forms defines the consistent anomaly $\omega_{p+1}^1$:

$$\delta_c \omega_{2n-1}^0(A) = d\omega_{2n-2}^1(A, \epsilon) .$$ (1.81)

It can be written in the form

$$\omega_{2n-2}^1(A, \epsilon) = \text{tr} d\epsilon \phi_{2n-3}(A) .$$ (1.82)

where the $2n-3$-form $\phi_{2n-3} = \phi_{2n-3}^a T_a$ is a polynomial in $A$ and $F$. For $n = 2, 3, 4$ this polynomial is given by

$$\phi_1 = -k_2 A ,$$  \hspace{1cm} (1.83)

$$\phi_3 = -\frac{k_3}{2} (FA + AF - A^3) ,$$  \hspace{1cm} (1.84)

$$\phi_5 = -\frac{k_4}{3} \left[ (F^2 A + FAF + AF^2) - \frac{4}{5} (A^3 F + FA^3) - \frac{2}{15} (A^2 FA + AFA^2) + \frac{3}{5} A^5 \right] .$$ (1.85)
CHAPTER 1. ANOMALIES

The coboundary of \( \omega_{2n-2}^1 \) defines \( \omega_{2n-3}^2 \):

\[
\delta_{\epsilon_1} \omega_{2n-2}^1(A, \epsilon_2) - \delta_{\epsilon_2} \omega_{2n-2}^1(A, \epsilon_1) - \omega_{2n-2}^1(A, [\epsilon_1, \epsilon_2]) = d\omega_{2n-3}^2(A, \epsilon_1, \epsilon_2) .
\]

(1.86)

For \( n = 2, 3, 4 \) it is given by

\[
\omega_1^2(A, \epsilon_1, \epsilon_2) = -2k_2 \text{tr} \epsilon_1 d\epsilon_2 ,
\]

(1.87)

\[
\omega_3^2(A, \epsilon_1, \epsilon_2) = -k_3 \text{tr} \{d\epsilon_1, d\epsilon_2\} A ,
\]

(1.88)

\[
\omega_5^2(A, \epsilon_1, \epsilon_2) = \frac{k_4}{15} \text{tr} (5F - 3A^2) [2A\{d\epsilon_1, d\epsilon_2\} - d\epsilon_1 Ad\epsilon_2 + d\epsilon_2 Ad\epsilon_1] .
\]

(1.89)

**Remark 1.** These 2-cocycles appear as “Schwinger terms” in the algebra of the gauge generators \( G_\epsilon = \int d^{2n-3}x \epsilon^a G_a \) for an anomalous gauge theory in \( 2n - 2 \) spacetime dimensions:

\[
[G_\epsilon_1, G_\epsilon_2] = G_{[\epsilon_1, \epsilon_2]} + \int d^{2n-3}x \omega^2(\epsilon_1, \epsilon_2) .
\]

(1.90)

In the case \( n = 2 \) they define a central extension of the gauge algebra (a Kac-Moody algebra). In higher dimension the extension is by a function of \( A \). The presence of the Schwinger term obstructs the definition of physical states as those states that are annihilated by the Gauss operator \( G_\epsilon|\psi_{\text{phys}}\rangle = 0 \). This is the manifestation of the anomaly at the canonical level.

**Remark 2.** It is clear from (1.81) and (1.86) that \( \omega_{2n-2}^1 \) and \( \omega_{2n-3}^2 \) are only defined up to a closed form. In particular one could add to \( \omega_{2n-2}^1 \) the closed form \( -d(\text{tr} \epsilon \phi(A)) \) and get

\[
\hat{\omega}_{2n-2}^1(A, \epsilon) = -\text{tr} \epsilon d\phi_{2n-3} ,
\]

(1.91)

which is another form of the consistent anomaly. Applying the coboundary to \( \hat{\omega}_{2n-2}^1 \) defines a different 2-cocycle \( \hat{\omega}_p^2 \):

\[
\delta_{\epsilon_1} \hat{\omega}_{2n-2}^1(A, \epsilon_2) - \delta_{\epsilon_2} \hat{\omega}_{2n-2}^1(A, \epsilon_1) - \hat{\omega}_{2n-2}^1(A, [\epsilon_1, \epsilon_2]) = d\hat{\omega}_{2n-3}^2(A, \epsilon_1, \epsilon_2) .
\]

(1.92)
1.5. THE DESCENT EQUATIONS

For \( n = 2, 3, 4 \)

\[
\hat{\omega}_2^2(A, \epsilon_1, \epsilon_2) = k_2 \text{tr} [\epsilon_1, \epsilon_2] A ,
\]

\[
\hat{\omega}_2^3(A, \epsilon_1, \epsilon_2) = \frac{1}{2} k_3 \text{tr} \left[ [\epsilon_1, \epsilon_2] (FA + AF - A^3) - \epsilon_1 dA \epsilon_2 A - \epsilon_1 A \epsilon_2 dA \right] ,
\]

\[
\hat{\omega}_2^4(A, \epsilon_1, \epsilon_2) = \frac{1}{3} k_4 \text{tr} \left\{ [\epsilon_1, \epsilon_2] \left[ (F^2 A + FAF + AF^2) - \frac{4}{5} \{ A^3, F \} - \frac{2}{5} \{ A, AFA \} + \frac{3}{5} A^5 \right] \right. \\
\left. - \frac{1}{5} [\epsilon_1, d\epsilon_2][F, A^2] - \frac{3}{5} (d\epsilon_1 A \epsilon_2 + \epsilon_2 A d\epsilon_1) (FA + AF - A^3) + \frac{1}{5} [\epsilon_2, d\epsilon_1][F, A^2] - \frac{3}{5} (d\epsilon_2 A \epsilon_1 + \epsilon_1 A d\epsilon_2) (FA + AF - A^3) \right\} (1.95)
\]

These are the cocycles one gets in the Gauss law algebra of an anomalous fermionic theory using the Bjorken-Johnson-Low procedure\(^5\) or in the gauged Wess-Zumino-Witten model at the canonical level.\(^6\) They differ from the cocycles \( \omega_{2n-3}^2 \) by a redefinition of the current. Note that even in two dimensions these cocycles do not define a central extension.

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