Notes on critical phenomena
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based in part on a forthcoming paper with A. Codello

Text in red was not covered in class
1 Critical Phenomena

1.1 Phase transitions

The macroscopic state of a pure substance is determined by volume \( V \), pressure \( P \) and temperature \( T \), of which only two are independent, the third being related to the others by the equation of state (see fig.1). Pure substances can be in three different states: gas, liquid or solid.\(^1\) These phases are separated by lines in the \( P-T \) plane, as shown in Fig.(2). The transition lines are characterized by a discontinuous jump in the density and when one crosses them, energy is either absorbed or released, without changing the temperature (latent heat).

It is also characteristic that at the transition point the material consists of spatially separated patches of the two phases. Thus, for example, when one crosses the line from the gas to the liquid, droplets form, until all gas has turned to liquid. In the other direction, bubbles form in the liquid until it has all evaporated. Unlike most points in the \( P-T \) diagram, that represent a unique state of the system, the points on the transition lines represent a one-parameter family of states, going from having only gas and no liquid to only liquid and no gas. This is seen better in the \( P-V \) diagram, Fig.(3), where the transition points of the \( P-T \) diagram corresponds to the horizontal segments of the isotherms (co-existence of liquid and vapor). As one moves along these segments, the internal energy of the system changes because latent heat is either absorbed or released.

Another characteristic feature of these transitions is that when the gas is cooled, it may remain in a gaseous state even when the temperature has decreased significantly below the transition temperature. In the absence of impurities (condensation nuclei) it may take a macroscopically long time for the material to change phase. This is called supercooling and represents a metastable state. The same happens when a very pure liquid is brought below the freezing point. (No such metastable state exists when melting a solid or evaporating a liquid).

All changes of phase occurring when one crosses a transition line have these general features and are said to be *discontinuous*, because the density has a discontinuity when one crosses the transition line. Equivalently one calls these phase transitions of *first order*.\(^2\) When one crosses the liquid-gas transition line at higher pressure and temperature, the discontinuity in the density decreases and the latent heat decreases. Eventually both become zero and the transition line ends at a point called the critical point. The critical state is characterized by the presence of fluctuations in density of all scales, making the fluid opaque (critical opalescence).\(^3\) For pressures exceeding the critical pressure, or temperatures exceeding the critical temperature, there is no qualitative difference between gas and liquid, in fact there is no phase transition to

\(^1\) actually, there may be several different solid phases, but we shall not be interested in them here.

\(^2\) This terminology was introduced by Ehrenfest, who defined a transition to be of \( n \)-th order if the lowest derivatives of the free energy that presents a discontinuity at the transition is the \( n \)-th. In the present case, the density is discontinuous, and the density is the (inverse of the) first derivative of the free energy with respect to pressure. A review of thermodynamic potentials is given in Appendix A.

\(^3\) see e.g. https://www.youtube.com/watch?v=GEr3NxsPt0A for CO\(_2\) traversing the critical point.
Fig. 1: Allowed states of a substance form a surface in the $P$-$V$-$T$ space. For large $T$ and $V$ the equation of state approximates that of a perfect gas.

Fig. 2: Phase diagrams of carbon dioxide (left) and water (right). Most substances expand when melting and contract when freezing. Water is anomalous in having the opposite behavior. This is reflected in the slope of the solid/liquid transition line.

Fig. 3: $P$-$V$ diagram with isothermal lines for a fluid.
Fig. 4: Magnetization as a function of the external field, for $T < T_c$. If the system comes from a previous state of positive magnetization, it will remain positively oriented even if a (small) negatively directed magnetic field is applied. This is called hysteresis and the state when $M$ and $H$ are oppositely oriented is metastable. Eventually the magnetization has a discontinuous jump. At thermodynamic equilibrium (i.e. if one waits for a sufficiently long time) this happens for infinitesimal $|H|$.

speak of. Exactly at the critical point, however, there is a transition that is qualitatively different from those further down the line. It is called a continuous phase transition, because the density has no discontinuity. However, the specific heat is discontinuous. Since the specific heat is a second derivative of the free energy, in the terminology introduced by Ehrenfest, this is a second order transition. Another notable feature is that the compressibility diverges. Altogether, in the $P-V$ plane, there is a line of first order transitions that ends at a second order transition.

Let us now consider a completely different physical system, namely a magnetic material with a single direction for magnetization. The state of the material depends on the temperature and on the external magnetic field $H$. One can measure the magnetization $M$ as a function of these external parameters. When $H = 0$ and the temperature is sufficiently low, there is a residual (or “spontaneous”) magnetization that depends upon the earlier history of the system (hysteresis). Thus for example if previously the magnetic field had been pointing up, then also the magnetization will point up. See Fig.(4). When one changes the sign of the external magnetic field, the magnetization also follows and has a discontinuous jump. Actually, also in this case the material may remain for some time in a metastable state with the direction of magnetization opposite to that of $H$.

As the temperature increases, the discontinuity in magnetization across the transition decreases, until the Curie temperature is reached, where the discontinuity vanishes, see Fig.(5). Above the Curie temperature there is no transition: the magnetization changes smoothly as a function of the magnetic field and in particular, $M = 0$ when $H = 0$. However, exactly at the critical temperature there is still a phase transition, but a continuous one. Once again, at the critical point the specific heat has a discontinuity and the magnetic susceptibility diverges. So, also in this case there is a line of first order transitions that end at a second order transition.
1 Critical Phenomena

Fig. 5: Spontaneous magnetization as a function of temperature. At a given $T$, the two possible states are energetically degenerate and which one the system is in depends on its earlier history.

1.2 Critical behavior

In spite of their very different nature, the phase transitions of the fluid and of the magnetic system exhibit many similarities, which can be highlighted by using a unified language for both cases. Above the critical temperature the system is said to be in the disordered phase, while below the critical temperature it is in an ordered phase. There are two ordered phases, that are characterized by different values of a quantity called the order parameter. For the ferromagnet the order parameter is the magnetization and for the fluid it is the density, or more precisely the difference of the density from the density at the transition. We note that for a fixed mass, the density is just the inverse of the volume, so in the $P$-$V$ plane one can also think of $V$ as the order parameter, with the discontinuity of $V$ going to zero at criticality. The variable driving the transition between the two ordered phases is called the external field, or “source”. For the ferromagnet it is the magnetic field and for the fluid it is the pressure, more precisely the difference of the pressure from the pressure at the transition. The derivative of the order parameter with respect to the source, at constant temperature and zero source, is called the isothermal compressibility in the case of the fluid and the magnetic susceptibility in the case of the magnetic system.

These correspondences are summarized in the following table.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Fluid</th>
<th>Ferromagnet</th>
</tr>
</thead>
<tbody>
<tr>
<td>order parameter</td>
<td>$\phi$</td>
<td>$V - V_i$</td>
</tr>
<tr>
<td>external field</td>
<td>$j$</td>
<td>$P - P_i$</td>
</tr>
<tr>
<td>isothermal compressibility or magnetic susceptibility</td>
<td>$\chi$</td>
<td>$-\frac{1}{V} \frac{\partial V}{\partial P}<em>{T=1} = \frac{1}{P} \frac{\partial P}{\partial T}</em>{T=1}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Phase Type</th>
<th>$T &lt; T_c$</th>
<th>$T &gt; T_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ordered phase 1</td>
<td>$\phi &gt; 0$</td>
<td>liquid</td>
</tr>
<tr>
<td>ordered phase 2</td>
<td>$\phi &lt; 0$</td>
<td>vapor</td>
</tr>
<tr>
<td>disordered phase</td>
<td>$\phi = 0$</td>
<td>supercritical fluid</td>
</tr>
</tbody>
</table>

Note that in both cases the order parameter is an extensive quantity and the external source is an intensive quantity, and in both cases the first law of thermodynamics contains a term of the form $jd\phi$ (see Table 1 in Appendix A). The analogy is further
strengthened by observing that in the phase diagram of the fluid, in the neighborhood of the critical point, the plot of the order parameter ($\Delta V$) as a function of temperature is very similar to Fig.(5).

Thus, the qualitative properties of the phase transitions of the two systems are quite similar. We now have to look also at quantitative properties. It turns out that the behavior of a material near a continuous phase transition is characterized by power laws. The exponents of these power laws are called critical exponents. It is convenient to express them as functions of the reduced temperature

$$t = \frac{T - T_c}{T_c}.$$ 

Note that $t > 0$ above the transition (disordered phase) and $t < 0$ below the transition (ordered phase). 4

The exponent $\alpha$ characterizes the behavior of the specific heat

$$C_V \sim |t|^{-\alpha}.$$ (1)

The exponent $\beta$ (which is only meaningful in the ordered phase) appears in the dependence of the order parameter on temperature

$$\phi \sim (-t)^\beta,$$ (2)

where $\phi$ is the density discontinuity, for water, and the spontaneous magnetization for the magnet, evaluated at the transition. The exponent $\gamma$ is defined by

$$\chi \sim |t|^{-\gamma}$$ (3)

where $\chi$ is either the isothermal compressibility or the magnetic susceptibility. The exponent $\delta$ relates the order parameter to the external control parameter $P$ (for the fluid) or $H$ (for the magnet), at $T = T_c$:

$$\Delta \rho \sim (P - P_c)^{1/\delta} \quad \text{or} \quad M \sim H^{1/\delta}.$$ (4)

Two more critical exponents are defined in terms of the two-point function. In water we consider the density-density correlator, in a magnet the spin-spin correlator. In any material under normal conditions the correlation function decays exponentially. The characteristic scale $\xi$, called the correlation length, is typically comparable to, or only slightly larger than, molecular distances. At first order phase transitions the correlation length is also generally finite. However, at a continuous transition the correlation length becomes infinite and the decay of the two-point function is only a power law:

$$G(r) \sim \frac{1}{r^{d-2+\eta}}.$$ (5)

---

4 In the literature a distinction is sometimes made between the exponents above and below the transition. We will not pay attention to this aspect in the following.
This relation defines the critical exponent $\eta$. (For $\eta = 0$ we have the standard Green’s function of the Laplacian in $d$ dimensions.) Finally, the behavior of the correlation length as a function of temperature, near the critical point, defines the exponent $\nu$:

$$\xi \sim |t|^{-\nu}.$$  

(6)

It is an experimental fact that the critical exponents of the liquid-vapor transition and of the uniaxial magnetic system are the same. Thus, in addition to the qualitative analogies pointed out earlier, we also have a striking quantitative correspondence. Given that the systems are so different, this is quite surprising.

Actually, this is just a special example of a much more general fact. There is a large number of materials, each exhibiting phase transitions. The microscopic dynamics of each material is different and it may seem impossible to find a common explanation for all their critical properties. However, critical phenomena have a striking feature: the critical exponents are not randomly distributed. Instead, many different continuous phase transitions have the same critical exponents, within experimental errors. This property is called universality and all the phase transitions having the same exponents are said to belong to the same universality class. Furthermore, it turns out that the universality classes are to a large extent characterized by just two properties: the dimension and the symmetries of the system. The microscopic Hamiltonian of a magnetic (spin) system, that we shall discuss in the next section, is even under the $\mathbb{Z}_2$ symmetry that consists of changing the sign of all the spins (and simultaneous changing the sign of the magnetic field). This manifests itself macroscopically in the symmetry that is seen in Fig.(5). This symmetry is less evident in the fluid system, but it is present and consists of changing the sign of $\Delta \rho$. Thus the universality class of the fluid and of the uniaxial magnet is also called the $\mathbb{Z}_2$ universality class. A three-dimensional magnetic system that is invariant under rotations has different critical exponents that characterize the $O(3)$ universality class, and so on.

These facts point to the existence of an underlying common explanation for many such phenomena, that must be independent of the microscopic details of the system. The oldest attempt to explain universality is the Landau-Ginzburg theory, to which we now turn.

### 1.3 The Ising model in the mean field approximation

#### 1.3.1 The Ising model

Given a microscopic model, one can sometimes obtain an approximate equation of state for the order parameter by using the so-called mean-field approximation. Let us illustrate this for the Ising model, which is a microscopic model for the uniaxial ferromagnet. It consists of spins pointing in a fixed direction and with values $S_i = \pm 1$ on a $d$-dimensional lattice, with Hamiltonian

$$\mathcal{H} = -J \sum_{i,j} S_i S_j - \mu H \sum_i S_i.$$  

(7)
where the first sum extends over all nearest neighbors. The magnetization is \( M = \langle \sum_i S_i \rangle \) and we define the magnetization per site \( m = M/N \), where \( N \) is the number of lattice sites. For \( J > 0 \) the nearest neighbor interaction favors the alignment of the spins. The state of lowest energy consists of all spins oriented in the same direction as the magnetic field. When the magnetic field is zero, there are two degenerate lowest energy states, with the spins aligned in either direction. Thermal fluctuations introduce some degree of disorder and for sufficiently high temperature destroy the order completely.

There is a version of the Ising model that can be applied to the liquid/gas transition. It has a lattice Hamiltonian

\[
H = -J \sum_{i,j} \rho_i \rho_j - \bar{\mu} \sum_i \rho_i \tag{8}
\]

where \( \rho_i = 0, 1 \) is the occupation number at site \( i \) (\( \rho_i = 0 \) means that the site \( i \) is empty, \( \rho_i = 1 \) that it is occupied by a molecule) and \( \bar{\mu} \) now stands for chemical potential. By making the change of variable \( \rho_i = (S_i + 1)/2 \), the Hamiltonian takes the form (8), with an “effective magnetic field”

\[
H = \frac{1}{\mu} \left( \frac{1}{2} \bar{\mu} + 2zJ \right).
\]

There will be a critical point describing a liquid-gas transition for \( \bar{\mu} = -4zJ \). Even though a “lattice gas” is a very crude approximation of a real fluid, thanks to universality it is enough to capture the main properties of the liquid/gas transition.

We continue the discussion in the language of the spin system.

1.3.2 Mean field approximation

Knowing that the material sometimes exhibits spontaneous magnetization, it is natural to write the spin as the sum of a mean field, which can be treated as constant on lattice scales, and a fluctuation field

\[ S_i = m + \delta S_i \]

When this is inserted in (8), the mean field approximation consists in neglecting the terms quadratic in the fluctuation.

\[
\mathcal{H}' = -J \sum_{i,j} (m^2 + 2m\delta S_i) - \mu H \sum_i S_i \\
= -J \sum_{i,j} (-m^2 + 2mS_i) - \mu H \sum_i S_i \tag{9}
\]

The double sum over \( S_i \) can now be written as a single sum times \( z \) (the number of nearest neighbors) times one half (to compensate double counting). Discarding a constant, we have

\[
\mathcal{H}' = \frac{1}{2} JzNm^2 - \mu H \sum_i S_i , \tag{10}
\]

\(^5\) See https://www.youtube.com/watch?v=wobhrl3I5a0 for numerical simulations.
Fig. 6: Graphical solution of (13). Plot of $\tanh \left( \frac{J_z}{T} m \right)$ for $J_z/T = 1/2$ (green), 1 (red) and 2 (blue).

where $N = \sum_i 1$ is the number of spins and

$$\tilde{H} = H + J_z m / \mu .$$

This is the Hamiltonian of $N$ decoupled spins interacting with an effective magnetic field $\tilde{H}$, which is the sum of the external field and a term representing the average magnetic field generated by all the other spins.

The partition function can now easily be calculated:

$$Z = e^{-\beta N z m^2 / 2} \left( 2 \cosh (\beta (\mu H + J_z m)) \right)^N .$$

and from here one gets the free energy density, or free energy per site

$$f = -\frac{\log Z}{\beta N} = \frac{J_z^2 m^2}{2} - \frac{1}{\beta} \log \left( 2 \cosh (\beta (\mu H + J_z m)) \right) .$$

Note that this expression holds for any type of lattice and in any dimension.

The magnetization is obtained by looking for the minimum of the free energy. This leads to the equation

$$m = \tanh \left( \frac{J_z}{T} m \right) .$$

This equation can be solved graphically, see Fig.(6). It has nonzero solutions for $m$ provided $T \leq J_z$. We thus identify $J_z$ with the critical temperature:

$$J_z = T_c .$$

We could go on and derive from here all the properties of the phase transition. For the sake of generality, however, it is more instructive to expand the free energy in $m$, $h$ and $T - T_c$, all of which are supposed to be small near the critical point. We obtain

$$f(m) = C - hm + \frac{1}{2} rm^2 + \frac{1}{3} sm^3 + \frac{1}{4} tm^4 + \ldots$$
where
\[ h = \frac{T_c}{T} \mu H; \quad r = T - T_c; \quad s = \left( \frac{T_c}{T} \right)^3 \mu H; \quad u = \frac{1}{3} \frac{T_c}{T^3}. \] (15)

This is the Landau free energy for a homogeneous system at equilibrium. The following features are crucial: the odd terms vanish when \( h = 0 \) and the term quadratic in \( m \) vanishes at the critical temperature.

### 1.4 Landau-Ginzburg theory

This was an instructive derivation because it allowed us to calculate thermodynamic properties of the material applying certain approximations to a microscopic model. In particular, it gives us relations between thermodynamic properties and the microscopic parameters \( J \) and \( z \).

However, we could have obtained the same general formula without any calculation, from the following general arguments. Knowing that the thermodynamic variables are continuous at the phase transition, the free energy must have a Taylor expansion around \( m = 0 \). For \( h = 0 \) and \( m = 0 \) the material is isotropic. In particular, it is invariant under the group \( \mathbb{Z}_2 \) of reflections along the axis of magnetization. Thus, for \( h = 0 \), the free energy must be an even function of \( m \) and the odd terms must be odd functions of \( h \). Finally, since the correlation length is related to the mass term in this expansion, and since we know that the correlation length goes to infinity at the critical point, \( r \) must be proportional to \( T - T_c \). This essentially fixes the free energy density to have the form (14), with \( r = r_0(T - T_c) \). Note that these arguments are not limited to an uniaxial magnet: the Landau-Ginzburg formula for the free energy holds also for many other transitions characterized by the same symmetry. In fact, the same formulae hold for any material in the same universality class of the Ising model.

So far we have restricted ourselves to the case when the order parameter is constant. If we take into account also long wavelength perturbations, the free energy will contain also a kinetic term. In this section we denote the order parameter \( \phi \) and the external field (or “source”) by \( j \). The free energy is
\[
F = \int d^d x \left[ \frac{1}{2} (\partial \phi)^2 + j \phi + \frac{1}{2} r \phi^2 + \frac{1}{4} u \phi^4 + \ldots \right].
\] (16)

It is assumed that \( r \) and \( u \) are analytic functions of temperature and that \( r \) has a zero at the critical temperature. To leading order we must have
\[
r = r_0(T - T_c) + \ldots, \quad u = u_0 + \ldots \quad (17)
\]

This form of the free energy is sufficient to explain all the qualitative properties of the system near the critical state, and to calculate the critical exponents.

The order parameter at equilibrium is determined by the condition of stationarity of \( F \):
\[
\phi(r + u \phi^2) + j = 0,
\] (18)

and we recall that \( u \) can be considered constant (independent of temperature) sufficiently near the critical point, while \( r \) is proportional to the reduced temperature. In
In order to have nonzero solutions for $\phi$ we must have $r < 0$. The number and nature of the solutions depends on the values of $j$, see Figure (7). For vanishing source one has the familiar symmetry-breaking potential with two degenerate minima and a maximum in the origin. For sufficiently small sources the two minima are no longer degenerate and the maximum moves away from $\phi = 0$. Eventually when the source becomes too strong, only one stable minimum remains.

In the $r$-$j$ plane, the region where there is more than one solutions is the cusp-shaped shaded region between the curves

$$j = \pm \left(\frac{-4r}{3}\right)^{3/2},$$

see figure (8).

For fixed $r < 0$, moving $j$ between sufficiently large positive and negative values will take the system through hysteresis cycles. These cycles become smaller and disappear when $r$ increases to zero. We see that the simple Lagrangian (16) successfully
Fig. 9: Solution of equation (18), as a function of the control parameters $r$ and $j$. Left: the metastable (green) and unstable (red) solutions. Right: the thermodynamic state space of stable minima of the free energy. All the lines $r = \text{const} < 0$ represent first order transitions, the line $r = 0$ the second order transition. The slope of these lines is the susceptibility/compressibility. It is finite at all points (including at the first order transitions) but infinite in the origin (the critical point).

describes the qualitative properties of the system near the critical point. The global structure of the space of solutions is shown in Figure (9).

Let us now calculate the critical exponents of the Ising universality class from (16). For $j = 0$, minimizing the free energy we find that nonzero solutions exist for $r < 0$ and are given by

$$\phi = \pm \sqrt{-\frac{r}{u}}.$$  \hfill (19)

Since $r$ is proportional to the reduced temperature, we see that $\beta = 1/2$.

In order to calculate the susceptibility, we differentiate (18) and obtain

$$\frac{\partial \phi}{\partial j} = -\frac{1}{r + 3u\phi^2}.$$  

At $j = 0$, $\phi$ is given by (19), so the susceptibility is $\frac{\partial \phi}{\partial j}|_{j=0} = 1/2r \sim t^{-1}$, therefore $\gamma = 1$.

At the critical temperature $r = 0$ and (18) gives $\phi \sim j^{1/3}$, which implies $\delta = 3$.

For the specific heat $c_V = \frac{1}{V} \frac{\partial U}{\partial T}|_{V}$, we first note that using (191) and (192) one has

$$U = -T^2 \frac{\partial}{\partial T} \left( \frac{1}{T} F \right)$$

and therefore

$$c_V = -T \frac{\partial^2 f}{\partial T^2},$$
where \( f = F/V \). Below the critical temperature, evaluating \( f \) at the solution (19) and with \( j = 0 \), we get \( f = -r^2/4u \sim (T - T_c)^2 \). Its second derivative with respect to \( T \) gives a constant and therefore \( \alpha = 0 \). Above the critical temperature, at equilibrium we have \( f = 0 \) independent of temperature, so again \( \alpha = 0 \).

In order to determine the remaining two exponents, we have to consider the two-point function for the Landau-Ginzburg theory (14). In coordinate space it behaves like

\[
G(\rho) \sim e^{-\rho/\xi} \frac{\rho^{d-2}}{\rho^{d-2}}
\]

At the critical temperature \( \xi \to \infty \), and the Green function has a power law behavior \( \rho^{-d+2} \), so \( \eta = 0 \). Furthermore, since \( \xi^2 \sim r^{-1} \sim t^{-1} \), the mass critical exponent is \( \nu = 1/2 \).

These results certainly explain universality. In fact, they turn out to be too universal: they are the same in all dimensions, which is certainly not true in practice. For example, it is known that the Ising model does not have a transition in \( d = 1 \), but mean field theory predicts one. In the following table we compare the mean field exponents of the Ising universality class with the ones obtained from the exact solution of the Ising model in \( d = 2 \) and from the conformal bootstrap in \( d = 3 \). One sees that the mean field approximation gets worse as the dimension decreases.

<table>
<thead>
<tr>
<th></th>
<th>Mean field</th>
<th>( d = 3 )</th>
<th>( d = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>0</td>
<td>0.11008(1)</td>
<td>0</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.5</td>
<td>0.326419(3)</td>
<td>1/8</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>1</td>
<td>1.237075(10)</td>
<td>7/4</td>
</tr>
<tr>
<td>( \delta )</td>
<td>3</td>
<td>4.78984(1)</td>
<td>15</td>
</tr>
<tr>
<td>( \nu )</td>
<td>0.5</td>
<td>0.629971(4)</td>
<td>1</td>
</tr>
<tr>
<td>( \eta )</td>
<td>0</td>
<td>0.036298(2)</td>
<td>1/4</td>
</tr>
</tbody>
</table>

### 1.5 The Ginzburg criterion

The basic assumption of mean field theory is that the field does not fluctuate too strongly. For self-consistency of this approximation, it must be the case that the expectation value of the squared fluctuation is much smaller that the square of the mean field when both are averaged over a volume \( V \) of linear size \( \xi \). We are going to assume that the mean field \( \bar{\phi} \) is constant over the distance scale of interest. Denoting \( \bar{\phi} = \langle \phi \rangle \), the averaged square fluctuation is

\[
\int d^d x \int d^d y \langle (\phi(x) - \bar{\phi})(\phi(y) - \bar{\phi}) \rangle \\
= \int d^d x \int d^d y \left( \langle (\phi(x)\phi(y)) \rangle - \bar{\phi}^2 \right) \\
= \int d^d x \int d^d y G_c(x,y) = V \int d^d x G_c(x)
\]

(20)

where \( G_c \) is the connected correlation function and in the last step we used translation invariance. The connected two-point function has an exponential decay with charac-
teristic scale $\xi$, so, neglecting numerical factors, by dimensional analysis the integral must behave like

$$V \int d^d x G_c(x) \sim V \int d^d r \frac{e^{-r/\xi}}{r^{d-2}} \sim V \xi^2 \sim \xi^{d+2}$$

Validity of the mean field approximation demands that this be much less than $\bar{\phi}^2 V^2 \sim \bar{\phi}^2 \xi^{d+2}$. Taking the ratio, we must have

$$\xi^{2-d} \ll \bar{\phi}^2.$$  \hspace{1cm} (21)

Now recalling that $\bar{\phi}^2 = -r/u$ and $\xi \sim (-r)^{1/2}$, we obtain

$$1 \gg u (-r)^{\frac{d-4}{2}} \sim (T_c - T)^{\frac{d-4}{2}}.$$  \hspace{1cm} (22)

For $T$ approaching $T_c$ from below, the r.h.s. goes to zero for $d > 4$ but inevitable blows up for $d < 4$. Thus we see that mean field theory can be a good approximation only in dimensions $d \geq 4$, which is called the *upper critical dimension* for this universality class. This fact explains the observation made earlier that the mean field becomes worse as the dimension decreases.

Below four dimensions mean field theory always fails when $T$ becomes sufficiently close to $T_c$. In practice there is in general a regime in which mean field theory is a reasonable approximation. How close to the transition this regime can get depends in large part on the range of the interactions. For short range interactions, such as in the Ising model, the mean field approximation fails at values of $t$ that are not very small and deviations from mean field behavior can be easily measured. For long range interactions, each degree of freedom feels the force of many other degrees of freedom and the mean field approximation is better. This is the case of ordinary superconductors, where the range is of the order of the size of the Cooper pair, typically 1000 lattice spacings. In this case the mean field approximation only breaks down when $t \approx 10^{-14}$, which is too small to be resolved experimentally.

### 1.6 The scaling hypothesis

Let us evaluate the Landau-Ginzburg functional (16) on a homogeneous field satisfying the equation (18):

$$f(t, h) = \frac{F}{V} \bigg|_{\phi = \bar{\phi}} = \begin{cases} -\frac{r^2}{4u} & \text{for } h = 0 \text{ and } t < 0; \\ -\frac{3}{4} \frac{h^{4/3}}{u^{1/3}} & \text{for } h \neq 0 \text{ and } t = 0. \end{cases}$$

Both relations can be explained by the single assumption that

$$f(t, h) = t^2 \Phi(h/t^{3/2}),$$

where the function $\phi$ has the asymptotic behavior

$$\lim_{x \to 0} \Phi(x) = -\frac{r^2 T_c^2}{4u}; \quad \lim_{x \to \infty} \Phi(x) = -\frac{3}{4} \frac{x^{4/3}}{u^{1/3}}.$$
One can now assume that going beyond mean field theory the free energy will still be a homogeneous function of the form:

\[ f(t, h) = |t|^{2-\alpha}\Phi\left(h|t|^{-\Delta}\right), \]  

where \( \alpha \) and \( \Delta \) are two exponents to be determined. Let us see what this hypothesis implies for the critical exponents.

Since \( \frac{\partial^2 f}{\partial t^2} \bigg|_{h=0} \sim |t|^{-\alpha} \), we see that \( \alpha \) can be identified with the exponent characterizing the behavior of the specific heat, introduced in (1). Since the spontaneous magnetization \( \frac{\partial f}{\partial h} \bigg|_{h=0} \) is proportional to \( |t|^{2-\alpha-\Delta} \), comparing with (2) we have

\[ \beta = 2 - \alpha - \Delta. \] (24)

Since the magnetic susceptibility \( \frac{\partial^2 f}{\partial h^2} \bigg|_{h=0} \) is proportional to \( |t|^{2-\alpha-2\Delta} \), we have

\[ \gamma = \alpha - 2 + 2\Delta. \] (25)

The magnetization is

\[ M = \frac{\partial f}{\partial h} = |t|^{2-\alpha-\Delta}\Phi^\prime\left(h|t|^{-\Delta}\right). \]

In order for \( M \) to have a finite limit for \( t \to 0 \), we must have \( \Phi^\prime(x) \sim x^{(2-\alpha-\Delta)/\Delta} \) for \( x \to \infty \). This implies that \( M \sim h^{(2-\alpha-\Delta)/\Delta} \), and comparing with (4) we find

\[ \delta = \frac{\Delta}{2 - \alpha - \Delta} = \frac{\Delta}{\beta}. \] (26)

We can eliminate \( \Delta \) from these relations to obtain two independent relations between the critical exponents

\[ \alpha + 2\beta + \gamma = 2; \]
\[ \beta + \gamma = \beta\delta. \] (27)

These identities were derived examining the behavior of the free energy as a function of a homogeneous field. There are two additional independent relations that derive from the properties of the correlation length. We assume that the correlation length is the most important length scale and determines the singularities of the thermodynamic functions. Then, observing that \( W = \log Z \) is dimensionless and extensive, it must behave like

\[ W(t, h) = \left(\frac{L}{\xi}\right)^d G(t, h) \]

where \( G \) is a dimensionless, regular function of its arguments. Since \( \beta \) is regular at the critical point, the singular behavior must be given by

\[ f(t, h) = -\frac{1}{\beta L^d} W(t, h) \sim \xi^{-d} \sim t^{d\nu}. \]
Comparing with (23) we obtain Josephson’s identity
\[ 2 - \alpha = d \nu . \] (28)

To derive the last relation we use (206) to write
\[ \chi \sim \int d^d x G_c(x) \]
and using the scaling behavior (5) we obtain
\[ \chi \sim \int d r r^{1-\eta} \sim \xi^{2-\eta} \sim t^{-\nu(2-\eta)} . \]
Comparing this with (3) we obtain
\[ \gamma = \nu(2 - \eta) . \] (29)

We now have four independent relations between six parameters, so all the critical exponents can be expressed as functions of just two of them, which are usually taken to be \( \nu \) and \( \eta \):

\[ \alpha = 2 - \nu d , \] (30)
\[ \beta = \nu \frac{d - 2 + \eta}{2} , \] (31)
\[ \gamma = \nu(2 - \eta) , \] (32)
\[ \delta = \frac{d + 2 - \eta}{d - 2 + \eta} . \] (33)

We note that the mean-field exponents satisfy these relations in the critical dimension \( d = 4 \).

2 The Renormalization Group

In this chapter we motivate the definition of Functional Renormalization (FR), derive a Functional Renormalization Group Equation (FRGE) and briefly discuss approximation schemes.

2.1 Wilson’s renormalization group

The renormalization group arose originally in perturbative QFT (Gell-Mann Low). A nonperturbative definition was given only later by Wilson. The functional renormalization techniques that we shall discuss here are an outgrowth of Wilson’s ideas. We shall therefore begin by discussing briefly the notion of Wilsonian renormalization group. Several crucial aspects of our approach can already be appreciated at this level and will be emphasized again in due course later.

In the generating functionals \( W \) and \( \Gamma \) all modes of the quantum field \( \phi \) are integrated out with a weight factor given by the exponential of the bare action. Wilson
proposed to calculate the functional integral by an iterative procedure where at each step the functional integral involves only modes with momenta contained in a given finite range. Suppose we start with a theory with a bare action $S$ at some UV cutoff $\Lambda_{UV}$. The modes of the field are classified by the wave vector or equivalently by the momentum $q^\mu$, and by definition $|q| < \Lambda_{UV}$. Consider the modes contained in the “momentum shell” $\Lambda_1 < |q| < \Lambda_{UV}$, with $\Lambda_1 = \Lambda_{UV}/b$ and $b > 1$. Denote $\phi_f$ the “fast” modes in the shell and $\phi_s$ the “slow” modes with $|q| < \Lambda_1$. The Wilsonian action at scale $\Lambda_1$, denoted $S_1[\phi_s]$, is defined by

$$e^{-S_1[\phi_s]} = \int (d\phi_f) e^{-S[\phi]} .$$

(34)

Whereas $S$ is generally chosen to be a simple local functional of the field, the action $S_1$ will contain all manner of complicated interactions induced by the quantum fluctuations. Nevertheless, by construction, it is such that if we integrate over the remaining modes $\phi_s$ with action $S_1$, the final result is the same partition function $Z$ we would have obtained by integrating over all modes with action $S$:

$$\int (d\phi_s) e^{-S_1[\phi_s]} = \int (d\phi_s) \int (d\phi_f) e^{-S[\phi]} = Z .$$

(35)

This action $S_1$ can be viewed as the new action valid for the slow modes $\phi_s$. The next step is the integration, weighted with the action $S_1$, over modes contained in the momentum shell $\Lambda_2 < |q| < \Lambda_1$, where $\Lambda_2 = \Lambda_1/b = \Lambda_{UV}/b^2$. This produces another Wilsonian action $S_2$ for the modes with $|q| < \Lambda_2$. The procedure is iterated until one reaches some infrared, macroscopic scale.

Iterating this procedure one obtains a sequence of actions $S, S_1, S_2 \ldots S_n$ that are valid at decreasing cutoffs $\Lambda_{UV}, \Lambda_1, \Lambda_2 \ldots \Lambda_n$ but all produce the same IR physics when the remaining modes are integrated out. This sequence of actions is the Wilsonian renormalization group. It is a “functional renormalization” in the sense that at each step one renormalizes the whole action functional and not just one or a few couplings.

For a numerical implementation a typical choice for the parameter $b$ would be 2. On a discrete square lattice, such transformations are usually referred to as Kadanoff blocking. However, by taking the number $b$ close to 1, one can make the difference between successive Wilsonian actions arbitrarily small. In this continuous formulation the Wilsonian action $S_\Lambda$ results from the functional integration over all the modes with $|q| > \Lambda$

$$e^{-S_\Lambda} = \int_{\Lambda < |q| < \Lambda_{UV}} e^{-S} .$$

(36)

and is such that its change with $\Lambda$ exactly compensates the change in the cutoff, so that the partition function is $\Lambda$–independent:

$$Z = \int_{0 < |q| < \Lambda} e^{-S_\Lambda} .$$

(37)

\(^6\text{Note that it would take infinitely many iterations to reach zero momentum. In practice any physical system has a finite size $L$ and there is no point to talk about modes with momenta below $1/L$.}\)
In this way one obtains a continuous family of actions $S_\Lambda$. The derivative
\[
\Lambda \frac{dS_\Lambda}{d\Lambda} = \lim_{b \to 1} \frac{S_i - S_{i+1}}{1 - 1/b}
\]
defines a “beta functional” describing the flow of the whole action. This flow can be described by relatively simple differential equations. The existence of these exact differential equations is what characterizes “functional renormalization”.

The final result of the Wilsonian procedure is the same as if we had integrated out all the modes at once, but this method has some advantages over the traditional approach. The following points are worth noting.

1) Each step of the renormalization involves the integration over a finite range of momenta and is therefore finite. The differences between the action $S_{i+1}$ and $S_i$ are finite. In particular, in the limit of continuous renormalization, the beta functional is finite, and so are the beta functions of all the couplings it contains.

2) In quantum field theory one often encounters the renormalization group as a consequence of the perturbative renormalization procedure. The Wilsonian way of calculating the functional integral one momentum shell at a time provides a non-perturbative definition of renormalization transformations and renormalization group. Thus one may hope to be able to obtain results that go beyond perturbation theory. The Wilsonian definition is also conceptually simpler and more direct than the perturbative one.

3) Imagine evaluating the difference $S_i - S_{i-1}$ in a loop expansion. It will consist of a sum of Feynman diagrams where the integration over the loop momenta is restricted to a momentum shell of thickness $\delta q = (1 - 1/b)q$. Thus, each loop will contribute some power of $\delta q/q = 1 - 1/b$. In the limit of continuous RG transformations $a \to 1$, higher loops become negligible (see [?] for a proof). Thus the RG equation for the Wilson action must have the form of a one-loop equation. The price one pays for this simplicity is that one has to follow the flow of infinitely many couplings at the same time.

Note that in equation (37) $\Lambda$ acts as the UV cutoff of the modes that remain to be integrated out (see fig. (?), left), but from formula (36) it can also be seen as the IR cutoff of the modes that have already been integrated out (see fig. (?), right). So if we use (36) as the definition of $S_\Lambda$ we can interpret the beta functional alternatively as the derivative of the Wilsonian action with respect to an IR cutoff. Since the functional integral over all the modes is insensitive to the artificial intermediate sliding cutoff, it is clear that the two points of view give the same flow.

The original definition based on (37) has the advantage that the functional integral that one talks about is regulated by the cutoff $\Lambda$ itself. In the definition based on (36) one has to introduce an additional cutoff $\Lambda_{UV}$ to make the integral well defined. In this way the Wilsonian action would seem to depend on two cutoffs. In practice, as we shall see, this is not a severe problem. In the following we will not be be interested in calculating the action itself but rather its beta functional $\Lambda \frac{dS_\Lambda}{d\Lambda}$. As already observed, any UV divergence that may exist in the definition of $S_\Lambda$ cancels when we take the derivative, so it is not necessary to specify $\Lambda_{UV}$. We shall see that using an IR cutoff has other advantages, so in the following we adopt this alternative point of view.
For the sake of simplicity, in the preceding discussion we have omitted to mention two important points. First, in Wilson’s definition of the RG, after each integration over a momentum shell one rescales physical momenta by a factor $1/(1 - b)$ and physical lengths by a factor $1 - b$. Second, one assumes that in the initial action $S$ the kinetic term is canonically normalized. Integration over a momentum shell will generally change the coefficient of the kinetic term. At each step one should therefore rescale the field $\phi$ by a suitable factor so that the kinetic term remains canonically normalized.

These two additional operations change the beta functional by simple calculable terms. In the following we shall find it convenient not to include these steps in the definition of the FRGE. Instead, they can be taken care of separately afterwards. The FRGE will therefore only contain the effect of the functional integration.

### 2.2 The RG equation for the effective action

In this section we will derive a FRGE that is close in spirit to Wilson’s approach, but with a few important differences. First, we will not introduce in the functional integral a UV cutoff but rather an IR cutoff. Second, the cutoff will be smooth, rather than sharp. Third, we are interested in a cutoff–dependent effective action, so the definition will involve a Legendre transform. And fourth, we will not include the rescaling of momenta and fields in the FRGE. We will now give all details of how to define formally an effective action depending on an IR cutoff, which, following an established tradition, we shall call $k$. In the next section we will discuss the properties of the smooth cutoff and in the subsequent one we derive the remarkably simple formula for the beta functional.

#### 2.2.1 Smooth cutoffs

Instead of sharply cutting off the integral over the slow modes as in (34), we will introduce a suppression factor for their contribution to the functional integral. When the suppression becomes infinitely strong, the effect is the same as not integrating over such modes at all. We will therefore modify the definition of the functional integral (193) introducing a smooth infrared cutoff, with a cutoff scale $k$. This is achieved by adding to the action a “cutoff” or “regulator” term $\Delta S_k[\phi]$ which in momentum space has the form $^7$

$$\Delta S_k[\phi] = \frac{1}{2} \int_q \phi(-q) R_k(q^2) \phi(q).$$

(38)

The function $R_k(q^2)$ is required to satisfy certain conditions:

1. for fixed $k$ it is a monotonically decreasing function of $q^2$;
2. for fixed $q^2$ its is monotonically increasing function of $k$;
3. $\lim_{k \to 0} R_k(q^2) = 0$;

$^7$ We recall the notation $\int_q \equiv \int \frac{d^d q}{(2\pi)^d}$.
4. for $q^2 > k^2$ it goes to zero sufficiently fast, typically as an exponential;

5. $R_k(0) = k^2$.

The first two conditions are obvious properties of a cutoff. The third guarantees that the EAA at $k = 0$ coincides with the standard EA: $\Gamma_0 = \Gamma$. The fourth condition ensures that high momentum modes are integrated out unsuppressed. We will discuss later the meaning of “sufficiently fast”. For now suffice it to say that this property guarantees the finiteness of the flow (i.e. of the beta functions). The fifth condition is a kind of normalization. As we shall see later, it ensures that the beta functions of dimensionless couplings have the familiar universal form.

Aside from these requirements, there is a large arbitrariness in the form of the cutoff. A cutoff that satisfies all these properties will be called a standard cutoff. For certain limited purposes one may sometimes forgo some of these conditions, in which case the cutoff will be said to be “nonstandard”. For dimensional reasons the cutoff can be written in the form

$$R_k(q^2) = k^2 r(y), \quad y = q^2 / k^2,$$

where $r$ is a dimensionless function of the dimensionless variable $y$.

Examples of standard cutoffs are

$$r(y) = \frac{y}{e^y - 1}; \quad r(y) = \frac{y^2}{e^{y^2} - 1}; \quad r(y) = (1 - y) \theta(1 - y).$$

The most popular choice is the “optimized” cutoff (42) [?] It is not analytic, and this may cause issues in certain cases, but it has other important advantages. Examples of nonstandard cutoffs are

$$r(y) = \frac{y}{e^{y^2} - 1}; \quad r(y) = y^{-\alpha}, \quad \alpha > 1 \quad \text{(44)}$$

The power–law cutoff (44), which violates conditions 4 and 5, has been used for example in [?]. One has to choose the exponent to be sufficiently large to guarantee convergence of the beta functions that one is interested in. The simplest possible IR cutoff is the mass cutoff (45), which violates condition 4 and does not guarantee finiteness of the flow. In Figures 10 and 11 we show plots of the standard and non-standard cutoff shape functions listed above.

It is important that the cutoff action $\Delta S_k$ is purely quadratic in the field, so that the vertices are not modified. The only effect of $\Delta S_k$ is to replace the free massless inverse propagator $q^2$ by the cutoff inverse propagator

$$P_k(q^2) = q^2 + R_k(q^2)$$

(46)
Fig. 10: Left: The standard cutoff profiles (40) (red), (41) (green) and (42) (black), in terms of the dimensionless function $r(y)$. Right: the corresponding propagators.

Fig. 11: Left: The nonstandard cutoff profiles (43) (red), (44) (green) and (45) (black), in terms of the dimensionless function $r(y)$. Right: the corresponding propagators.
and the massless propagator $q^{-2}$ by the cutoff propagator

$$G_k(q^2) = \frac{1}{q^2 + R_k(q^2)}.$$  \hfill (47)

Also note again that this is an infrared cutoff: its effect is to give a mass of order $k$ to the modes with $|q| < k$, and no mass to the modes with $|q| > k$, so it introduces a mass gap in the excitation spectrum of the field. However, curing IR divergences is not its primary purpose: rather, it is a way of introducing explicitly a $k$ dependence in the functional integral. We also notice that when $k \to 0$, $\Delta S_k \to 0$ and therefore $Z_k$ reduces to the ordinary effective action $Z$, where all fluctuations have been integrated out unsuppressed.

### 2.2.2 The Effective Average Action

Introducing the cutoff action (38) in the functional integral we obtain the scale dependent generalization of (193)

$$Z_k[J] = e^{W_k[J]} = \int (d\phi) e^{-S[\phi] + \Delta S_k[\phi] + \int J\phi}.$$ \hfill (48)

We then define the Legendre transform of $W_k$ by

$$\tilde{\Gamma}_k[\varphi] = -W_k[J_\varphi] + \int J_\varphi \varphi,$$ \hfill (49)

where\footnote{Explain if $\varphi$ depends on $k$.}

$$\varphi_J(x) = \frac{\delta W_k}{\delta J(x)}$$ \hfill (50)

and $J_\varphi$ in the r.h.s. of (49) is to be regarded as a function of $\varphi$, obtained by inverting (50). Finally we define the Effective Average Action (EAA) $\Gamma_k$ by subtracting from $\tilde{\Gamma}_k$ the suppression factor that was introduced in the beginning

$$\Gamma_k[\varphi] = \tilde{\Gamma}_k[\varphi] - \Delta S_k[\varphi].$$ \hfill (51)

Following the same steps that led to (202) we can derive an integro-differential equation for the EAA

$$e^{\Gamma_k[\varphi]} = \int (d\chi) e^{-S[\varphi + \chi] + \Delta S_k[\varphi + \chi] + \int \left( \frac{\delta S_k[\varphi]}{\delta \varphi} + \frac{\delta \Delta S_k[\varphi]}{\delta \varphi} \right) \chi + \Delta S_k[\varphi]}.$$ \hfill (52)

Equation (52) has to be considered together with the condition of vanishing vacuum expectation value of the fluctuation field $\langle \chi \rangle = 0$. Using the fact that the cutoff action is quadratic in its argument we can rearrange the cutoff terms in (52)

$$-\Delta S_k[\varphi + \chi] + \Delta S_k[\varphi] + \int \frac{\delta \Delta S_k[\varphi]}{\delta \varphi} \chi = -\Delta S_k[\chi],$$ \hfill (53)
to obtain
\[ e^{-\Gamma_k[\varphi]} = \int (d\chi) \exp \left( -S[\varphi + \chi] - \Delta S_k[\chi] + \int \frac{\delta \Gamma_k[\varphi]}{\delta \varphi} \delta \varphi \right). \] (54)

Relation (54) is the integro-differential equation that is satisfied by the EAA. Note that we can also use (54) as the starting point to define the EAA in place of (51). We see that (54) is just (202) with the replacement \((d\chi) \to (d\chi) e^{-\Delta S_k[\chi]}\) as expected.

### 2.2.3 The Wetterich equation

The major virtue of the EAA is that it is possible to write down an exact equation describing how it varies when the cutoff scale \(k\) is changed. We will now describe three routes to this fundamental result.

**Motivation:** One-loop RG improvement. It is instructive to begin from the one-loop approximation to the EAA. Recall that for a theory with bare action \(S\), the one-loop EA is given by
\[ \Gamma^{(1)}[\varphi] = S[\varphi] + \frac{1}{2} \text{Tr} \log \left( \frac{\delta^2 S[\varphi]}{\delta \varphi \delta \varphi} \right). \] (55)

In order to obtain the one-loop approximation to the EAA (51), we only have to replace \(S\) by \(S + \Delta S_k\), and subtract \(\Delta S_k\), to obtain:
\[ \Gamma^{(1)}_k[\varphi] = S[\varphi] + \frac{1}{2} \text{Tr} \log \left( \frac{\delta^2 S[\varphi]}{\delta \varphi \delta \varphi} + R_k \right). \] (56)

Note that the explicit appearance of the cutoff action \(\Delta S_k\), has been canceled by the second term in (51), so that the cutoff only appears through the Hessian. This provides some rationale for the definition (51). If we differentiate (56) with respect to \(k\), we are led to
\[ k \frac{d\Gamma^{(1)}_k[\varphi]}{dk} = \frac{1}{2} \text{Tr} \left( \frac{\delta^2 S[\varphi]}{\delta \varphi \delta \varphi} + R_k \right)^{-1} k \frac{dR_k}{dk}. \] (57)

This is the one-loop flow equation that the EAA satisfies. In the r.h.s. of (57) there appears the bare cutoff inverse propagator. The “RG improvement” of this equation, namely the equation obtained by replacing \(S\) by \(\Gamma_k\) in the r.h.s., is close in spirit to Wilson’s idea of integrating out modes “one momentum shell at the time” and updating the action at each iteration. One may guess that in this way one obtains a more accurate description of the physics. We shall now show that this RG improved one-loop equation is actually exact.

**Derivation I: Scale derivative of \(W_k\).** Let us begin by deriving the functional \(W_k\)
\[ k \frac{dW_k}{dk} = - \left\langle k \frac{d}{dk} \Delta S_k \right\rangle = - \frac{1}{2} \text{Tr} \langle \varphi \varphi \rangle k \frac{dR_k}{dk}, \]
where the trace is an integration over coordinate and momentum space. (For other fields it would imply also a finite-dimensional trace over internal or spacetime indices.)
Note that $\phi$ does not depend on $k$. Then using (194)
\[
\frac{d\Gamma_k}{dk} = -k \frac{dW_k}{dk} - k \frac{d\Delta S_k}{dk} \\
= \frac{1}{2} \text{Tr}(\langle \phi\phi \rangle - \phi\phi) \frac{dR_k}{dk} \\
= \frac{1}{2} \text{Tr} \frac{\delta^2 W_k}{\delta \phi \delta \phi} \frac{dR_k}{dk}
\]
(58)

The $k$-dependent functional $W_k$ and its Legendre transform satisfy the analog of (204)
\[
\frac{\delta^2 W_k}{\delta \Gamma_k \delta \phi \delta \phi} = (\frac{\delta^2 \Gamma_k}{\delta \phi \delta \phi})^{-1}
\]
(59)

Using this identity and (51,38) we arrive at the FRGE
\[
\frac{d\Gamma_k}{dk} = \frac{1}{2} \text{Tr} \left( \frac{\delta^2 \Gamma_k}{\delta \phi \delta \phi} + R_k \right)^{-1} \frac{dR_k}{dk}
\]
(60)

This is also called the Exact RG Equation, or the Wetterich equation. This derivation was given in [?] and independently in [?].

**Derivation II: Scale derivative of $\Gamma_k$.** This equation is central to all developments discussed in the following, so it will be worth giving also an alternative derivation. Differentiating the integro-differential equation (54) with respect to $t$ we obtain
\[
e^{-\Gamma_k[\phi]} \frac{d\Gamma_k[\phi]}{dt} = \int (d\chi) \left( k \frac{d\Delta S_k[\chi]}{dk} - \int k \frac{d\delta \Gamma_k[\phi]}{d\phi} \chi \right) e^{-S[\phi+\chi]-\Delta S_k[\chi]+\int \frac{\delta \Gamma_k[\phi]}{\delta \phi} \chi}.
\]
(61)

Expressing the terms on the r.h.s. as expectation values, using the fact that the field $\chi$ has vanishing vacuum expectation value and using equation (38), we can rewrite this as
\[
\frac{d\Gamma_k[\phi]}{dk} = \left\langle k \frac{d\Delta S_k[\chi]}{dk} \right\rangle = \frac{1}{2} \text{Tr} \left\langle \chi\chi \right\rangle \frac{dR_k}{dk}.
\]
(62)

Using again $\langle \chi \rangle = 0$, the two-point function in (62) can be considered to be the connected one. We are thus back at equation (58), from which equation (60) follows as before.

**Remarks.** The Wetterich equation (60) is the main technical tool employed within the functional RG approach to quantum field theory. It deserves several comments.

It is customary to rewrite the Wetterich equation in a slightly more condensed way. We define the RG parameter $t = \log(k/k_0)$, then the derivatives are usually written
\[
\frac{d}{dk} = \frac{d}{dt} \equiv \partial_t.
\]

Furthermore, the second functional derivative of the EAA with respect to the field is usually denoted $\Gamma_k^{(2)}$. Then the equation can be written
\[
\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \frac{\partial_t R_k}{\Gamma_k^{(2)} + R_k}.
\]
(63)
Note that the numerator and denominator have to be thought of as infinite dimensional matrices. This notation is ambiguous as to the order of the factors, but this does not matter due to the cyclic property of the trace. We also note that if \( t \) is treated as the independent variable, it would be more appropriate to write \( \Gamma_t \) instead of \( \Gamma_k \). The notation \( \partial_t \Gamma \) is universally adopted in the literature and we shall stick to it.

Let us note again that the Wetterich equation is identical to the one loop equation (57) except for the replacement of the bare action by the EAA in the r.h.s.. We had already mentioned in the end of section 1.2 that the one loop RG becomes exact in the continuum limit of the Wilsonian RG. Now we see this fact very explicitly in the flow of the EAA.

The RG flow is driven by some nontrivial interactions. If these interactions come from a fixed classical action \( S \), the equation is only approximate, but if they correspond to the running couplings of the full EAA the equation is exact. This simple one loop structure can be traced to the fact that the cutoff action (38) is quadratic in the field. If it were not, we would have found higher order vertices of the EAA on the rhs of (60) and this would spoil the one-loop structure of the flow equation.

The derivation of the FRGE given above is formal: all the functionals involved were defined by functional integrals and we did not specify an UV regularization. The resulting flow equation, however, is perfectly well defined. Due to the fall-off property of the cutoff (property 4 in section 1.2.1), the term \( \partial_t R_k \) under the trace on the r.h.s. of (60) goes to zero fast enough, when its argument is greater than \( k^2 \), to make the trace UV-convergent. With reference to figure 1.1 (right) one can understand this intuitively by realizing that in the difference of two functionals \( \Gamma_k \) and \( \Gamma_k - dk \), any UV divergent parts cancel exactly. The presence of the cutoff kernel \( R_k(z) \), behaving like a mass term in the propagator for \( z \to 0 \), also makes the result IR convergent for \( k \neq 0 \), even in the presence of massless particles.

Having obtained a finite equation for the derivative of the EAA, we can now reverse the original logic and use the FRGE to define the EAA. Namely, given any initial “bare” EAA at a scale \( k \), we define the EAA at any other scale by integrating the FRGE. In particular, by taking the limit \( k \to 0 \) one can calculate the EA of the theory. On the other hand, by taking the limit \( k \to \infty \) one can explore the UV behavior of the theory. We will discuss these aspects in more detail in the next section.

We also note that the FRGE does not contain any reference to a classical action: it is written entirely in terms of \( \Gamma_k \). If we think of the classical action as the “bare” action at some UV scale, this means that the equation contains no reference to UV physics: its is written entirely in terms of quantities at the scale \( k \). In fact, the classical action that enters in the functional integral is a rather unphysical notion. All the observables in a QFT contain the effect of quantum fluctuations and it is somehow artificial to try and split a quantity into a classical and quantum part\(^9\).

Although this was not used in the derivation, we observe that the effective action \( \Gamma = \lim_{k \to 0} \Gamma_k \) is obviously independent of \( k \). This parallels the fact that in the Wilso-

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\(^9\) Such a procedure is only justified in a perturbative context when one can make a good guess of an action that describes a particular phenomenon. The bulk of the effect is then captured by this “classical” part and the remaining “quantum” effects are only small corrections.
nian definition of the RG considered in section 2.1 the partition function is independent of the cutoff.

Finally we emphasize two properties of the FRGE: its generality and its exactness. It is general because it applies to any QFT, independent of its renormalizability properties. We shall see in the next section how UV complete theories and effective field theories differ in this context. Of course one cannot go far with exact results and one has to make approximations. We shall see in section 1.5 that the FRGE lends itself to a great variety of approximations.

2.3 Approximation schemes

The FRGE is an exact statement that holds in any QFT. One can imagine two ways of using it: as a tool to prove an exact property of a theory, or as a tool to calculate some measurable quantity. We will have little to say about the first use and concentrate here on the second. It is generally impossible to calculate anything exactly in the QFT’s that are used to model the real world. One of the positive aspects of working with the FRGE is that it lends itself to a large number of approximate treatments. We will discuss here some of the main ideas.

2.3.1 Truncations

Let us begin with the hardest possible case, when there is no obvious small parameter that can serve as a basis for an expansion. In such cases, faced with infinitely many equations for infinitely many couplings, the first thing one can try to do is to solve only a finite subset of equations, putting all the other couplings to zero. We call such a procedure a “truncation”. Thus one considers an EAA of the form:

\[ \Gamma_k[\phi] = \sum_{i=1}^{N_{tr}} g_i(k) \mathcal{O}_i[\phi], \]

where \( N_{tr} \) is the number of terms that one considers. There is in general little \textit{a priori} justification for such calculations. The flow equations are generally all coupled. If a coupling \( g \) is set to zero at a particular scale, its beta function will generally not be zero and so \( g \) will be turned on as soon as one starts integrating the flow. By neglecting it at all scales, one may be making an important error. On the other hand it is possible that in certain cases, for reasons that may not be apparent due to the complicated form of the equations, the flow of a subset of couplings that is physically important is relatively insensitive to all the others. In such cases a truncation may be useful.

Aside from successfully describing experimental facts, which is a rare occurrence, the quality of a truncation can only be judged \textit{a posteriori} when better approximations are available. Sometimes one is able to deal with a finite number of couplings, ordered in some natural way. One can then calculate a physical quantity keeping the first \( N_{tr} \) couplings and plot the result as a function of \( N_{tr} \). If the result is insensitive to \( N_{tr} \) or seems to approach a limit as \( N_{tr} \) increases, one has some reason to think that the truncation captures some significant aspect of the system.
There is another criterion that can be used to measure the quality of a truncation: a physically measurable quantity must be independent of the shape of the cutoff kernel $R_k$. To test the cutoff-dependence of a truncation, one can calculate the quantity with a one-parameter family of cutoffs, for example

$$r(y) = \frac{ay}{e^{ay} - 1} \quad \text{or} \quad r(y) = \frac{y^b}{e^{y^b} - 1},$$

which generalize (40) and (41). One then plots the result for a range of values of the parameters. If the result of the calculation changes significantly (by factors of order one) when the parameter changes by a factor of order one, the truncation is not trustworthy. On the other hand if the resulting curves are relatively flat for a large range of values of the parameter, one has reason to believe that the truncation captures some truth. This argument can also be turned into a “principle of minimum sensitivity”, saying that the best value of the parameter is a stationary point for the parameter dependence.

In practice when one uses a truncation one almost always keeps the terms of lowest dimension. These are the terms that one would keep in a perturbative effective field theory treatment. There is therefore an implicit hope that the ordering provided by the expansion around a free theory is not too wrong and selects the terms that are physically most significant.

2.3.2 Vertex expansion

This is widely used in particle physics, where one usually deals with weak fields. The vertex expansion consists in expanding the action in powers of the field

$$\Gamma_k[\phi] = \sum_n \int_{x_1} \ldots \int_{x_n} \Gamma_k(x_1 \ldots x_n) \phi(x_1) \ldots \phi(x_n),$$

where $\int_x$ stands for $\int dx$, or in momentum space

$$\Gamma_k[\phi] = \sum_n \int_{p_1} \ldots \int_{p_n} \Gamma_k(p_1 \ldots p_n) \tilde{\phi}(p_1) \ldots \tilde{\phi}(p_n),$$

where $\tilde{\phi}$ is the Fourier transform of $\phi$, $\int_p$ stands for $\int \frac{d^4p}{(2\pi)^4}$ and $\Gamma_k(p_1 \ldots p_n)$ contains the usual delta function $\delta(p_1 + \ldots + p_n)$ due to translation invariance.

The FRGE amounts to an infinite tower of coupled differential equations for the $n$-point correlation functions $\Gamma_n$.

2.3.3 Derivative expansion

When one is interested in low energy phenomena, one can expand the action in powers of derivatives. For a single scalar field this means

$$\Gamma_k[\phi] = \int dx \left( V_k(\phi) + \frac{1}{2} Z_k(\phi) (\partial \phi)^2 + O(\partial^4) \right)$$
where $V_k$ and $Z_k$ are arbitrary functions of the field.

We shall see that the FRGE can be used to derive partial differential equations (in the variables $\phi$ and $k$) for the flow of the functions $V_k$ and $Z_k$. Expanding $V_k$ and $Z_k$ in Taylor series these reduce to ordinary differential equations (in the variable $k$) for the Taylor coefficients. These are just the standard beta functions. Thus, in the derivative expansion one can keep track of infinitely many couplings in one stroke.

We observe that the derivative expansion is complementary to the vertex expansion: In the former one expands in momenta keeping the full field-dependence whereas in the latter one expands in the field keeping the full momentum-dependence.

### 2.4 Theory space and beta functions

We begin by defining the theory space to be the space of all functionals of $\phi$. We will not attempt to give a precise definition of this functional space. For many practical purposes one can consider semi-local functionals\(^{10}\) of the form

$$\Gamma_k[\phi] = \sum_i g_i(k) O_i[\phi]$$  \hspace{1cm} (69)

where $O_i[\phi]$ are integrals of monomials in the field and its derivatives, and $g_i(k)$ are running couplings. Taking the derivative of (69) we have

$$\frac{d\Gamma_k}{dt} = \sum_i \frac{dg_i}{dt} O_i[\phi]$$  \hspace{1cm} (70)

while expanding the trace on the rhs of the Wetterich equation in the same basis defines the (dimensionful) beta function

$$\frac{1}{2} \text{Tr} \frac{\partial_t R_k}{\Gamma_k^{(2)} + R_k} = \sum_i \beta_i O_i$$  \hspace{1cm} (71)

thus the flow projected on the basis $O_i$ is given be the set of coupled non-linear ODEs

$$\frac{dg_i}{dt} = \beta_i(g_j, k).$$  \hspace{1cm} (72)

One can compare equations (71) and (60). If we are able to expand the trace in the r.h.s. of (60) on the basis of operators $O_i$, then the coefficient of the operator $O_i$ is the beta function of $g_i$. In this way, with some work, one can extract explicit beta functions from the abstract form of the Wetterich equation. We shall see several examples of this procedure in the sequel.

Now we have to address the issue of the rescalings of momenta and fields, which was mentioned in the end of the preceding section. The first of these, the rescaling of

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\(^{10}\) We call “local” a functional containing at most finitely many derivatives and “semi-local” one containing infinitely many derivatives. For example $\phi \frac{1}{\Box + m^2} \phi$ is semi-local, since the fraction can be expanded as a geometric series containing only positive powers of $\Box$. An example of a “non-local” functional is $\phi \frac{1}{\Box} \phi$. Non–local operators can arise in the IR as the $k \to 0$ limit of semi–local operators.
physical lengths and momenta (and more generally of all dimensionful quantities, each with a power of \( \delta k / k \) equal to its mass dimension) is solved very simply by working with dimensionless variables. If the coupling \( g_i \) has mass dimension \( d_i \), we define the dimensionless parameter
\[
\tilde{g}^i = k^{-d_i} g_i.
\]
(73)

It is just the coupling \( g_i \) measured in units of the cutoff \( k \). From now on we are going to take the \( \tilde{g}^i \) as coordinates of theory space. Note that the index \( i \) in \( \tilde{g}^i \) has been written as a superscript rather than as a subscript. This complies with the standard usage for coordinates in differential geometry, and will be of some relevance in the rest of this chapter.

These variables avoid the need to perform explicitly a rescaling of physical lengths and momenta, because they would not be affected by such a rescaling. However, their beta functions are different from the ones of the dimensionful couplings \( g_i \). From dimensional analysis we can write \( \beta_i (g_j, k) = k^{d_i} \alpha_i (\tilde{g}^j) \) where \( \alpha_i (\tilde{g}^j) = \beta_i (g_j k^{-d_j}, 1) \). The beta functions of the dimensionless variables are given by
\[
\tilde{\beta}_i (\tilde{g}^j) = -d_i \tilde{g}^i + \alpha_i (\tilde{g}^j),
\]
(74)
where the first term contains the classical scaling and the second contains the contribution of quantum fluctuations. Note that \( \alpha_i \) can only depend on \( k \) (or \( t \)) implicitly via its arguments. This is an additional advantage of working with dimensionless variables. The RG flow equation in theory space is then
\[
\frac{d\tilde{g}^i}{dt} = \tilde{\beta}_i (\tilde{g}^j(t)).
\]
(75)

Since the beta functions do not depend on \( t \) explicitly, this is an autonomous system of equations. We observe that \( \tilde{\beta}^i \) can be viewed as a vectorfield on theory space (it has a contravariant index, see Equation (94) below), and (75) has the standard form of a first order differential equation describing a dynamical system on a manifold.

There is then the issue of the normalization of the field. Usually a QFT contains a kinetic term of the form \( \frac{1}{2} Z_\phi (\partial \phi)^2 \) and by a simple rescaling of the field (and appropriate redefinitions of all the other couplings) one can set \( Z_\phi = 1 \). In the Wilsonian definition of the RG one rescales the fields at each step in order to maintain this normalization condition. This makes the fields \( k \)-dependent. In the Wetterich equation, the fields are \( k \)-independent and \( Z_\phi (k) \) is treated like any one of the couplings \( g_i \). From the FRGE one can read off the beta function of \( Z_\phi \), which is usually reexpressed as an anomalous dimension
\[
\eta_\phi = - \frac{d \log Z_\phi}{d \log k}.
\]
(76)

If one wants, one can introduce powers of \( \sqrt{Z_\phi} \) in the definition of the \( \tilde{g}^i \) in such a way that the canonical normalization is automatically maintained. This would introduce
additional terms proportional to $\eta_\phi$ in (74).

Discussion of redundant couplings. The normalization of the kinetic term is just one aspect of a much more general issue. One says that a parameter in the action is “redundant” or “inessential” if it can be eliminated from the action by a field redefinition. The wave function renormalization constant is just the simplest example, but there are infinitely many ways of redefining the field and therefore in the action (69) there are infinitely many redundant couplings. For some purposes it is convenient to define the physical theory space as the quotient of the space of the functionals $\Gamma_k$ by the field reparametrizations. In this case the $\tilde{g}_i$ are assumed to be coordinates in this quotient space.

2.5 Fixed points and scaling regions

When one integrates a trajectory towards the UV or the IR, several types of behavior are possible in principle. Generically one would expect that the RG trajectory tends to infinity (i.e. that at least one of the coordinates $\tilde{g}_i$ diverges) as $k \to \infty$. There are also known cases when the trajectory goes to infinity for finite $k$. An example of such a behavior is the Landau pole of QED or $\phi^4$ theory in four dimensions. In such cases the theory only makes sense for finite UV scale $\Lambda$ and must be regarded as an effective field theory. Alternatively, the trajectory could flow towards a fixed locus, a subset of theory space that is left invariant by the flow. The simplest, zero-dimensional example of fixed locus is a fixed point (FP), namely a point $g^*$ where

$$\tilde{\beta}^i(\tilde{g}^i_*) = 0$$  \hspace{1cm} (77)

for all essential couplings $\tilde{g}^i$. Note that inessential couplings are not required to tend to a fixed value, since they do not affect physical observables. In general, near a fixed point they will obey a scaling relation. For example the wave function renormalization satisfies

$$Z_\phi(t) = e^{\eta_* t}$$  \hspace{1cm} (78)

where $\eta_*$ is the value of the anomalous dimension (76) at the fixed point.

A one-dimensional fixed locus would be a limit cycle and in principle there may exist higher dimensional analogues of any dimension and codimension, including the possibility of the RG trajectory ergodically filling theory space or a subset thereof. We will not discuss such exotic possibilities here and focus only on fixed points.

A trajectory that reaches a FP in the UV is called a (non-perturbatively) renormalizable or asymptotically safe trajectory. One would expect that all physical observables are well defined in this limit.

The fixed points determine the qualitative properties of the flow and therefore an understanding of their properties is the first step in the study of the flow. After having determined the position of all fixed points, the next step is to try and understand the

\[12\] It is understood that the redefinitions of the fields must be local. If one allowed nonlocal field redefinitions, then all couplings would be redundant.
nature of their IR and UV basins of attraction. One calls IR (resp. UV) critical surface of a FP the set of all points that flow to the FP for \( t \to 0 \) (resp. \( t \to \infty \)). In general one cannot say much on these surfaces, but one can determine their tangent space at the FP by studying the linearized flow, as we shall discuss next.

### 2.5.1 The linear approximation

In the neighborhood of a fixed point we can expand the beta functions in a Taylor series. If \( y^i \equiv \delta \tilde{g}^i = \tilde{g}^i - \tilde{g}^i_* \) parametrize the deviation from the fixed point, we have

\[
\tilde{\beta}^k(\tilde{g}_* + y) = \sum_i M^k_i y^i + O(y^2),
\]

where we defined the stability matrix

\[
M^i_j \equiv \frac{\partial \tilde{\beta}^i}{\partial \tilde{g}^j} \bigg|_*.
\]

The linearized flow equation then reads

\[
\frac{dy^k}{dt} = \sum_i M^k_i y^i.
\]

We will only be concerned with cases in which either the matrix \( M^i_j \) is already diagonal, or its left and right spectra coincide (implying that the spectrum is unique and unambiguous). It is then convenient to use the eigenvectors of \( M \) as basis. We define \( z^a = (S^{-1})^i_a y^i \) where the transformation \( S \) diagonalizes \( M \):

\[
(S^{-1})^i_a M^i_j S^j_b = \lambda_a \delta^a_b.
\]

(Summation over repeated indices is implied.) In terms of the new variables the linearized RG equation reduces to a set of independent equations:

\[
\frac{dz^a}{dt} = \lambda_a z^a,
\]

whose solutions are

\[
z^a(t) = C^a e^{\lambda_a t} = C^a \left( \frac{k}{k_0} \right)^{\lambda_a},
\]

where \( C^a \) are arbitrary constants. Because of this simple behavior, \( z^i \) are called scaling fields or scaling variables and the opposite of the eigenvalues are called the scaling exponents

\[
\theta_a = -\lambda_a.
\]

- The directions \( z^a \) that correspond to positive eigenvalues (negative scaling exponents) are repelled by the FP for growing \( t \): they are UV repulsive and IR attractive. Such infinitesimal deformations of the FP theory are said to be irrelevant.
The directions $z^a$ that correspond to negative eigenvalues (positive scaling exponents) are attracted by the FP for growing $t$: they are UV attractive and IR repulsive. Such infinitesimal deformations of the FP theory are said to be relevant.

A deformation with eigenvalue zero is said to be marginal and in order to establish whether it is marginally relevant or irrelevant one has to go beyond the linearized analysis. Except for the case of the Gaussian FP, discussed below, there are in general no marginal deformations.

There follows from these definitions that the tangent space to the UV critical surface at the FP is the space spanned by the relevant directions, whereas the tangent space to the IR critical surface is the space spanned by the irrelevant directions. In particular, the dimension of the UV critical surface is equal to the number of negative eigenvalues of the matrix $M$ and that of the IR critical surface is equal to the number of positive eigenvalues.

**Remark 1.** It is important to note that in general a fixed point cannot be said to be a “IR fixed point” or an “UV fixed point”. There will be directions from which it can be approached in the IR and other directions from which it can be approached in the UV.

**Remark 2.** We are studying here the RG abstractly as a mathematical problem. Insofar as we can think of it as a first order system of differential equations, the solutions can be integrated indifferently in either direction of $t$. However, one has to bear in mind that the RG, viewed physically as a coarse-graining procedure, proceeds from the UV towards the RG, i.e. in the direction of decreasing $t$. For this reason, in many texts, the RG parameter is defined to be $t' = -t$.

**Remark 3.** Under a discrete blocking transformation with parameter $b$ (as discussed in section 2.1)

\[ k' = \frac{k}{b}, \quad b > 1 \]

denoting $z^a = z^a(t)$ and $z'^a = z^a(t')$ we find from (84):

\[ z'^a = b^{-\lambda_\alpha} z^a. \quad (86) \]

and after $n$ blocking iterations

\[ z'^a = b^{-n\lambda_\alpha} z^a, \quad \text{or} \quad z'^a = b^n \theta_\alpha z^a. \quad (87) \]

The relevant scaling fields ($\theta_\alpha > 0$) increase with the RG flow and conversely the irrelevant ones go to zero (this being actually the origin of this terminology).

### 2.5.2 The Gaussian Fixed Point

A free theory has vanishing beta functions and therefore must be a FP of the RG flow. It is called the Gaussian FP. In the neighborhood of the Gaussian FP one can use perturbation theory. Let us Taylor expand the second term of (74) around the Gaussian FP:

\[ \alpha'(g_i) = \alpha^i g^i + \alpha^i g^j g^k + \ldots . \quad (88) \]
Note that there cannot be a constant term otherwise $\tilde{g}_i = 0$ would not be a FP. When used in equation (80) we find

$$M^i_j = -d_i \delta^i_j + \alpha^i_j. \quad (89)$$

The first term comes from the classical scaling, the second from quantum effects. It is often the case that the expansion (88) begins with the quadratic terms; then the matrix $M_{ij}$ is diagonal with eigenvalues $-d_i$. Even if this does not happen, it follows from the structure of the perturbative corrections that a coupling of dimension $d_i$ only enters in the beta functions of couplings of dimension $d_j$ for $d_j \leq d_i$. (We can see an explicit example of this in equations XXXXX.) There follows that the matrix $\alpha_{ij}$ only has entries for $j > i$, so the scaling exponents are simply

$$\theta_i = d_i. \quad (90)$$

There follows that the UV–attractive (relevant) couplings are those that have positive mass dimension. So, near the origin, the UV critical surface is simply the space spanned by the renormalizable couplings and the IR critical surface is spanned by the non-renormalizable couplings.

**Remark.** Usually, the mass is the most relevant parameter. This means that if we flow towards the Gaussian FP, in order to have a small mass in the IR (where “small” means “small with respect to the UV cutoff”) one has to accurately tune the initial conditions. This is one way of seeing the hierarchy problem.

### 2.5.3 The quadratic approximation

Now we extend the Taylor expansion (79) to second order:

$$\bar{\beta}^k(g_* + y) = \sum_i M^k_i y^i + \sum_{i,j} N^k_{ij} y^i y^j + O(y^3), \quad (91)$$

where

$$N^k_{ij} = \frac{1}{2} \left. \frac{\partial^2 \bar{\beta}^i}{\partial \bar{g}^j \partial \bar{g}^k} \right|_{g_*} \quad (92)$$

is symmetric in the last two (lower) indices. In the basis of eigenvectors of $M$ the components of this tensor are

$$C^a_{bc} = (S^{-1})^a_i N^i_{jk} S^j_b S^k_c, \quad (93)$$

We shall see that some of these coefficients are universal and are related to the coefficients of the Operator Product Expansion (OPE).

### 2.5.4 Transformation properties under reparametrizations

The scaling exponents $\theta_a$ are important, in principle measurable quantities. They should therefore be independent of immaterial details of the parametrization of the system. We will now show that this is indeed the case.
Suppose we change the definition of the couplings $g_i$:

$$\tilde{g}^{\ell i} = \tilde{g}^{\ell i} (\tilde{g}) .$$

(93)

Such a transformation may arise for example from the comparison between two different regularization and renormalization procedures, but in more complicated theories we may also simply envisage redefinitions such as $g \rightarrow 1/g$.

We can regard (93) as a coordinate transformation in the space of all couplings. The beta functions can be regarded as a vectorfield in this space, and their transformation under (93) is

$$\tilde{\beta}^{\ell i} = (J^{-1})^{j i} \tilde{\beta}^j .$$

(94)

where

$$j^i_k = \frac{\partial \tilde{g}_i}{\partial \tilde{g}_k} ; \quad (J^{-1})^{i j} = \frac{\partial \tilde{g}^{\ell i}}{\partial \tilde{g}^{\ell j}} .$$

(95)

Then, the matrix $M$ transforms as

$$M^{\ell i}_j = (J^{-1})^{i k} M^k_j + j^i_k \frac{\partial^2 \tilde{g}^{\ell i}}{\partial \tilde{g}^k \partial \tilde{g}^\ell} \tilde{\beta}^k .$$

(96)

At a FP the second term vanishes, so the matrix $M$ transforms as

$$M^{\ell i}_j \bigg|_* = (J^{-1})^{i k} M^k_j \bigg|_* .$$

(97)

The eigenvalues of a matrix are invariant under similarity transformations, so this implies that the scaling exponents are independent of the choice of coordinates in the space of couplings. Of course, if a coordinate transformation is singular at the FP this need not be the case. (This happens for example if we choose to describe the Gaussian FP with a coordinate $g' = 1/g$.)

Things are more complicated for the second order Taylor coefficients. A direct computation gives

$$N^{\ell i j}_k = (J^{-1})^{i l} J^m_j J^n_k \left[ N^{\ell m n}_n + \frac{1}{2} M^\ell_m \frac{\partial^2 \tilde{g}^{\ell j}}{\partial \tilde{g}^m \partial \tilde{g}^{\ell i}} (J^{-1})^r_m (J^{-1})^s_n - \frac{1}{2} M^\ell_n \frac{\partial^2 \tilde{g}^{\ell j}}{\partial \tilde{g}^n \partial \tilde{g}^{\ell i}} (J^{-1})^r_n (J^{-1})^s_m \tilde{\beta}^r \right] .$$

(98)

Now inverting (82) we can reexpress the matrices $M$ in terms of their eigenvalues $-\theta_a$ as

$$M^\ell_t = -\theta_a (S^{-1})^{\ell a} S^a_t ,$$

etc., where a summation over $a$ is again implied.

13 Transformations between two mass-independent schemes may result in simple transformations, but in general they can be highly non-linear [?].
On the other hand, from equations (82) and (97) we see that if the linear transformation \( S \) diagonalizes the matrix \( M \), the linear transformation \( S' = J^{-1} S \) diagonalizes the matrix \( M' \). We can then define

\[
C'^{ab}_{c} = (S'^{-1})^{a}_{i} N^{i}_{jk} S'^{i}_{b} S'^{k}_{c}, \tag{99}
\]

Inserting the previously obtained formula for \( N'_{ijk} \) we find

\[
C'^{ab}_{c} = C^{ab}_{c} - \frac{1}{2} (\theta_a - \theta_b - \theta_c) (S^{-1})^{a}_{\ell} \frac{\partial^{2} S^{\ell}_{r}}{\partial S^{i}_{n} \partial S^{j}_{s}} (J^{-1})^{r}_{m} (J^{-1})^{s}_{n} S^{m}_{b} S^{n}_{c}. \tag{100}
\]

We see that the coefficients \( C^{a}_{bc} \) are invariant in one of two cases: either certain components of the Hessian of the transformation vanish, which may happen for particular transformations, or

\[
\theta_a - \theta_b - \theta_c = 0 \tag{101}
\]

in which case these particular coefficients are universal and must have a physical meaning. We will discuss this in the next section.

### 2.6 Conformal Field Theories

Fixed points of the renormalization group are scale-invariant QFT’s, and it is generally believed that unitary scale-invariant QFT’s are also conformally invariant. This is proven in two dimensions, see [1] for higher dimensions. The conformal group consists of diffeomorphisms of flat space \( x \rightarrow x' \) such that

\[
\eta_{\rho \sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Omega^2(x) \eta_{\mu \nu}. \nonumber
\]

By taking the determinant of both sides we see that \( \Omega = \left| \det \left( \frac{\partial x'^{\mu}}{\partial x^{\nu}} \right) \right|^{1/d} \).

Infinitesimal conformal transformations are generated by vectorfields \( v^{\mu} \) satisfying the conformal Killing equation

\[
\partial_{\mu} v_{\nu} + \partial_{\nu} v_{\mu} = 2 \omega g_{\mu \nu} \quad \text{where} \quad \omega = \frac{1}{d} \partial_{\rho} v^{\rho} \nonumber
\]

In \( d \)-dimensional Minkowski space, the conformal group is isomorphic to \( SO(d, 2) \). In addition to the Poincaré generators, that have \( \omega = 0 \), its algebra also contains the generator of dilatations \( D = x^{\mu} \partial_{\mu} \), that has \( \omega = 1 \) and the generators of special conformal transformations

\[
K_{\mu} = 2 x_{\mu} x^{\nu} \partial_{\nu} - x^{2} \partial_{\mu} \nonumber
\]

that have \( \omega = 2 x_{\mu} \).
A QFT that is invariant under the conformal group is called a Conformal Field Theory (CFT). We recall here some relevant results. In a CFT there is a class of local operators called primary operators, among which are the fields themselves, such that

$$\Phi'_a(x') = \Omega^{-\Delta_a} \Phi_a(x) .$$

(102)

or infinitesimally, for \( \Omega = 1 + \omega \)

$$\delta \Phi_a = -(x^\mu \partial_\mu + \Delta_a \omega) \Phi_a .$$

(103)

\( \Delta_a \) is called the dimension of the operator \( \Phi_a \). We can relate these dimensions to the scaling exponents introduced earlier. Writing

$$O_a[\phi] = \int d^d x \Phi_a(x) .$$

we find that in a CFT, \( O_a \) must have dimension \(-d + \Delta_a \). On the other hand, at a FP, the operators that have a definite dimension are the scaling operators. Let \( O_a \) be a scaling operator. Since the corresponding coupling \( z^a \) has scaling dimension \( \theta_a \), we must have

$$\theta_a = d - \Delta_a .$$

(104)

Recall that since \( g^a \) has canonical dimension \( d^a \), \( O_a \) has canonical dimension \(-d^a \) and therefore \( \Phi_a \) has canonical dimension \( d^a = d - d^a \). The true scaling dimension of \( \Phi_a \) consists of the classical dimension plus an “anomalous” quantum term:

$$\Delta_a = d^a + \gamma_a .$$

Invariance under conformal transformations is a very powerful constraint and is enough to completely determine the form of the two- and three-point functions. The two-point function of two (suitably normalized) operators \( \Phi_a \) and \( \Phi_b \) must satisfy

$$\langle \Phi_a(x) \Phi_b(y) \rangle = \frac{\delta_{ab}}{|x - y|^{2\Delta_a}}$$

(105)

In particular, it must vanish when the scaling dimensions of the operators is different. The three-point function has the form

$$\langle \Phi_a(x) \Phi_b(y) \Phi_c(z) \rangle = \frac{c_{abc}}{|x - y|^\Delta_a + \Delta_b - \Delta_c |y - z|^\Delta_b + \Delta_c - \Delta_a |z - x|^\Delta_c + \Delta_a - \Delta_b}$$

(106)

and \( c_{abc} \) are the coefficients of the OPE, which in the present case can be written

$$\langle \Phi_b(y) \Phi_c(x) \rangle = \sum_{a} \frac{c_{abc}^{a}}{|y - x|^\Delta_b + \Delta_c - \Delta_a} \langle \Phi_a(x) \rangle .$$

(107)

Note that \( c_{abc} \) are pure numbers.

Conformal perturbation theory consists in studying the infinitesimal neighborhood of the FP. To linear order the action near a FP is

$$\Gamma_k = \Gamma_* + \sum_a k^a z^a \int d^d x \Phi_a + O(z^2) .$$

(108)
When a CFT is perturbed, the beta functions in the neighborhood of the FP can be written, in terms of the scaling variables, as

$$\beta^a = (\Delta^a - d) z^a + c^a_{\ b c} z^b z^c + O(z^3) \ .$$

(109)

This provides a link to the RG techniques introduced earlier. The form (109) for the beta functions holds in a specific scheme that we may call the “CFT scheme”.

Comparison of the first term with (83) confirms that $\Delta^a - d$ has to be identified with $\lambda^a = -\theta^a$. From the analysis in the preceding section we know that the coefficients of the quadratic expansion of the beta functions in the basis of the scaling operators, equation (99), are universal only if they satisfy equation (101). Therefore these particular coefficients $C^a_{\ bc}$ can be identified with the corresponding OPE coefficients.

2.7 Global properties of the flows

So far we have discussed in some detail the behavior of the RG flow in the neighborhood of a fixed point. In the case of the Gaussian FP, this reproduces the standard results of perturbation theory. For conformal fixed points, it agrees with conformal perturbation theory. In addition, the nonperturbative RG also gives a picture of the flow away from fixed points. One has to keep in mind that this picture is in general highly non-universal, but qualitative (topological) features, such as trajectories joining certain fixed points, should be scheme-independent.

We can then classify all RG trajectories (i.e. all theories) according to their behavior in the UV and IR limits. In both limits the trajectory can either go to infinity or tend to a fixed point, so that we have four possible classes of trajectories.

2.7.1 IR behavior

We have defined earlier the Gaussian FP as a theory where all the coordinates $\tilde{g}_i$ are zero. This includes also the dimensionless mass $\tilde{m} = m/k$. For $\tilde{m}$ to be zero for finite $k$, also the dimensionful mass $m$ must be zero. Thus the Gaussian FP corresponds to a massless theory in the conventional sense. However, a Gaussian theory could also have a mass term. Being free, its beta functions are trivial and therefore the dimensionful mass $m$ must be scale-independent. This means that the dimensionless mass $\tilde{m} = m/k$, must run from the Gaussian FP in the UV to $\tilde{m} \to \infty$ in the IR. This is a divergence that is clearly not pathological from the physical point of view. A theory that approaches this particular limit in the IR is said to be “gapped” (i.e. to have a mass gap) and IR-free.

More generally, the decoupling theorem \footnote{This theorem states that in a massive theory the beta functions become equal to zero when the RG scale becomes smaller than the physical mass. It implies that in this limit the beta functions become equal to zero when the RG scale becomes smaller than the physical mass.} implies that in a massive theory the beta functions become equal to zero when the RG scale becomes smaller than the physical mass. (We shall see explicit examples of this threshold behavior in the sequel.) All the couplings $g_i$ cease to run and their dimensionless counterparts $\tilde{g}_i$ then have purely classical scaling. This implies that if $d_i$ (the dimension of $g_i$) is negative, $\tilde{g}_i$ will tend to zero and if $d_i$ is positive (typically this is only the case for the mass), $\tilde{g}_i$ will go to infinity. In principle, couplings with $d_i = 0$ can remain stuck at nonzero values in this limit, so we would have interacting gapped limits.
If an IR-free theory has no massive physical states, there is no threshold and the running continues down to arbitrarily low momentum scales. Massless QED would be an example of such a behavior, with very slow, logarithmic, running.

The least familiar type of theory corresponds to a trajectory that reaches an interacting FP in the IR. An example is the Banks-Zaks FP in gauge theories \[?\]. Such theories do not look at all like ordinary QFTs. However, they are the ones that are useful for the description of critical phenomena, as we shall discuss in the next section.

2.7.2 UV behavior

There are famous examples of theories where some coupling goes to infinity for finite \(k\): this corresponds to a Landau pole and signals a breakdown of the theory. \(^{14}\) Trajectories of this type correspond to effective field theories. (If the dimensionless couplings go to infinity in the limit \(k \to \infty\) it is also very unlikely that the theory can be used.)

Conversely, a trajectory that reaches a FP in the UV is said to be “renormalizable” or “asymptotically safe”. Such a trajectory corresponds to an UV complete theory and all physical observables should remain under control up to infinite energy. For example, a cross-section \(\sigma\) must be a function of the couplings \(g_i\) and of the external momenta. The couplings depend on a scale \(k\) and one can identify \(k\) with one of the characteristic momenta of the process. Then, from dimensional analysis, \(\sigma = k^{-2} \tilde{\sigma}(\tilde{g_i}, X)\), where \(X\) denotes dimensionless kinematical variables such as angles and ratios of energies. One does not expect \(\tilde{\sigma}\) to have a singularity precisely at \(\tilde{g}_i = \tilde{g}_i^*\), so on a renormalizable trajectory all physical observables should be well defined in the UV limit.

The condition of renormalizability can be used as a criterion to select interesting theories. In the old days of QFT it was thought that this was in fact a necessary condition for a theory to be useful. The techniques of effective field theory have made this point of view obsolete: we now know that also non-renormalizable theories can be useful and predictive within their domain of applicability. However, EFT leaves us too much freedom and it would be desirable to have a stricter selection criterion. Renormalizability, in the general sense defined above, could provide such a guideline. The main motivation not be the desire to have an UV complete theory, since we shall never be able to test any theory up to infinite energy. Instead, the main motivation is to have a more predictive framework.

In order to use UV completeness as a selection criterion that gives more predictive theories, we must make an additional demand on the FP, namely that its UV critical surface be finite dimensional. If this surface is \(D\)-dimensional, UV completeness (at the chosen FP) leaves only \(D - 1\) free parameters in the theory (equal to the dimension of the UV critical surface minus one, that corresponds to the RG scale \(t\)). The critical surface is hard to study in general, but its tangent space can be determined by studying the linearized flow, as in section 2.2.1. The dimension of the UV critical surface is just the number of relevant operators at the FP.

\(^{14}\) Of course, it signals a breakdown of perturbation theory. However both for scalar theory and QED there is significant evidence that the Landau pole that is seen in perturbation theory is a genuine non-perturbative feature.
This is to be contrasted to the case of an EFT, where the local terms in the action all have arbitrary coefficients and only non-local effects, due to the low-energy part of Feynman diagrams, can be calculated unambiguously. Instead, in and asymptotically safe theory only $D - 1$ parameters in the action are free and all other local terms can be calculated as functions of these.

The most familiar example of an asymptotically safe theory is QCD. In this case the prediction is that all non-renormalizable terms in the action vanish when one goes to high energy. However, QCD is rather special since the FP is the Gaussian one. Asymptotic safety at a Gaussian FP is known as asymptotic freedom. It can be studied by perturbative methods.

Non-Gaussian examples of asymptotically safe theories have been known for some time in dimensions just above two or in three dimensions. In such models the non-trivial FP arises when quantum effects, represented by the term $\alpha$ in (??), balance the classical scaling. This is why such phenomena are easier to study near two dimensions: the classical scaling term can then be made arbitrarily small and the fixed point can be generated within the domain of perturbation theory.

More recently, a class of models of vector, spinor and scalar fields with gauge and Yukawa interactions, in four dimensions, has been proven to possess a non-trivial FP that can be used to construct asymptotically safe theories. In such models the FP arises from a balance between one- and two-loop contributions. The small parameter is related to the deviation from the point where asymptotic freedom is lost, as a function of the numbers of colors and flavors. The existence of these models shows that asymptotic safety may be more common than previously thought.
3 Critical phenomena and the RG

Mean field theory explains universality but it predicts values for the critical exponents that are not quantitatively accurate especially in one and two dimensions. We need a better tool, and this tool is the Renormalization Group, in the general form in which it has been described in section 2: one works in a general theory space and the RG defines trajectories all sharing the same large scale, or low energy, properties.

The simple explanation of universality is the fact that trajectories originating from very different regions of theory space converge to the same FP in the IR. Quantitatively: the number of relevant directions of a FP is typically finite (and usually small), so there are infinitely many parameters whose initial values do not matter for the large scale behavior. However, the relevant couplings are pushed away from the FP when one goes towards the IR, so in order to end up at the FP one has to choose their initial values carefully. This is related to the tuning of parameters that is necessary to reach the critical state. For example, in the case of the uniaxial ferromagnet the critical state is achieved by tuning the temperature and the magnetic field. Thus one expects that this critical theory is described by a fixed point with two relevant directions. The coordinates on this surface must be related in some way to the parameters that an experimentalist has to tune in order to reach the critical point.

3.1 The scaling relations

3.1.1 Derivation

In order to see how the critical exponents are related to the properties of the fixed point, let us consider again the example of the ferromagnet. The fixed point has two relevant scaling variables $z_t$ and $z_h$ that we associate to temperature and magnetic field. They are distinguished by being even and odd under $\phi \to -\phi$, respectively. The scaling variables must vanish at the critical point $t = h = 0$ and for symmetry reasons they must be related to the reduced temperature and magnetic field by a relation of the form

$$z_t = t/t_0 + O(t^2, h^2); \quad z_h = h/h_0 + O(th).$$

The free energy per site $f$ is a function of all the couplings. If we are sufficiently close to the fixed point that the linearized flow is a good approximation, we can think of $f$ as a function of the scaling variables. Since the irrelevant ones go to zero, we can neglect them and think of $f$ as a function of the relevant variables $z_t$ and $z_h$ only. A discrete “blocking” RG transformation with parameter $b$ transforms the couplings as in (87). It also increases the volume of the blocks by $V' = b^d V$ and hence decreases the total number of blocks by $N' = b^{-d} N$. The RG transformations are such that the partition function $Z$ is left invariant. Therefore

$$Z = e^{-Nf(z_a)} = e^{-N'f(z'_a)}$$

up to an additive term that can be neglected for our purposes. This implies that

$$f(z_t, z_h) = b^{-d} f(b^h z_t, b^h z_h).$$
After $n$ iterations, the transformation of the free energy is the same, with $b$ replaced by $b^n$. We can eliminate $b$ and replace it by some initial value along the trajectory, for example $b^n\theta_l z_t = \bar{z}_t$. The point $\bar{z}_t$ must obviously be still within the range of validity of the linearized approximation. We can then write

$$f(z_t, z_h) = |z_t/\bar{z}_t|^{d/\theta_l} f\left(\bar{z}_t, |z_t/\bar{z}_t|^{-\theta_h/\theta_l} z_h\right).$$ (110)

Dropping the spurious dependence on $\bar{z}_t$ and rewriting in terms of the reduced physical variables one obtains

$$f(t, h) = |t/\bar{t}|^{d/\theta_t} \Phi\left(|t/\bar{t}|^{-\theta_h/\theta_l} h/\bar{h}\right),$$

which has the same form as (23).

### 3.1.2 Consequences

From this scaling relation one can read off various critical exponents, exactly as we did in section 1.6. There, the critical exponents were all expressed as functions of the two parameters $\alpha$ and $\Delta$ appearing in (23). Now they will be expressed as functions of the scaling exponents $\theta_l$ and $\theta_h$ appearing in (110). Actually, it will not be necessary to repeat all the steps. It is enough to compare directly (110) to (23) to learn that

$$2 - \alpha = d/\theta_l$$ (111)

and

$$\Delta = \theta_h/\theta_l.$$ (112)

We can then use Josephson’s relation (28) to infer that

$$\nu = 1/\theta_l$$ (113)

On the other hand starting from the scaling relation (29) and using (25) and (113) one gets

$$\eta = 2 - \frac{\gamma}{\nu} = 2 - \theta_l(\alpha - 2 + 2\Delta)$$

and finally using (111)

$$\eta = d + 2 - 2\theta_h.$$ (114)

Since all the critical exponents were expressed as functions of $\nu$ and $\eta$ in (30-33), we can now write them as functions of the scaling exponents $\theta_l$ and $\theta_h$. We report for completeness:

$$\alpha = 2 - d/\theta_l$$ (115)

$$\beta = \frac{d - \theta_h}{\theta_l}$$ (116)

$$\gamma = \frac{2\theta_h - d}{\theta_l}$$ (117)

$$\delta = \frac{\theta_h}{d - \theta_h}.$$ (118)
We find that using the RG we could not only prove the scaling relation (23) and all its consequences, but also express all the critical exponents in terms of properties of the linearized RG at a FP.

### 3.1.3 Gaussian exponents

In order to calculate the critical exponents from the RG we must now choose a fixed point. As a first try, we can check what happens at the Gaussian FP. We use the Landau-Ginzburg theory as a representative in the Ising universality class. At the Gaussian FP the most relevant operator is the mass, and it is associated to the reduced temperature via the relation (17). It has dimension \( \theta_t = 2 \) and therefore \( \nu = 1/2 \).

The magnetic coupling \( h \) multiplies the operator

\[
\int d^d x \phi
\]

and therefore has dimension \( \theta_h = (d + 2)/2 \). When this is inserted in (114) we obtain \( \eta = 0 \). These are the values for the mean field approximation. Consequently also all the other critical exponents agree with the mean field values.

From what was said in section 1.5, the Gaussian FP seems to be the correct FP to describe the Ising universality class in dimensions \( d \geq 4 \). But what is the correct fixed point in dimensions \( d < 4 \)?

### 3.2 The Wilson–Fisher fixed point

#### 3.2.1 \( \epsilon \)-expansion

A first hint comes from the following observation. Consider actions of the form

\[
S[\phi] = \int d^d x \left\{ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right\} .
\]

As is well-known, the beta function of \( \lambda \) in \( d = 4 \) is

\[
\beta = \frac{3 \lambda^2}{(4\pi)^2} ,
\]

If we go to \( d = 4 + \epsilon \), the dimension of the field changes slightly to \([\phi] = d/2 - 1 = 1 + \epsilon/2\) and therefore also the dimension of \( \lambda \) changes to \([\lambda] = 4 - d = -\epsilon\). In order to study the flow we must therefore consider the beta function of the dimensionless coupling \( \tilde{\lambda} = \lambda k^\epsilon \), whose beta function is

\[
\tilde{\beta} = \epsilon \tilde{\lambda} + \left( \frac{3}{(4\pi)^2} + O(\epsilon) \right) \tilde{\lambda}^2 ,
\]

To first order in \( \epsilon \), we see that there is a nontrivial fixed point at

\[
\tilde{\lambda} = -\frac{16\pi^2}{3} \epsilon .
\]
critical phenomena and the RG

This is known as the Wilson-Fisher fixed point. It occurs at positive $\tilde{\lambda}$ and is IR attractive in $d < 4$. One could try to use it as the IR fixed point describing the Ising universality class. For the calculation of $\nu$ and $\eta$ to first order in $\epsilon$ see Cardy sect.5.5

Its existence can be rigorously established in this way only in the limit $\epsilon \to 0$, but there is overwhelming evidence from many other calculations that it exists for all $2 \leq d < 4$. Below we shall study this fixed point using the FRGE.

### 3.2.2 Two coupling truncation of the FRGE

The FRGE holds in any dimension. Let us see what it gives for the running of $\lambda$ and $m^2$. We thus make an ansatz for the EAA of the form

$$
\Gamma_k[\phi] = \int d^d x \left\{ \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{m_k^2}{2} \varphi^2 + \frac{\lambda_k}{4!} \varphi^4 \right\} .
$$

Note that this is quite different from assuming that the bare action is of the form (119). Equation (122) is much stronger because it is assumed to hold for all $k$. We know that as soon as we begin integrating out field modes, the EAA will contain all possible terms allowed by the symmetry $\varphi \mapsto -\varphi$. Equation (122) is thus a drastic approximation.

We will calculate below some of the terms that are being missed by this assumption. The Hessian is

$$
\delta^2 \Gamma_k = \int d^d x \delta \varphi \left\{ -\partial^2 + m_k^2 + \frac{\lambda_k}{2} \varphi^2 \right\} \delta \varphi ,
$$

so the flow equation becomes

$$
\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \frac{\partial_t R_k}{-\partial^2 + m_k^2 + \frac{\lambda_k}{2} \varphi^2 + R_k} = \frac{1}{2} V \int \frac{\partial_t R_k(q^2)}{q^2 + m_k^2 + \frac{\lambda_k}{2} \varphi^2 + R_k(q^2)} ,
$$

where on the last step we took the field constant. The l.h.s of the flow equation is then

$$
\partial_t \Gamma_k = V \left[ \partial_t E_k + \partial_t m_k^2 \frac{\varphi^2}{2} + \partial_t \lambda_k \frac{\varphi^4}{4!} \right] .
$$

where $V = \int d^d x$ is the volume of space. Setting the field constant and expanding the r.h.s is in powers of it

$$
\partial_t \Gamma_k = \left( \frac{1}{2} \int_q G_k(q^2) \partial_t R_k(q^2) \right) V + \left( -\frac{\lambda_k}{2} \int_q G_k(q^2) \partial_t R_k(q^2) \right) V \frac{\varphi^2}{2} \\
+ \left( 3\lambda_k^2 \int_q G_k(q^2) \partial_t R_k(q^2) \right) V \frac{\varphi^4}{4!} + O(\varphi^6) ,
$$

where $G_k(q^2) = [q^2 + m_k^2 + R_k(q^2)]^{-1}$ is the regularized propagator in this case. Com-
paring r.h.s to l.h.s finally gives the beta functions

\[ \partial_t E_k = \frac{1}{2} \int_q G_k(q^2) \partial_t R_k(q^2) \]
\[ \partial_t m_k^2 = -\frac{\lambda_k}{2} \int_q G_k^2(q^2) \partial_t R_k(q^2) \]
\[ \partial_t \lambda_k = 3\lambda_k^2 \int_q G_k^3(q^2) \partial_t R_k(q^2) . \]

(123)

The integrand is rotationally symmetric and so we can write

\[ \int_q \frac{S_d}{(2\pi)^d} \int_0^\infty dq q^{d-1} \rightarrow \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty dz z^{d/2-1} , \]

(124)

where \( S_d = 2\pi^{d/2}/\Gamma(d/2) \) is the volume of the \( d \)-sphere and we defined \( z = q^2 \). To obtain more explicit expressions we now choose a cutoff shape function. With the choice (42), i.e. \( R_k(z) = (k^2 - z)\theta(k^2 - z) \), we can perform the momentum integrals analytically. In fact one has the useful property that \( \partial_t R_k(z) = 2\theta(k^2 - z) \) so that the relevant integral is trivial

\[ \int_0^\infty dz z^{d/2-1} G_k^m(z) \partial_t R_k(z) = \left( \frac{1}{k^2 + m_k^2} \right)^m \int_0^{k^2} dz z^{d/2-1} = \frac{2k^{d-2m+1}}{d(2 + m_k)!!} . \]

(125)

Using this in (123) we finally obtain

\[ \partial_t m_k^2 = -c_d k^{d-2} \frac{\lambda_k}{(1 + m_k^2/k^2)^2} \]
\[ \partial_t \lambda_k = 6c_d k^{d-4} \frac{\lambda_k^2}{(1 + m_k^2/k^2)^3} , \]

(126)

where we defined

\[ c_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2 + 1)} . \]

(127)

This constant appears in all the beta functions (and could be easily eliminated by a redefinition of the couplings). Its value in the lowest dimensions is

\[ c_1 = \frac{1}{\pi} \quad c_2 = \frac{1}{4\pi} \quad c_3 = \frac{1}{6\pi^2} \quad c_4 = \frac{1}{32\pi^2} . \]

(128)

In terms of dimensionless variables\(^{15}\), \( \bar{m}_k^2 = m_k^2/k^2 \) and \( \bar{\lambda}_k = \lambda_k/k^{4-d} \), the flow is

\[ \partial_t \bar{m}_k^2 = -2\bar{m}_k^2 - c_d \frac{\bar{\lambda}_k}{(1 + \bar{m}_k^2)^2} \]
\[ \partial_t \bar{\lambda}_k = (d - 4)\bar{\lambda}_k + 6c_d \frac{\bar{\lambda}_k^2}{(1 + \bar{m}_k^2)^3} , \]

(129)

\(^{15}\) Which implement the similarity of the RG transformation.
which is the main result of this section. If we expand the denominators in (119) and retain term up to order $\tilde{m}_k^2$ we find

$$
\partial_t \tilde{m}_k^2 = -2\tilde{m}_k^2 - c_d \tilde{\lambda}_k + 2c_d \tilde{m}_k^2 \tilde{\lambda}_k + ... \\
\partial_t \tilde{\lambda}_k = (d - 4) \tilde{\lambda}_k + 6c_d \tilde{\lambda}_k^2 + ...,
$$

(130)

which is the perturbative result.

The Wilson-Fisher FP appears at

$$
\tilde{m}_k^2 = \frac{d - 4}{16 - d} \quad \tilde{\lambda}_k = \frac{288}{c_d} \frac{d - 4}{(d - 16)^3}.
$$

(131)

We linearize the RG flow around the fixed points to understand the qualitative behavior and to extract the critical exponents

$$
\partial_t \left( \begin{array}{c} \delta \tilde{m}_k^2 \\ \delta \tilde{\lambda}_k \\ \end{array} \right) = \left( \begin{array}{cc} \partial_{\tilde{m}_k^2} \tilde{\beta} & \partial_{\tilde{\lambda}_k} \tilde{\beta} \\ \partial_{\tilde{\lambda}_k} \tilde{\beta} & \partial_{\tilde{\lambda}_k} \tilde{\beta} \\ \end{array} \right) \left( \begin{array}{c} \delta \tilde{m}_k^2 \\ \delta \tilde{\lambda}_k \\ \end{array} \right) \equiv \mathbf{M} \left( \begin{array}{c} \delta \tilde{m}_k^2 \\ \delta \tilde{\lambda}_k \\ \end{array} \right)
$$

The stability matrix $\mathbf{M}$ at the Gaussian fixed point is:

$$
\mathbf{M}_G^T = \left( \begin{array}{cc} -2 & -c_d \\ 0 & -4 + d \\ \end{array} \right),
$$

(132)

with eigenvalues $(-2, -4 + d)$, while at the Wilson–Fisher fixed point is

$$
\mathbf{M}_{WF}^T = \left( \begin{array}{cc} -\frac{d + 2}{3} & -c_d \frac{(d - 16)^2}{144} \\ -\frac{72}{c_d} \left( \frac{d - 4}{d - 16} \right)^2 & -c_d \frac{(d - 16)^2}{4 - d} \\ \end{array} \right)
$$

(133)

with eigenvalues

$$
\theta_{\pm}^{WF}(d) = \frac{1}{6} \left( 10 - 4d \pm \sqrt{484 - 200d + 22d^2} \right),
$$

(134)

The correlation length critical exponent is related to the negative eigenvalue of the Wilson–Fisher fixed point

$$
v(d) = -1/\theta_-(d).
$$

(135)

We find the values $v(2) = 0.53, v(3) = 0.54, v(4) = 0.5, v(5) = 0.38$ and so on.

Since we know the critical exponent (defined in chapter 1) as a function of the dimension we can make contact with the $\epsilon$–expansion by setting $d = 4 - \epsilon$ (or $\epsilon = 4 - d$) and expanding to first order in $\epsilon$ to obtain:

$$
v = \frac{1}{2} + \frac{\epsilon}{12} + O(\epsilon^2)
$$

(136)

and

$$
\eta = 0 + O(\epsilon^2).
$$

(137)
which agree with those of the epsilon expansion [Wilson]. Using the scaling relations we can determine all other critical exponents
\[
\alpha = \frac{\epsilon}{6} + O(\epsilon^2) \quad \beta = \frac{1}{2} - \frac{\epsilon}{6} + O(\epsilon^2) \quad \gamma = 1 + \frac{\epsilon}{6} + O(\epsilon^2) \quad \delta = 3 + \epsilon + O(\epsilon^2).
\]

Let’s now focus on three dimensions and discuss the phase diagram and the application to phase transitions. It is possible to find an analytic solution for the flow if one considers (119) in the approximation in which the mass in the denominator on the r.h.s. is neglected
\[
\partial_t \tilde{m}_k^2 = -2 \tilde{m}_k^2 - \frac{\tilde{\lambda}_k}{6\pi^2} \quad \partial_t \tilde{\lambda}_k = -\tilde{\lambda}_k + \frac{\tilde{\lambda}_k^2}{\pi^2}.
\]

The values of the two fixed points are slightly shifted
\[
\tilde{m}_*^2 = -\frac{1}{12} \quad \tilde{\lambda}_* = \pi^2 \quad \text{(IR – Wilson–Fisher)}
\]
\[
\tilde{m}_*^2 = 0 \quad \tilde{\lambda}_* = 0 \quad \text{(UV – Gaussian)}
\]

but their properties are not. The analytic solution of the system in (138) is
\[
\tilde{m}_k^2 = \frac{C_2}{k^2} - \frac{C_1}{6k} + \frac{C_1^2}{6k^2} \log(C_1 + k)
\]
\[
\tilde{\lambda}_k = \frac{C_1 \pi^2}{C_1 + k}.
\]
where $C_1$ and $C_2$ are integration constants with, respectively, dimension of mass and mass squared. Notice that the UV fixed point is reached, for $k \to \infty$, independently of the values of these constant, since it is attractive in both directions. On the other hand, the IR fixed point is reached, for $k \to 0$, only if a particular relation between the integration constants holds, i.e. only if we are on the critical trajectory.

In this truncation, all trajectories can be parametrized by the value of the renormalized mass

$$m_R^2 = \lim_{k \to 0} k^2 \tilde{m}_k^2 = C_2 + \frac{1}{6} C_1^2 \log(C_1).$$

(140)

The unique trajectory that flows between the two fixed points is defined by the condition

$$m_R^2 = 0.$$  

(141)

Using this condition in (139) gives the following solutions

$$\tilde{m}_k^2 = -\frac{C_1}{6k} + \frac{C_1^2}{6k^2} \log \left( \frac{C_1 + k}{C_1} \right)$$

$$\tilde{\lambda}_k = \frac{C_1 \pi^2}{C_1 + k}.$$  

(142)

Equation (142) describes the unique trajectory in the $(\tilde{m}_k^2, \tilde{\lambda}_k)$ plane that flows from the Gaussian fixed point in the UV to the Wilson–Fisher fixed point in the IR. In Figure 12 this finite trajectory is shown in red. All non critical trajectories are characterized by the condition $m_R^2 \neq 0$. Positive values of $m_R^2$ label trajectories that flow towards the symmetric phase, while negative values of $m_R^2$ label trajectories that flow towards the broken phase. To see this we consider the renormalized effective potential $V_R(\phi) = m_R^2 \phi^2 + \frac{\lambda_R}{4!} \phi^4$ and look for vacuum solutions $V_R'(\phi_*) = 0$. We find the solutions

$$\phi_* = \left\{ \begin{array}{ll}
0 & m_R^2 \geq 0 \quad \text{symmetric} \\
\pm \sqrt{-\frac{6m_R^2}{\lambda_R}} & m_R^2 < 0 \quad \text{broken}
\end{array} \right.$$  

(143)

characterizing the symmetric and broken phases. The effective potential is not yet convex!

Viceversa, from a statistical mechanics point of view, at fixed $\tilde{\lambda}_\Lambda > \tilde{\lambda}_*$, there is a $\tilde{m}_\Lambda^2$ such that in the IR we hit the FP. For values of $\tilde{m}_\Lambda^2 > \tilde{m}_c^2$ the flow leads to the symmetric phase, while it leads to the broken in the opposite case. Here $\tilde{m}_c^2$ is related to the reduced critical temperature of the microscopical spin model characterized by the microscopic interaction $\tilde{\lambda}_\Lambda$. Different microscopic models have different $\tilde{\lambda}_\Lambda$ and thus different critical temperatures, i.e. $\tilde{m}_c^2(\tilde{\lambda}_\Lambda)$.

### 3.2.3 Local potential approximation

Having checked that the functional RG easily reproduces the known perturbative beta functions for the mass and for the quartic scalar coupling, we would like now to take advantage of the functional nature of the flow and derive beta functions for a large,
possibly infinite, number of couplings. We will arrive at this goal in stages. It is gener-
ally quite difficult to follow exactly the flow of the EAA through theory space and
to find the relative fixed-point functionals: approximations are needed. The derivative
expansion is reliable in studying the IR behavior of a theory. For example, in statistical
models, it is a good way of studying phase transitions.

Derivation I: Ansatz. For a single real scalar field, assuming invariance under \( \phi \to -\phi \), the first term of the derivative expansion is just the potential. In order to have
also propagation we include a kinetic term, but we do not consider its renormalization
(which would produce a wave function renormalization \( Z \) and an anomalous dimen-
sion \( \eta \)). Thus the ansatz for the EAA is

\[
\Gamma_k[\phi] = \int d^d x \left\{ \frac{1}{2} (\partial \phi)^2 + V_k(\phi) \right\} .
\]  

This is called the Local Potential Approximation (LPA). We will now insert this ansatz
(144) in the FRGE and extract the beta functions of the potential. The inverse propaga-
tor corresponding to (2) is

\[
\frac{\delta^2 \Gamma_k[\phi]}{\delta \phi(x) \delta \phi(y)} = (-\partial^2 + V''_k) \delta(x - y) ,
\]  

where a prime denotes the derivative with respect to \( \phi^2 \). In the definition of the func-
tional \( \Gamma_k \) we modify the inverse propagator by adding to it the cutoff kernel \( R_k(-\partial^2) \),
which in momentum space is simply a function of \( q^2 \). With these definitions the FRGE
becomes

\[
\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \left[ \frac{\partial_t R_k(-\partial^2)}{-\partial^2 + R_k(-\partial^2) + V''_k} \right] .
\]  

The trace involves an integration over spacetime and over momenta. For any function
\( W \) we have

\[
\text{Tr} f(-\partial^2) = \int d^d x \int \frac{d^d q}{(2\pi)^d} f(q^2) = \frac{1}{(4\pi)^{d/2}} \int d^d x Q_{\frac{d}{2}}[f] .
\]  

In the last step we have performed the angular integration, using as before

\[
\frac{1}{(2\pi)^d} \text{Vol}(S^{d-1}) = \frac{1}{(4\pi)^{d/2}} \frac{2}{\Gamma(d/2)} ,
\]  

where \( \text{Vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \) is the volume of the \((d-1)\)-dimensional sphere, and we
defined the \( Q \)-functionals

\[
Q_n[f] = \frac{1}{\Gamma(n)} \int_0^{+\infty} dz z^{n-1} f(z) ,
\]  

\[16\] Because of the invariance, the potential must actually be a function of \( \phi^2 \), but the notation \( V(\phi^2) \)
would complicate certain formulae later, for example instead of (??) we would have \( 2V' + 4\phi^2 V'' \).

\[17\] Also known as Mellin transform.
where $f$ is viewed as a function of $z = |q|^2$. The $Q$–functionals contains the integration over the modulus of the momentum. Writing the r.h.s. of the flow equation in terms of a $Q$–functional is more explicit than writing it in terms of “Tr”, but it is not yet completely explicit. We refrain from evaluating the $Q$–functional at this stage because that would require committing ourselves to a specific choice of $R_k$. It is sometimes useful to retain this freedom as long as possible.

If we now restrict ourselves to constant scalar fields, we can remove a volume factor from both sides of the FRGE and we obtain the $k$-dependence of the potential as

$$\partial_t V_k = \frac{1}{2} \frac{1}{(4\pi)^\frac{d}{2}} \int Q \frac{\partial^2}{\partial k^2} G_k \partial_t R_k,$$

(148)

where, as before,

$$G_k(z, \phi) = \frac{1}{z + R_k(z) + V_k''(\phi)}.$$

(149)

The functional $Q$ is left here in a general, unevaluated form. The precise form of the r.h.s. as a function of $V_k''$ depends on the choice of the cutoff profile $R_k$. In the next section we will show how to use this result to easily evaluate a large number of beta functions and later we will also explicitly evaluate the r.h.s of (148) for some cutoff.

**Explicit flow equation for $V_k$** One can also write an explicit form for the equation (148). Using the cutoff (42), the $Q$–functional can be evaluated analytically; then the exact functional RG equation for the effective potential becomes

$$\partial_t V_k(\phi) = c_d k^d \frac{1}{1 + V''_k(\phi)/k^2},$$

(150)

where $c_d$ is as in (127). Again, one has to go to dimensionless variables. We define the dimensionless field $\tilde{\phi} = k^{\frac{2-d}{2}} \phi$ and the dimensionless potential

$$\tilde{V}(\tilde{\phi}) = k^{-d} V_k(k^{d/2-1} \tilde{\phi}).$$

(151)

Then we have

$$k \frac{d\tilde{V}_k}{dk} = -d \tilde{V}_k + \left( \frac{d}{2} - 1 \right) \tilde{\phi} \tilde{V}_k' + k^{-d} \partial_t V_k.$$

(152)

Therefore

$$k \frac{d\tilde{V}_k}{dk} = -d \tilde{V}_k + \left( \frac{d}{2} - 1 \right) \tilde{\phi} \tilde{V}_k' + c_d \frac{1}{1 + \tilde{V}_k''},$$

(153)

$$\partial_t v_k = -d v_k + \frac{d-2}{2} \phi v_k' + c_d \frac{1}{1 + v_k''},$$

(154)

This is the equation that determines the flow of the effective potential as a function of $k$. It will play an important role later.

---

18 Use the cutoff $R_k(z) = a(k^2 - z) \theta(k^2 - z)$ and the limits $a = 1$ (Litim) and $a = \infty$ (Morris).
3.2.4 Series expansion and beta functions

There are various ways of studying the flow equation (148). Let us first consider potentials that admit a Taylor expansion of the form

\[ V_k(\phi) = \sum_{n=0}^{\infty} \frac{\lambda_{2n,k}}{(2n)!} \phi^{2n}. \] (155)

We are considering an even potential \( V_k(-\phi) = V_k(\phi) \). The coupling constants can be extracted from the potential by

\[ \lambda_{2n} = \frac{(2n)!}{n!} \left. \frac{\partial^n V_k}{\partial(\phi^2)^n} \right|_{\phi=0}. \] (156)

and the beta functions can be extracted from the “beta functional” (148) by

\[ \beta_{2n} = \frac{(2n)!}{n!} \left. \frac{\partial^n}{\partial(\phi^2)^n} \partial_t V_k \right|_{\phi=0}. \] (157)

This shows that the r.h.s. of (148) is the generating function of beta functions. Explicitly, the first few beta functions are given by

\[
\begin{align*}
\beta_0 &= \frac{1}{(4\pi)^{d/2}} \frac{1}{2} Q_2 \{G_k \partial_t R_k\} \\
\beta_2 &= \frac{1}{(4\pi)^{d/2}} \left\{ -\frac{1}{2} \lambda_4 Q_2 \{G_k^2 \partial_t R_k\} \right\} \\
\beta_4 &= \frac{1}{(4\pi)^{d/2}} \left\{ 3\lambda_2^2 Q_2 \{G_k^2 \partial_t R_k\} - \frac{1}{2} \lambda_6 Q_2 \{G_k^2 \partial_t R_k\} \right\} \\
\beta_6 &= \frac{1}{(4\pi)^{d/2}} \left\{ -45 \lambda_2^4 Q_2 \{G_k^4 \partial_t R_k\} + 15 \lambda_4 \lambda_6 Q_2 \{G_k^2 \partial_t R_k\} - \frac{1}{2} \lambda_8 Q_2 \{G_k^2 \partial_t R_k\} \right\} \\
\beta_8 &= \frac{1}{(4\pi)^{d/2}} \left\{ 1260 \lambda_2^4 Q_2 \{G_k^5 \partial_t R_k\} - 630 \lambda_6 \lambda_4 Q_2 \{G_k^3 \partial_t R_k\} + 28 \lambda_4 \lambda_8 Q_2 \{G_k^3 \partial_t R_k\} \\
&\quad + 35 \lambda_8^2 Q_2 \{G_k^3 \partial_t R_k\} - \frac{1}{2} \lambda_{10} Q_2 \{G_k^2 \partial_t R_k\} \right\}.
\end{align*}
\] (158)

where the regularized propagator is

\[ G_k(z) = \frac{1}{z + R_k(z) + \lambda_2}. \] (159)

Some comments are in order at this point. First we observe that each term in the r.h.s. can be represented as a diagram with \( 2n \) external legs, one loop and an insertion of the cutoff on one of the internal lines. This explicitly confirms that the FRGE has the form of a one loop equation. However, here the internal lines are the full propagators of the theory and the vertices are the full vertices.
Then we note that it is not consistent to truncate the potential to a finite polynomial, except in the quadratic case. Indeed, if all the $\lambda_{2n} = 0$ for $n > 2$, the theory is free and all the beta functions vanish identically. However, as soon as one of the couplings is nonzero, then all the other ones will be turned on by the flow. Thus, considering polynomial potentials is inconsistent in the sense that one ignores terms in the action that have no reason to be small. Nevertheless, for the sake of simplicity we will consider truncations of this type and we shall see a posteriori that they do also yield some useful information.

The value of the $Q$–functionals depends in general on the shape of the cutoff function $R_k$. There are however some $Q$–functionals which are independent of it and hence carry special significance. We will identify them in the next section.

When looking for fixed points, and more generally when doing numerical work, one has to reduce everything to dimensionless variables. We assume that units are such that $\bar{h} = 1$, $\bar{c} = 1$, so that everything has dimension of a power of mass. Then, we take $k$ as unit of mass and we measure everything else in units of $k$. Thus, we define the dimensionless variables

$$\tilde{\lambda}_{2n} = k^{(d-2)n} \lambda_{2n},$$

which are the couplings measured in units of $k$. Their beta functions are

$$\partial_t \tilde{\lambda}_{2n} = ((d-2)n - d) \tilde{\lambda}_{2n} + k^{(d-2)n-2} \beta_{2n}.$$  

Likewise the cutoff can be written as $R_k(z) = k^2 r(y)$, with $y = z/k^2$. The beta functions depend on $Q$–functionals of the general form

$$Q_n[G_k^n \partial_t R_k] = \frac{2}{\Gamma(n+1)} k^{2(n-\ell+1)} \int_0^\infty dy \, y^{n-1} \left( \frac{r(y) - yr'(y)}{(y + r(y) + \bar{\lambda}_2)^\ell} \right),$$

where $\bar{\lambda}_2 = \lambda_2/k^2$. We observe that if $\bar{\lambda}_2 = 0$ and $\ell = n + 1$ the integrand is a total derivative

$$y^{n-1} \frac{r(y) - yr'(y)}{(y + r(y))^{n+1}} = \frac{1}{n} \frac{d}{dy} \left( \frac{y}{y + r(y)} \right)^n.$$  

The general conditions on the cutoff $R_k$ discussed in the preceding chapter imply that $\lim_{y \to \infty} r(y) = 0$ and $r(0) = 1$. Therefore

$$Q_n[G_k^n \partial_t R_k] = \frac{2}{\Gamma(n+1)} \left( \frac{d}{dy} \left( \frac{y}{y + r(y)} \right)^n \right)$$

independently of the shape of the cutoff. These are the only cutoff independent $Q$–functionals. Notice that they are also the only dimensionless ones. All the dimensionful ones depend on the shape of the cutoff and in order to evaluate them we will choose the so-called “optimized cutoff function” (Litim 2001)

$$R_k(z) = (k^2 - z) \theta(k^2 - z); \quad r(y) = (1 - y) \theta(1 - y).$$

It corresponds to the black curve in Figure XXX. This cutoff has the advantage that one can calculate the $Q$–functionals in closed form. In fact, with this cutoff $\partial_t R_k = \ldots$
2k^2\theta(k^2 - z). Since the integrals are all cut off at \( z = k^2 \) by the theta function in the numerator, we can simply use \( P_k(z) = k^2 \) in the integrals. For \( n > 0 \) we find

\[
Q_n[G_k^2 \partial_t R_k] = \frac{2}{\Gamma(n + 1)} \left( \frac{1}{1 + \alpha} \right)^{k^2(n - \ell + 1)}.
\]  

(166)

From this we obtain the beta functions of the dimensionless couplings \( \tilde{\lambda}_{2n} \).

Alternatively, for a potential of the form (155), we have

\[
\bar{V}_k(\phi) = \sum_{n=1}^{N} \frac{\tilde{\lambda}_{2n}}{(2n)!} \phi^{2n}.
\]  

(167)

The beta functions of \( \tilde{\lambda}_{2n} \) can also be obtained directly from this equation by:

\[
\tilde{\beta}_{2n} = \frac{(2n)!}{n!} \frac{1}{\bar{V}(\phi)} \left. \frac{\partial^n \bar{V}_k}{\partial \phi^n} \right|_{\phi = 0}.
\]  

(168)

We obtain

\[
\tilde{\beta}_0 = -d\tilde{\lambda}_0 + c_d \frac{1}{1 + \tilde{\lambda}_2}
\]

\[
\tilde{\beta}_2 = -2\tilde{\lambda}_2 - c_d \frac{\tilde{\lambda}_4}{(1 + \tilde{\lambda}_2)^2}
\]

\[
\tilde{\beta}_4 = (d - 4)\tilde{\lambda}_4 + 6c_d \frac{\tilde{\lambda}_6^2}{(1 + \tilde{\lambda}_2)^3} - c_d \frac{\tilde{\lambda}_6}{(1 + \tilde{\lambda}_2)^2}
\]

\[
\tilde{\beta}_6 = (2d - 6)\tilde{\lambda}_6 - 90c_d \frac{\tilde{\lambda}_4^3}{(1 + \tilde{\lambda}_2)^4} + 30c_d \frac{\tilde{\lambda}_4 \tilde{\lambda}_6}{(1 + \tilde{\lambda}_2)^3} - c_d \frac{\tilde{\lambda}_8}{(1 + \tilde{\lambda}_2)^2}
\]

\[
\tilde{\beta}_8 = (3d - 8)\tilde{\lambda}_8 + 2520c_d \frac{\tilde{\lambda}_4^3}{(1 + \tilde{\lambda}_2)^5} - 1260c_d \frac{\tilde{\lambda}_4 \tilde{\lambda}_6}{(1 + \tilde{\lambda}_2)^4} + 70c_d \frac{\tilde{\lambda}_6^2}{(1 + \tilde{\lambda}_2)^3}
\]

\[
+ 56c_d \frac{\tilde{\lambda}_4 \tilde{\lambda}_6^2}{(1 + \tilde{\lambda}_2)^3} - c_d \frac{\tilde{\lambda}_{10}}{(1 + \tilde{\lambda}_2)^2}
\]

\[
\tilde{\beta}_{10} = \ldots
\]

(169)

where \( c_d \) is the coefficient defined in (127).

We note once again that if \( \tilde{\lambda}_{2n} = 0 \) for \( n > 2 \) all the beta functions vanish except for \( \tilde{\beta}_2 = -2\tilde{\lambda}_2 \), which implies the classical scaling of the squared mass \( \tilde{\lambda}_2 = \lambda_2/k^2 \) with constant \( \lambda_2 \). If one of the \( \tilde{\lambda}_{2n} \) with \( n > 2 \) is nonzero, then all the others will be turned on as soon as one starts integrating the flow equations.

We note the ubiquitous appearance of the denominators \( 1 + \tilde{\lambda}_2 \). These mark the occurrence of a threshold at \( k^2 = \lambda_2 \). In fact, for \( k^2 \gg \lambda_2 \) the denominators become equal to one, whereas for \( k^2 \ll \lambda_2 \) the denominators become large and the beta functions go to zero. This is a manifestation of the phenomenon of decoupling: at energies lower than the mass of a certain particle species, the loops containing that particle are suppressed. In scale-free regularization schemes such as dimensional regularization, threshold phenomena are not seen and have to be accounted for by hand by switching
off the contribution of massive particles below their mass. In the FRGE, thresholds are automatically taken into account.

Finally we may ask which ones of the coefficients in the beta functions are universal. From (164) we see that universal coefficients only occur in even dimensions, in the high energy limit when the mass can be neglected. For example the first term in each beta function is universal in \( d = 2 \) and the last term is universal in \( d = 2n \). However the beta functions generally involve, in addition to some universal term, also non-universal ones. The only fully universal beta function is the one of \( \phi^4 \) theory in \( d = 4 \), in the high energy limit. Indeed, if we put all the couplings to zero except for \( \lambda_4 = \tilde{\lambda}_4 \), we recover the familiar one loop result

\[
\beta_4 = \frac{3\lambda_4^2}{16\pi^2} .
\]

This is one way of understanding why this beta function is so special.

### 3.2.5 The WF fixed point in polynomial truncations

We now focus on the case \( d = 3 \) and give a more accurate description of the Wilson–Fisher fixed point, which we encountered before in section 3.2.2.

One way to improve the approximation is to take into account the effect of higher couplings. This means truncating the potential to a polynomial of higher order. Let us see in detail the result of the truncation \( N = 3 \), where we keep \( \phi^6 \) interactions. The beta functions are

\[
\begin{align*}
\partial_t \tilde{\lambda}_2 &= -2\tilde{\lambda}_2 - \frac{1}{6\pi^2} \frac{\tilde{\lambda}_4}{(1 + 2\tilde{\lambda}_2)^2} \\
\partial_t \tilde{\lambda}_4 &= -\tilde{\lambda}_4 - \frac{1}{6\pi^2} \frac{\tilde{\lambda}_6}{(1 + 2\tilde{\lambda}_2)^2} + \frac{1}{\pi^2} \frac{\tilde{\lambda}_4^2}{(1 + 2\tilde{\lambda}_2)^3} \\
\partial_t \tilde{\lambda}_6 &= \frac{5}{\pi^2} \frac{\tilde{\lambda}_4 \tilde{\lambda}_6}{(1 + 2\tilde{\lambda}_2)^3} - 15 \frac{\tilde{\lambda}_4^3}{\pi^2 (1 + 2\tilde{\lambda}_2)^4} .
\end{align*}
\]

We have now included also the power-counting renormalizable (dimensionless) coupling \( \tilde{\lambda}_6 = \lambda_6 \). There are again two fixed points, the Gaussian one and a nontrivial one at

\[
\tilde{\lambda}_2 = -\frac{1}{7} \approx -0.1429 ; \quad \tilde{\lambda}_4 = \frac{432\pi^2}{343} \approx 12.43 ; \quad \tilde{\lambda}_6 = \frac{93312\pi^4}{16807} \approx 540.8 .
\]

The eigenvalues of the stability matrix at the nontrivial fixed point are \(-1.686, 1.133\) and 12.22. Note that the first two are quite close to the values in the \( N = 2 \) truncation, and the new one is very different from the canonical expectation, which would be zero.

When we proceed to the next truncation, which includes the nonrenormalizable coupling \( \tilde{\lambda}_8 \), something unexpected happens: in addition to the Gaussian fixed point one now finds two nontrivial fixed points, one at

\[
P_1 = (-0.1760, 14.15, 889.3, 82494)
\]

and the other at

\[
P_2 = (-0.01824, 2.082, -92.36, -6720).
\]
The question arises whether both should be regarded as true fixed points of the theory, and if not which one is the true one? This question is nearly impossible to answer with the present methodology. However, one observes that in the two lower truncations \( N = 2 \) and \( N = 3 \) the value of \( \tilde{\lambda}_2 \) went from \(-0.07\) to \(-0.14\) and the value \( \tilde{\lambda}_2 \approx -0.17 \) at \( P_1 \) continues this pattern. Likewise the value of \( \tilde{\lambda}_4 \) went from 7.7 to 12 and the value \( \tilde{\lambda}_4 \approx 14 \) at \( P_1 \) also continues this pattern. The point \( P_1 \) could thus be interpreted as the continuation to \( N = 4 \) of the single nontrivial fixed point found in the lower truncations, whereas \( P_2 \) seems to be unrelated to it. This guess is reinforced by comparing the eigenvalues of the stability matrix, that are \(-1.586, 0.9565, 8.824, 32.52\) at \( P_1 \) and \(-2.008, -1.307, 0.2188, 4.813\) at \( P_2 \). Again, the values at \( P_1 \) fit smoothly with the ones of the lower truncations while those at \( P_2 \) do not. One may thus guess that \( P_1 \) is a true fixed point of the theory, whereas \( P_2 \) may be an artifact of having approximated an entire function by a polynomial.

One can continue this procedure to higher and higher truncations. As the degree of the polynomials grows, one finds more and more fixed points, and it is always possible to identify among them one that is close, both in the values of the \( \tilde{\lambda}_{2n} \) and in the eigenvalues, to the one of the preceding truncation. Table 2.1 give the position of the fixed point for truncations up to \( N = 8 \). From the table one observes that the inclusion of higher order operators in the potential has very little influence on the fixed point values of the lower couplings, in spite of the fact that the new couplings are very large.

The critical exponent \( \nu = -1/\lambda_1 \) is plotted as a function of \( N \) in figure 7.

### 3.2.6 Scaling solution

In the preceding section we have found fixed points for polynomial truncations of the potential. We have done this by solving a finite number of equations of the system (169). Given that we had system of \( N \) equations for \( N \) unknowns, it is not surprising
Fig. 14: First five eigenvalues of the stability matrix as a function of the truncation order. One clearly observe convergence. Only one negative (IR attractive) eigenvalue is present, confirming that the WF fixed point as one control parameter. *Compare with Gaussian values!*

Fig. 15: The critical exponent $\nu$ as a function of the order of the truncation. The LPA predicts a value close to 0.65.
that we have found a number of solutions growing with $N$. One of these had rather stable properties and we tentatively identified it with a FP for the complete system. With this method, however, we will never be able to prove that there exists a FP for the whole potential. Furthermore, we will not be able to prove that the additional solutions of the polynomial truncation are unphysical. We need a better method, which can give us a FP solution for the whole potential.

To motivate what follows, let us consider again the infinite system of flow equations of which (169) are the first five. We see that the beta functions $\beta_{2n}$ depends on all the couplings $\tilde{\lambda}_2, \ldots, \tilde{\lambda}_{2n+2}$, the last one appearing linearly. It is not difficult to prove that this is a general property: just look at the flow equations for the proper–vertices. The flow of $\Gamma^{(n)}$ contains vertices up to $\Gamma^{(n+2)}$ and this last enters linearly.

Now we observe that if we fix an arbitrary value $\tilde{\lambda}_2 = \sigma$, the equation $\tilde{\beta}_2 = 0$ determines $\tilde{\lambda}_4$, then $\tilde{\beta}_4 = 0$ determines $\tilde{\lambda}_6$ as a function of $\tilde{\lambda}_2$ and $\tilde{\lambda}_4$, but since the latter is a function of $\sigma$, also $\tilde{\lambda}_6$ is determined as a function of $\sigma$. Next, $\tilde{\beta}_4 = 0$ determines $\tilde{\lambda}_8$ and so on iteratively. In particular, with the optimized cutoff we find:

$$
\tilde{\lambda}_4^*(\sigma) = -2c_d^{-1}\sigma(1 + \sigma)^2
$$

$$
\tilde{\lambda}_6^*(\sigma) = -2(d - 4)c_d^{-2}\sigma(1 + \sigma)^4 + 24c_d^{-2}\sigma^2(1 + \sigma)^3
$$

$$
\tilde{\lambda}_8^*(\sigma) = ...
$$

(172)

It seems that for any value of $\sigma$ we have a full solution of the infinite system of beta functions. The problem is that most of these Taylor series correspond to functions $\tilde{V}(\tilde{\phi})$ that have a pole at some finite value of $\tilde{\phi}$. The question arises whether this procedure determines one or more functions that are well defined for all values of $\tilde{\phi}$.

With the insight gained so far, we shall now turn to a slightly different method. Instead of trying to determine the fixed point for the Taylor coefficients of the potential, we will try to determine directly the potential itself by solving the fixed point equation.
for the flow (154):
\[ \partial_t \tilde{V}_s(\tilde{\varphi}) = 0 \] (173)

Solutions of this equation will be called scaling solutions: they correspond to RG fixed points in the functional space of effective potentials. We now return to general dimension \( d \).

From (154) we see that a scaling solution satisfies the following ordinary differential equation (ODE):
\[
-d v_s' (\varphi) + \frac{d}{2} - 2 \varphi v_s' (\varphi) + c_d \frac{1}{1 + v_s''(\varphi)} = 0.
\] (174)

The \( \mathbb{Z}_2 \)-symmetry of the effective potential requires that its first derivative vanishes at the origin \( \tilde{V}_s'(0) = 0 \); and thus, at \( \tilde{\varphi} = 0 \), (174) then implies \( \tilde{V}_s(0) = \frac{c_d/\tilde{V}_s''(0)}{1 + \tilde{V}_s''(0)} \). We can set \( \sigma = \tilde{V}_s''(0) \) where the real parameter \( \sigma \) is the convexity of the dimensionless effective potential at the origin. Since equation (174) is a second order non-linear ODE, we need to use numerical methods to solve it\(^{19}\). It’s easy to set up the initial value problem as a function of the parameter \( \sigma \) using the two initial conditions just given.

In accordance with the preceding discussion of the Taylor expansion, one immediately observes that for most values of the parameter \( \sigma \) the solution ends up in a singularity at a finite value of the dimensionless field. Following Morris, for every \( d \) and \( \sigma \) we can call this value \( \tilde{\varphi}_d^s (\sigma) \). Requiring a scaling solution to be well defined for any \( \tilde{\varphi} \in \mathbb{R} \) restricts the admissible initial values of \( \sigma \) to a discrete set \( \{ \sigma_{d,i} \} \) (labeled by \( i \)). One can now construct a numerical plot of the function \( \tilde{\varphi}_d^s (\sigma) \) to find the \( \sigma_{*,i} \) as those values where the function \( \tilde{\varphi}_d^s (\sigma) \) has a “spike”, since a singularity in \( \tilde{\varphi}_d^s (\sigma) \) implies that the relative scaling solution, obtained by integrating the ODE (174), is a well defined function for every \( \tilde{\varphi} \in \mathbb{R} \). For any \( d \), the function \( \tilde{\varphi}_d^s (\sigma) \) gives us a snapshot of theory space, where dimensionless effective potentials are parametrized by \( \sigma \) and where RG fixed-points, i.e. scaling solutions, appear as spikes. By studying \( \tilde{\varphi}_d^s (\sigma) \) we will be able to follow the evolution of universality classes as we vary the dimension.

---

\(^{19}\) All the numerical analysis has been performed employing standard routines for solving ODE present in symbolic manipulation software packages.
A Thermodynamical variables

Here we recall the basic definitions and properties of the thermodynamic potentialss from a purely thermodynamical point of view, without making reference to the statistical interpretation. Consider a simple thermodynamical system whose macroscopic equilibrium state can be described by the mass, volume $V$, internal energy $U$, entropy $S$ and possibly other extensive parameters. The main example will be a fixed mass of a single-component fluid. The first principle of thermodynamics states that $dU = \delta Q - \delta w$, where where $\delta w$ is the infinitesimal work done by the system and $\delta Q$ is the infinitesimal amount of heat entering the system. For reversible processes, $\delta Q = T dS$ and $\delta w = P dV$, so

$$dU = T dS - P dV .$$

(175)

Since $U$, $S$ and $V$ are thermodynamic functions of state, this relation holds also for irreversible processes. From (175) it is natural to think of $U$ as a function of the variables $V$ and $S$, while the intensive variables $T$ and $P$ are given by

$$T = \frac{\partial U}{\partial S} \bigg|_V , \quad P = -\frac{\partial U}{\partial V} \bigg|_S .$$

(176)

An isolated system is one that cannot exchange mass nor heat nor work with the external environment. For an isolated system with fixed mass and internal energy, the second principle of thermodynamics states that the entropy is maximal at equilibrium:

$$\frac{\partial S}{\partial V} \bigg|_U = 0 , \quad \frac{\partial^2 S}{\partial V^2} \bigg|_U < 0 .$$

(177)

A closed system is one that can exchange heat and work with the environment, but not mass. For such a system it is possible to perform reversible transformations that keep the entropy constant. If a closed system has constant entropy, $\frac{\partial S}{\partial U} = 0$, then $\frac{\partial S}{\partial V} = 0$ for any $U$, and therefore the internal energy must be stationary at equilibrium:

$$0 = dS = \frac{\partial S}{\partial U} \bigg|_V dU + \frac{\partial S}{\partial V} \bigg|_U dV = \frac{1}{T} dU + \frac{\partial S}{\partial V} \bigg|_U dV$$

and therefore the internal energy must be stationary at equilibrium:

$$\frac{\partial U}{\partial V} \bigg|_S = -T \frac{\partial S}{\partial V} \bigg|_U = 0 .$$

The last equality follows from the second principle, see Fig. 9. This means that at equilibrium the unconstrained variable $V$ will adjust itself so as to minimize $U$, given the fixed value of $S$. Thus the energy at equilibrium, $U_0$, will be a function of $S$ only.

We consider mostly an ideal fluid, but other thermodynamical systems will be described by other coordinates and generalized forces, as listed in Table 1. This leads to slight reformulations of the first principle. If the system is open, or if the fluid contains more than one component and chemical reactions can happen, one should also add to the r.h.s. of (175) a term $\sum_i \mu_i dN_i$, where $N_i$ is the number of particles of species $i$ and
Fig. 17: Plot of the function $S(V, U)$. At fixed $U$, the entropy reaches a maximum at equilibrium (blue curve). The same point is a minimum of $U(V, S)$ at constant $S$ (red curve).

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<td>Wire</td>
<td>Force</td>
<td>$F$</td>
<td>Length</td>
<td>$L$</td>
<td>$FdL$</td>
</tr>
<tr>
<td>Surface</td>
<td>Surface tension</td>
<td>$S$</td>
<td>Area</td>
<td>$A$</td>
<td>$SdA$</td>
</tr>
<tr>
<td>Dielectric</td>
<td>Electric field</td>
<td>$E$</td>
<td>Polarization</td>
<td>$\vec{P}$</td>
<td>$\vec{E} \cdot d\vec{P}$</td>
</tr>
<tr>
<td>Magnet</td>
<td>Magnetic field</td>
<td>$\vec{H}$</td>
<td>Magnetization</td>
<td>$\vec{M}$</td>
<td>$\vec{H} \cdot d\vec{M}$</td>
</tr>
<tr>
<td>Reversible cell</td>
<td>Electromotive force</td>
<td>$V$</td>
<td>Charge</td>
<td>$Q$</td>
<td>$VdQ$</td>
</tr>
</tbody>
</table>

Tab. 1: Generalized force and generalized coordinate for various thermodynamical systems. The last column gives the formula for the work done on the system when the generalized coordinate is shifted.
μi is its chemical potential. In the following we mostly neglect the last two terms. They can be treated in a similar way to the others.

In the preceding treatment it is natural to think of U as a function of S, V. Since U, V, S, are all extensive, U is a homogeneous function of its argument of degree one. Then, using Euler’s theorem, we can write

\[ U(S, V) = TS - PV . \] (178)

One can define various other thermodynamic potentials, that are useful to study specific types of transformations. For a transformation that occurs at constant pressure, from equation (175) the amount of heat exchanged is \(\delta Q = TdS = d(U + PV)\) and it is therefore useful to introduce the thermodynamic potential

\[ H = U + PV , \] (179)
called the enthalpy. Its variation in a process at constant pressure is equal to the amount of heat absorbed by the body, as can also be seen from the differential relation

\[ dH = TdS + VdP . \]

Likewise the work done in a transformation at constant temperature is \(\delta W = PdV = d(U - TS)\) and it is useful to define

\[ F = U - TS , \] (180)
which is called the Helmholtz free energy. Its differential is given by

\[ dF = -SdT - PdV . \]

Its variation in an isothermal process is equal to the work done on the body.

Finally one can define the Gibbs free energy

\[ G = U - TS + PV = H - TS = F + PV . \] (181)
whose differential is

\[ dG = -SdT + VdP . \] (182)

It is most useful to view these thermodynamic potentials as Legendre transforms of U. The idea is to replace one (or more) of the extensive variables with the corresponding intensive variable. For example, we can replace S by T by solving \(U(V, S)\) and the first relation in (176) to give \(S(V, T)\) and then use this in (180). Then it is natural to think of the Helmholtz free energy as a function of T and V, and

\[ F(T, V) = \max_S (U(S, V) - TS) . \] (183)

In this definition T has to be regarded a priori as an independent variable, and the maximization with respect to S tells us that T has to be identified with temperature, as defined in (176).
As already mentioned, the Helmholtz free energy is useful when studying isothermal processes. Recall from the principle of minimum energy, that for processes at constant entropy \( \bar{S} \), the energy at equilibrium is

\[
U_0(\bar{S}) = \min_V U(\bar{S}, V) .
\]

Thus the free energy at equilibrium is

\[
F_0 = \max_{\bar{S}} (U_0(\bar{S}) - T_0 \bar{S})
= \max_{\bar{S}} (\min_V U(\bar{S}, V) - T_0 \bar{S})
= \min_V \max_{\bar{S}} (U(\bar{S}, V) - T_0 \bar{S}) = \min_V F(T_0, V) .
\]

In this calculation \( T_0 \) denotes the temperature at equilibrium and we assumed that the order of extremizations can be interchanged. We have found that for a system in equilibrium that is in contact with a heat bath (and hence at constant temperature \( T \)) the volume will be such as to minimize \( F \), for given \( T \). If one starts away from equilibrium, the volume will change so as to decrease \( F \), and \( F \) reaches a minimum at equilibrium. \(^{20}\) It is this that makes \( F \) especially useful when one studies isothermal processes: to find the equilibrium state one has to minimize \( F \) with respect to a single variable \( V \), whereas using \( U \) one would have to vary both \( V \) and \( S \).

In a similar way we can define the enthalpy as the Legendre transform of \( U \), replacing \( V \) by its conjugate intensive variable \( P \):

\[
H(S, P) = \max_V (U(S, V) + PV) .
\]

In this definition \( P \) has to be regarded a priori as an independent variable, and the maximization with respect to \( V \) tells us that \( P \) has to be identified with pressure, as defined in (176). An argument similar to the one given above for \( F \) shows that for a system at constant pressure the enthalpy decreases and reaches a minimum at equilibrium.

The Gibbs free energy is the Legendre transform of \( U \) with respect to \( V \) and \( S \) and therefore its natural arguments are \( T \) and \( P \). Since the equilibrium state of a single component fluid is completely determined by \( T \) and \( P \), the Gibbs free energy is not a very useful variable in this case. Let us consider instead the case when the fluid has various components with chemical potentials \( \mu_i \), and chemical reactions can happen. Then we have

\[
dU = TdS - PdV + \sum_i \mu_idN_i
\]

and

\[
G(S, P, N_i) = \max_S \max_V (U(S, V, N_i) - TS + PV) .
\]

Its differential is

\[
dG = -SdT + VdP + \sum_i \mu_idN_i .
\]

\(^{20}\) For other systems described by other thermodynamic coordinates, it will be the some other unconstrained extensive coordinate that will adjust itself to minimize \( F \).
In this case it is found that for a transformation occurring at constant temperature and pressure, the $N_i$ will adjust so as to minimize $G$. If one starts away from equilibrium, $G$ will decrease and reaches a minimum at equilibrium. Since many chemical reactions occur at constant temperature and pressure, this is especially useful to determine the equilibrium state of a multicomponent fluid.

We often discuss magnetic systems. For such systems the total mass and volume are usually fixed, so we can think of $U$ as a function of the extensive variables $S$ and $\vec{M}$:

$$dU = TdS + \vec{H} \cdot d\vec{M}.$$  

(188)

The free energy for such a material is

$$F(T, \vec{M}) = \max_S (U - TS)$$  

(189)

and differentiating

$$dF = -SdT + \vec{H} \cdot d\vec{M}.$$  

For a system at fixed temperature the magnetization at equilibrium corresponds to a minimum of $F$.

For a system with a large but finite number of degrees of freedom at temperature $T$, the partition function is

$$Z \equiv e^W = \sum_r e^{-\beta E_r}$$  

(190)

where $E_r$ is the energy of the state $r$ and $\beta = 1/kT$. The “internal energy” of the system is

$$U = \langle E \rangle = -\frac{\partial}{\partial \beta} \log Z.$$  

(191)

To find the relation between $F$ and $Z$ let $j$ be an external source and $X$ the corresponding generalized force, such that the energy contains a term $-jX$. In this way $Z$ and $W$ become functions of $j$. The thermal average of $X$ is $\langle X \rangle = \frac{1}{\beta} \frac{\partial W}{\partial j}$. Then

$$dW = \frac{\partial W}{\partial \beta} d\beta + \frac{\partial W}{\partial j} dj = -Ud\beta + \beta Xdj = -d(\beta U) + \beta dU + \beta Xdj$$

Thus

$$dU = \frac{1}{\beta} d(W + \beta U) - Xdj.$$  

On the other hand in the thermodynamic limit

$$dU = TdS - Xdj,$$  

so

$$S = k \log Z + \frac{U}{T} + \text{const}.$$  

The constant can be seen to be zero by taking the limit $T \to 0$. In this case, denoting $r = 0$ the ground state and $\Omega_0$ its degeneracy, $Z = \Omega_0 e^{-\beta U_0}$, and $S = k \log \Omega_0$. Then, $S = k \log Z + U/T$. 

So we see that $U - TS = -kTW$, or

$$F = -\frac{1}{\beta} \log Z.$$  \hfill (192)

### B Functional integral basics

In the preceding appendix we have considered only systems at equilibrium. Here we take into account the effect of thermal fluctuations. It is remarkable that the same formalism applies also to quantum fluctuations.

In this appendix we recall the basic notions of functional integration, mainly to set up our notation.

The Euclidean functional integral for a scalar field $\phi$ coupled linearly to an external source $J$ is

$$Z[J] \equiv e^{W[J]} = N \int (d\phi) e^{-S[\phi]} + \int J \phi.$$  \hfill (193)

Here $S$ is the classical or bare action and $N$ is a normalization constant (independent of $J$) that will be discussed later.\(^{21}\) The formalism can be generalized quite straightforwardly to multiplets of bosonic or fermionic fields, and also to theories with gauge invariances.

The functional integral (193) is directly relevant in statistical physics where $Z[0]$ is called the partition function. The functional $Z[J]$ is the generating functional of the correlation functions of the theory, and $W = \log Z$ is the generating functional of the connected correlation functions. In particular the expectation value of the field is

$$\phi(x) \equiv \langle \phi(x) \rangle = \frac{\int (d\phi) \phi(x) e^{-S[\phi]}}{\int (d\phi) e^{-S[\phi]}} = \frac{\delta W}{\delta f(x)} \bigg|_{J=0}$$  \hfill (194)

and the connected two-point function or propagator is

$$G(x, y) \equiv \langle \phi(x) \phi(y) \rangle - \varphi(x) \varphi(y) = \frac{\delta^2 W}{\delta f(x) \delta f(y)} \bigg|_{J=0}.$$  \hfill (195)

The connected $n$-point function $G(x_1, \ldots, x_n)$ (we omit a traditional subscript “$c$” because we will not consider disconnected $n$-point functions) is\(^{22}\)

$$G(x_1, \ldots, x_n) = \frac{\delta^n W}{\delta f(x_1) \cdots \delta f(x_n)} \bigg|_{J=0}. \hfill (196)$$

---

\(^{21}\) We will write $\int_x \equiv \int d^d x$ for spacetime integration and $\int_q \equiv \int \frac{d^d q}{(2\pi)^d}$ for momentum space integration.

\(^{22}\) When one thinks of the dimensional analysis of this and similar formulae, one has to bear in mind that the functional derivative with respect to a variable $j(x)$ does not simply have the dimension of the inverse of $j$. This is because of the fundamental rule $\frac{\delta (f(x))}{\delta j(y)} = \delta^d (x - y)$, and the fact that $\int d^d x \delta(x - y) = 1$. Thus the dimension of $\frac{\delta (f(x))}{\delta j(y)}$ is equal to minus the dimension of $j$ minus $d$. In this specific example this gives $-1 - 3d/2$. Since the functionals $S$, $Z$, $W$, $\Gamma$ are dimensionless, one finds that $\varphi$ has the same dimension as $\phi$, $G$ has the dimension of $\phi^2$ etc.
If we do not set the source to zero in (194), the expectation value of the field will depend on the source, and we can define
\[ \phi_J(x) \equiv \langle \phi(x) \rangle_J = \frac{\delta W}{\delta J(x)} . \] (197)

This functional dependence of \( \phi \) on \( J \) can be inverted, in the sense that for any given function \( \phi \), one can find a source \( J_\phi \) such that the expectation value of \( \phi \) in the presence of the source is \( \phi \). The Effective Action (EA) is the Legendre transform of \( W \):
\[ \Gamma[\phi] = -W[J_\phi] + \int J_\phi \phi . \] (198)

From here, varying with respect to \( \phi \) we obtain
\[ \frac{\delta \Gamma}{\delta \phi} = -\int \frac{\delta W}{\delta J} \frac{\delta J}{\delta \phi} + \frac{\delta J}{\delta \phi} \phi + J = J . \] (199)

At zero source we obtain the equation
\[ \frac{\delta \Gamma}{\delta \phi} = 0 . \] (200)

This equation is called the quantum action principle and it shows that the effective action plays in the quantum theory a role that is similar to the action in the classical theory. Solutions \( \phi_* \) to (200) are the vacuum states of the theory and also represent the on-shell average field.

Inserting (198) and (199) in (193) one finds
\[ e^{-\Gamma[\phi]} = N \int (d\phi) e^{-S[\phi]} + \int \frac{\delta \Gamma}{\delta \phi} \phi - \phi . \] (201)

Assuming that the functional measure is translation invariant one can shift the integration variable to \( \chi = \phi - \phi \). The shifted quantum field \( \chi \) is characterized by having vanishing expectation value. Using this, (201) can be written equivalently in the form
\[ e^{-\Gamma[\phi]} = N \int (d\chi) e^{-S[\phi+\chi]} + \int \frac{\delta \Gamma}{\delta \phi} \chi . \] (202)

This is an integro-differential equation that can be used, in principle, to compute the effective action. If we set equation (202) on-shell we obtain that the on-shell effective action is the logarithm of the partition function \( \Gamma[\phi_*] = \log Z[0] \), i.e. it is the free energy.

Comparing the equations
\[ \frac{\delta^2 \Gamma}{\delta \phi \delta \phi} = \frac{\delta J}{\delta \phi} , \quad \frac{\delta^2 W}{\delta J \delta J} = \frac{\delta \phi}{\delta J} , \] (203)

one concludes that the second variation of the effective action is the inverse propagator
\[ \frac{\delta^2 \Gamma}{\delta \phi \delta \phi} = \left( \frac{\delta^2 W}{\delta J \delta J} \right)^{-1} . \] (204)
More generally, the on-shell $\Gamma$ is the generating functional of one-particle-irreducible (1PI) correlators
\begin{equation}
\frac{\delta^n \Gamma}{\delta \varphi(x_1) \ldots \delta \varphi(x_n)} = \Gamma^{(n)}(x_1, \ldots, x_n) .
\end{equation}

The usefulness of this functional is due to the fact that the perturbative expansion of a correlation function calculated with the action $S$ is equivalent to the perturbative expansion containing only tree level diagrams calculated with the action $\Gamma$. It is therefore a very efficient way of summarizing the information about the quantum theory.

We conclude with the following remark. If the source $j$ is constant, also the expectation value of the field will be constant. We can then define the ordinary derivative $\frac{\partial \varphi}{\partial j}$. This is related to the two-point function as follows. Since
\begin{equation}
\frac{\partial \varphi(x)}{\partial j} = \int d^d y \frac{\delta \varphi(x)}{\delta j(y)} = \int d^d y G_c(x, y) .
\end{equation}

By translation invariance, both sides of this relation must be independent of $x$, so
\begin{equation}
\frac{\partial \varphi}{\partial j} = \int d^d y G_c(y) .
\end{equation}