RADIATIVE PROCESSES IN ASTROPHYSICS

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Welcome!

• Hallo, I hope everyone is feeling fine! I am glad to meet you!
• This is my first attempt to use slides in my lectures: please, interrupt me if something is unclear!
• **Prerequisites**: basics of Quantum Mechanics, Special Relativity.
• I wish you'll learn a lot. Many exercises will be done together, at the blackboard. Some will be assigned as «homework»: working together is encouraged.
• Attendance is mandatory.
• At the end of the course, a **written exam** asked to assign you 4 credits.
• **ENJOY!**
Summary

Part I. Continuum radiation
• Fundamentals of radiative transfer
• Black Body and thermal radiation
• Statistical mechanics and thermodynamic equilibrium, Part 1
• Thermal radiation from cosmic dust
• Bremsstrahlung
• Synchrotron
• Scattering (diffusion processes)

Part II. Line radiation
• Statistical mechanics and thermodynamic equilibrium, Part 2
• Atomic and molecular structure
• Einstein coefficients (selection rules, transition rates, line profiles and curve of growth)
• Radiative transitions (Bound bound, Bound free transition)
• Collisional excitation and critical density
Textbooks

Fundamentals of radiative transfer

- The electromagnetic spectrum
- The panchromatic view
- Astrophysical observables
- Emission and absorption coefficients
- Radiative transfer equation

Main ref.: Rybicki, G., and Lightman, A., 1979, Radiation processes in astrophysics, John Wiley and Sons, Ch.1.
The electromagnetic spectrum

Light is characterized by:

- Wavelength $\lambda$
- Frequency $\nu$
- Energy $E$
- Temperature $T$

They are related by:

$$\lambda \nu = c$$
$$E = h \nu$$
$$T = E/k$$

Fundamental constants:

- $c \approx 2.998 \times 10^{10}$ cm s$^{-1}$
- $h \approx 6.626 \times 10^{-27}$ erg s
- $k \approx 1.381 \times 10^{-16}$ erg K$^{-1}$

Useful units

- in CGS: 1 erg = 1 g cm$^2$ s$^{-2}$
- 1 eV = $1.6 \times 10^{-12}$ erg
- 1 erg = $10^{-7}$ Joules
- 1 eV = $1.6 \times 10^{-19}$ J
The electromagnetic spectrum

X-rays with high photon energies above 5–10 keV (below 0.2–0.1 nm wavelength) are called hard X-rays, while those with lower energy (and longer wavelength) are called soft X-ray.
Atmospheric opacity

The atmosphere is mostly transparent (0% opacity) in the visible band (380-750 nm) and mostly opaque in the IR.

Radio window: 10 MHz ($\lambda \approx 30$ m) to 1 THz ($\lambda \approx 0.3$ mm) in optimal terrestrial observation sites.

IR window: 8-14 μm, depending on water vapor absorption.
The electromagnetic spectrum

- Radiation is produced in astrophysical sources by many processes: blackbody emission, bremsstrahlung, synchrotron, Compton scattering, as well as line emission from atoms and molecules.

<table>
<thead>
<tr>
<th>Region</th>
<th>Wavelength (Å)</th>
<th>Wavelength (cm)</th>
<th>Frequency (Hz)</th>
<th>Energy (eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radio</td>
<td>$&gt; 10^9$</td>
<td>$&gt; 10$</td>
<td>$&lt; 3 \times 10^9$</td>
<td>$&lt; 10^{-5}$</td>
</tr>
<tr>
<td>Microwave</td>
<td>$10^9 - 10^6$</td>
<td>10 – 0.01</td>
<td>$3 \times 10^9 - 3 \times 10^{12}$</td>
<td>$10^{-5} - 0.01$</td>
</tr>
<tr>
<td>Infrared</td>
<td>$10^6 - 7000$</td>
<td>$0.01 - 7 \times 10^{-5}$</td>
<td>$3 \times 10^{12} - 4.3 \times 10^{14}$</td>
<td>$0.01 - 2$</td>
</tr>
<tr>
<td>Visible</td>
<td>7000 – 4000</td>
<td>$7 \times 10^{-5} - 4 \times 10^{-5}$</td>
<td>$4.3 \times 10^{14} - 7.5 \times 10^{14}$</td>
<td>$2 - 3$</td>
</tr>
<tr>
<td>Ultraviolet</td>
<td>4000 – 10</td>
<td>$4 \times 10^{-5} - 10^{-7}$</td>
<td>$7.5 \times 10^{14} - 3 \times 10^{17}$</td>
<td>$3 - 10^3$</td>
</tr>
<tr>
<td>X-Ray</td>
<td>10 – 0.1</td>
<td>$10^{-7} - 10^{-9}$</td>
<td>$3 \times 10^{17} - 3 \times 10^{19}$</td>
<td>$10^3 - 10^5$</td>
</tr>
<tr>
<td>Gamma Ray</td>
<td>$&lt; 0.1$</td>
<td>$&lt; 10^{-9}$</td>
<td>$&gt; 3 \times 10^{19}$</td>
<td>$&gt; 10^5$</td>
</tr>
</tbody>
</table>

The panchromatic view

Multiwavelength images of M31, the Andromeda Galaxy. Quite clearly, different wavelengths reveal various details that are unseen in visible light alone. Planck Mission Team / NASA / ESA
The panchromatic view

Centaurus A galaxy. By including information from different instruments and wavebands, we can obtain a much more complete image of what we are observing.
The radio sky

The sky at 408 Mhz, mapped at Jodrell Bank radio telescope. Nearly all of the radio emission from normal galaxies is synchrotron radiation from relativistic electrons and free-free emission from HII regions.
The radio sky

Hercules-A black hole jets captured in a high-resolution image by LOFAR radiotelescope, 10–240 MHz

Giant radiogalaxies found with MeerKAT radiotelescope, 544 MHz-3.5 GhZ

The two giant radio galaxies found with the MeerKAT telescope. In the background is the sky as seen in optical light. Overlaid in red is the radio light from the enormous radio galaxies, as seen by MeerKAT. Left: MGTC J095959.63+024608.6. Right: MGTC J100116.84+031339.0. Credit: I. Heywood (Oxford/Rhodes/SARAO)
The radio sky

Spoiler alert! The radio emission is due to the synchrotron process. The observed structure in radio emission is determined by the interaction between twin jets and the external medium, modified by the effects of relativistic beaming.

Beyond the central point source, there is X-ray emission extending for hundreds of kiloparsecs, and matching the radio emission almost perfectly. The radio-emitting electrons in the jet are also giving some of their energy to photons of the Cosmic Microwave Background, turning them into the X-rays observed by Chandra. (Inverse Compton effect)
The microwave sky - All-sky image of the microwave sky from Planck satellite, covering the electromagnetic spectrum from 30 GHz to 857 GHz.
The microwave sky - All-sky image of the microwave sky from Planck satellite, covering the electromagnetic spectrum from 30 GHz to 857 GHz.
Rotational transition lines of CO are one of the major tracers used to study star forming regions and Galactic structures. A large number of observations of CO rotational lines are covering the galactic plane.
The mm and submm sky - ALMA array between 35 GHz (68 mm) and 125-163 GHz (1.8-2.4 mm)

ALMA identified faint galaxies that are not seen with the Hubble Space Telescope’s deepest view of the Universe 10 billion light-years away. Here, a comparison of Hubble and ALMA observations.

Composite image of the young star HL Tauri (Hubble space Telescope) and its surrounding region, a protoplanetary disk (Atacama Large Millimeter/submillimeter Array)
The infrared and visible sky

Reflected starlight and dust associated to the birth of stars in our Galaxy

Stars and dust in the spiral arms of external galaxies
The **Trifid Nebula** is an HII region. It is a combination of an open cluster of stars, an emission nebula (the relatively dense, reddish-pink portion), a reflection nebula (the mainly blue portion), and a dark nebula (the apparent 'gaps' in the former that cause the trifurcated appearance, also designated **Barnard 85**).

The **Helix Nebula**. Planetary nebula, region of dust and gas from the cast-off layers of a dying star. Oxygen atoms excited by the UV of the central star. Forbidden lines @500 nm of OIII is the blue-green Red color is due to H-alpha ad Nitrogen.
H-alpha line at 656 nm (optical, red)

Horsehead nebula in Hα

Lagoon nebula in Hα

Emission nebulae called HII regions

Andromeda in Hα

Hα is a Balmer series line. When the electrons and protons recombine in a HII region, they generally recombine to upper energy levels (large n), and then cascade down, emitting Balmer lines.
H-alpha line at 656 nm (optical, red)

The distribution of ionized hydrogen, HII in the parts of the Galactic interstellar medium visible from the Earth’s northern hemisphere, as observed with the Wisconsin Hα Mapper (Haffner et al. 2003).
The Visible/UV Sky

NASA's Galaxy Evolution Explorer (GALEX) satellite UV image of Andromeda. Star formation in Spiral arms.
**The X-ray Sky** eROSITA X-ray telescope onboard the Russian-German SRG spacecraft: 0.5-11 keV (soft and hard)

- Blue: highest energies, followed by green, yellow and red.
- The red foreground glow comes from hot gas near the solar system.
- Milky Way plane: gas and dust in the disk absorb all but the most energetic X-rays.
- Supernova remnant: Cassiopeia A and Vela, and a star system called Scorpius X-1.
- Orion Nebula (Star forming region), Cygnus Superbubble.
- A mysterious arc of X-rays called the North Polar Spur.
- Extra galactic, LMC and Shapley Supercluster.
- Supermassive black holes accreting in the centers of other galaxies.

credits: https://www.sciencenews.org/article/xray-map-sky-erosita-telescope-milky-way
credits: https://svs.gsfc.nasa.gov/11545
Gamma rays released from supernovae
Material falling into supermassive black holes accounts for the majority of gamma ray sources detected by Fermi.
Gamma rays emitted from pulsars
Radiation is produced in astrophysical sources by many processes: blackbody emission, bremsstrahlung, synchrotron, Compton scattering, as well as line emission from atoms and molecules.

### Astrophysical radiation emission processes

<table>
<thead>
<tr>
<th>Region</th>
<th>Wavelength (Å)</th>
<th>Main Sources</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radio</td>
<td>$&gt; 10^9$</td>
<td>Supernova remnants, galaxies, galactic center, HII regions, quasars</td>
</tr>
<tr>
<td>Microwave</td>
<td>$10^6 - 10^6$</td>
<td>Sun, Crab nebula, microwave background, Milky Way, radio galaxies, quasars</td>
</tr>
<tr>
<td>Infrared</td>
<td>$10^6 - 7000$</td>
<td>Sun, planets, Galactic center, nebulae, dust, quasars, active galaxies</td>
</tr>
<tr>
<td>Visible</td>
<td>7000 – 4000</td>
<td>Sun, planets, stars, galaxies, quasars</td>
</tr>
<tr>
<td>Ultraviolet</td>
<td>4000 – 10</td>
<td>Sun, hot stars, active galaxies, quasars</td>
</tr>
<tr>
<td>X-Ray</td>
<td>10 – 0.1</td>
<td>Sun, compact binaries, black holes, hot gas in galaxy clusters, AGN, quasars</td>
</tr>
<tr>
<td>Gamma Ray</td>
<td>$&lt; 0.1$</td>
<td>crab pulsar, vela pulsar, galactic disk, matter annihilation, extragalactic sources(?)</td>
</tr>
</tbody>
</table>
Astrophysical observables

Energy flux.

When the scale of a system is much larger than the radiation wavelength, we can consider radiation to travel in straight lines (rays). The energy flux $F$ of the radiation represents the energy passing through a given area in the unit time:

\[ dE = F \, dA \, dt \]

The flux is a measurable quantity (the energy carried out by the intercepted photons).

An isotropic source is a source emitting equal amounts of energy in all directions. Energy conservation implies that the flux through two shells around an isotropic source is the same (inverse square law):

\[ 4 \pi r_1^2 \, F(r_1) = 4 \pi r_2^2 \, F(r_2) \]

\[ F(r) = \frac{\text{constant}}{r^2} \]

CGS Flux units: \(\text{erg s}^{-1} \, \text{cm}^2\).
Astrophysical observables

Specific intensity.

The flux is a measure of the energy carried by all rays passing through a given area \( A \). More information lies in the energy carried along a specific direction of an individual ray or, better, the energy carried by a beam.

The energy crossing \( dA \) in the time \( dt \) and in frequency range \( dv \), carried by the beam (all the rays whose direction lies within the solid angle \( d\Omega \) with respect to the normal), is given by:

\[
\text{dE} = I_\nu \text{dA} \text{dt} \text{d}\Omega \text{dv}
\]

\((I_\nu \text{ is a } I_\nu \text{ (n), it has units in cgs: erg cm}^{-2} \text{ s}^{-1} \text{ sr}^{-1} \text{ Hz}^{-1})\)

\(I_\nu\) is the specific intensity or brightness. It is an intrinsic property of the radiation, and depends on location in space, on direction and on frequency, but is constant along the line of sight (if in free space).

- In an isotropic radiation field, \( I_\nu = \text{const} \) for all directions.
- Relation between intensity and flux collected by an observer in a given direction (normal to the surface):

\[
F_\nu = \int I_\nu \cos \theta \; d\Omega_{\text{obs}}
\]

CGS \( F_\nu \) units: erg s\(^{-1}\) Hz\(^{-1}\) cm\(^{-2}\).

Assuming rays arrive \( \perp \), \( \cos \theta = 1 \), and, in solid angles in which the specific intensity is not varying, \( F_\nu = I_\nu \Delta \Omega_{\text{obs}} \)
**Astrophysical observables**

**Flux at the surface of a uniform brightness sphere**

Calculate the flux at an arbitrary distance from a sphere of uniform brightness (or *specific intensity*) $I_{ν}$.

All rays leaving the sphere have the same brightness. This is an isotropic source.

The (specific) intensity is conserved along rays in free space, for energy conservation.

At the observer's position P (distance $r$ from source), the intensity is $B$ if the ray intersects the sphere, and zero otherwise:

$$F_{ν}(r) = \int I_{ν} \cos \theta \, dΩ_{obs} = I_{ν} \int_0^{2π} dφ \int_0^{θ_{c}} \sin \theta \cos \theta \, dθ$$

units: erg s⁻¹ Hz⁻¹ cm⁻²

where $R = r \sin θ_{c}$, so $θ_{c}$ is the angle at which a ray from P is tangent to the sphere.

The energy collected in 1 cm² of the *observer* area, per unit frequency, is

$$F_{ν}(r) = \pi I_{ν} (1-\cos²θ_{c}) = \pi I_{ν} \sin²θ_{c} \quad \text{or} \quad F_{ν}(r) = \pi I_{ν} (\frac{R}{r})^{2}$$

Setting $r = R$, we get the flux *emerging from the unit surface area.*

$$F_{ν, \text{surface}} = \pi I_{ν} \quad \text{intrinsic source property!}$$

The flux collected by the unit area at the observer position is then:

$$F_{ν}(r) = F_{ν, \text{surface}} \left(\frac{R}{r}\right)^{2}$$
Luminosity of a source

The flux arising from the surface of a source is not anymore dependent on the observer or on the observing distance. It is an intrinsic property of the source, which allows to define another intrinsic source property: its luminosity.

Remember: at the surface of the emitting source,

\[ F_{\nu, \text{surface}} = \pi I_\nu \quad \text{erg s}^{-1} \text{ Hz}^{-1} \text{ cm}^{-2} \]

Similarly, integrating over the whole spectrum,

\[ F_{\text{surface}} = \int F_\nu \, d\nu = \pi \int I_\nu \, d\nu = \pi I \quad \text{erg s}^{-1} \text{ cm}^{-2} \quad \text{(Watt m}^{-2} \text{ in the ISU)} \]

Assuming the source is a sphere of radius \( R \), homogeneously emitting, we define its luminosity as the total (=solid angle integrated) energy radiating from its surface per unit frequency interval:

\[ L_\nu = 4\pi R^2 F_{\nu, \text{surface}} \quad \text{in CGS,} \quad \text{erg s}^{-1} \quad \text{(Watt , in the ISU)} \]

An observer at distance \( r \) from the source, can infer the intrinsic source luminosity from the observed flux:

\[ F_\nu(r) = F_{\nu, \text{surface}} \left( \frac{R}{r} \right)^2 = \frac{L_\nu}{4\pi R^2} \left( \frac{R^2}{r^2} \right) = \frac{L_\nu}{4\pi r^2} \]

Integrating over the frequency spectrum, \( F(r) = \frac{L}{4\pi r^2} \) = power collected from the unit area at the observer distance.

**Problem 1.** Knowing that the bolometric solar luminosity is \( L_\odot = 3.832 \times 10^{26} \) W, the Sun radius is \( 6.96 \times 10^5 \) km, and the distance Sun-Earth is about 93 million kilometers, calculate (in ISU):

1) the flux received by a square meter of area on Earth.
2) the flux emitted by a square meter of the Sun's surface. How many LED light bulbs (typical wattage, 10 Watts) would you need to get the same flux?
Radiative energy density

For a certain direction, and given volume element, the energy density is defined via

\[ dE = u_v(\Omega) \, dV \, d\Omega \, dv \]

where the volume element for light is \( dV = c \, dA \, dt \). It follows that

\[ u_v(\Omega) = \frac{I_v(\Omega)}{c} \]

\( u_v(\Omega) \) units in cgs: erg cm\(^{-3}\) sr\(^{-1}\) Hz\(^{-1}\)

The total energy density at a given frequency is

\[ u_v = \int u_v(\Omega) \, d\Omega = \frac{1}{c} \int I_v(\Omega) \, d\Omega \equiv \frac{4\pi}{c} J_v \]

\( u_v \) units in cgs: erg cm\(^{-3}\) Hz\(^{-1}\)

where \( J_v \) represents the mean intensity of the radiation field:

\[ J_v = \frac{1}{4\pi} \int I_v(\Omega) \, d\Omega \]

For an isotropic field, \( J_v = I_v \)

Also \( J_v \) has units (in cgs) of erg cm\(^{-2}\) s\(^{-1}\) sr\(^{-1}\) Hz\(^{-1}\).

The total emission is simply

\[ u = \int u_v \, dv = \frac{4\pi}{c} \int J_v \, dv \]

cgs units: erg cm\(^{-3}\)
Glossary of observational quantities

• Spectral/Monochromatic Luminosity (for isotropically emitting source) :
  \[ (*) \quad L_\nu = 4 \pi D^2 F_\nu (D) \quad [ \text{W Hz}^{-1}] / [\text{erg s}^{-1} \text{Hz}^{-1}] \]

• Power (Bolometric / Absolute Luminosity) :  \[ L = \int_0^\infty L_\nu \, d\nu \quad [\text{W}] / [\text{erg s}^{-1}] \]

• Flux density: \[ F_\nu (D) = L_\nu / 4 \pi D^2 \quad [\text{W Hz}^{-1} \text{m}^{-2}] / [\text{erg s}^{-1} \text{Hz}^{-1} \text{cm}^{-2}] \]
  \[ 1 \text{ Jy} = 10^{-26} \text{W Hz}^{-1} \text{m}^{-2} \]

• (\( I_\nu \equiv B_\nu \) at the source surface, surface Brightness) \[ B_\nu = F_\nu \, (\text{Source radius}) \Delta \Omega \quad [\text{W Hz}^{-1} \text{m}^{-2} \text{ster}^{-1}] / [\text{erg s}^{-1} \text{Hz}^{-1} \text{cm}^{-2} \text{ster}^{-1}] \]
  (That is \( I_\nu \), assuming \( \cos \Theta = 1 \), rays arriving perpendicular to the collecting g area).
  (Flux density per unit solid angle).

• Specific emissivity \[ \varepsilon_\nu = \frac{L_\nu}{dV} \quad [\text{W Hz}^{-1} \text{m}^{-3}] \ [\text{erg s}^{-1} \text{cm}^{-3} \text{Hz}^{-1}] \]

(*) If we consider a star as the source of radiation, the flux emitted by the star into a solid angle \( \omega \) is \( L = \omega D^2 F \), where \( F \) is the flux density observed at a distance \( D \) from the star (it is also usual to refer to the total flux from a star as the luminosity, \( L \)). If the star radiates isotropically then radiation at a distance \( D \) will be distributed evenly on a spherical surface of area \( 4\pi D^2 \) and hence \( L = 4\pi D^2 F \).

For an extended luminous object such as a nebula or galaxy, we use the surface brightness, i.e. the flux density per unit solid angle.
Radiative transfer

If a ray passes through matter, energy may be added or subtracted from it by emission or absorption, and the specific intensity will not in general remain constant. Also «scattering» can change the beam intensity, by deflecting photons off the beam or adding photons to it.

Spontaneous emission.

The spontaneous emission coefficient $j_ν$ is defined as the energy emitted per unit time per unit solid angle, unit frequency and per unit volume:

$$dE = j_ν dV dΩ dt dν$$

where $j_ν$ has units of erg s$^{-1}$ cm$^{-3}$ Hz$^{-1}$ ster$^{-1}$

Remember the definition of specific intensity, $dE = I_ν dA dΩ dt dν$; in going a distance $ds$, a beam of cross section $dA$ travels through a volume $dV = dA ds$, thus adding intensity to the beam: $dI_ν = j_ν ds$

Also often used, the emissivity, defined as energy spontaneously emitted per unit time, unit frequency and unit volume:

$$dE = ε_ν dV dt dν \rightarrow ε_ν = \int j_ν dΩ$$

and if emission is isotropic, $j_ν = \frac{ε_ν}{4π}$; $ε_ν$ has units of erg s$^{-1}$ cm$^{-3}$ Hz$^{-1}$
Absorption

Radiation can also be absorbed. Consider a medium with given particle (microscopical absorbers) density \( n \) (cm\(^{-3}\)), with each particle having a given microscopic absorbing area \( \sigma_v \) (cm\(^2\)).

The total absorbing area is: \( n \sigma_v \text{d}A \text{d}s \).

The energy absorbed out of per unit the beam solid angle per unit crossed area, per unit time and frequency is

\[
dE = -dI_v \text{d}A \text{d}\Omega \text{d}v = -I_v n \sigma_v \text{d}A \text{d}\Omega \text{d}v \Rightarrow \text{d}I_v = I_v n \sigma_v \text{d}s
\]

where \( \alpha_v \equiv n \sigma_v \) is the absorption coefficient. In cgs units, it is in cm\(^{-1}\).

The energy subtracted from the beam is \( dE = -\alpha_v I_v \text{d}s \text{d}A \text{d}\Omega \text{d}v \) out of the beam.

Often used, the mass absorption coefficient, or opacity \( \kappa_v \), given by \( \alpha_v = \rho \kappa_v \), with \( \rho \) the mass density. The opacity has units cm\(^{-2}\) g.

Thus, obviously \( \frac{\rho}{n} = \frac{\sigma_v}{\kappa_v} \) \( \mu = \text{mass per absorber} = \frac{\sigma_v}{\kappa_v} \).

We define the mean free path as the mean physical distance traveled in a homogeneous medium, \( \ell_v = \frac{1}{\alpha_v} \).

Note: this is a phenomenological law. We'll see later on the course that the absorption coefficient has a deeply quantistic nature.
Equation of radiative transfer

In presence of emission of absorption, the energy is no more conserved, and, neglecting scattering,

\[
\frac{dI_\nu}{ds} = - \alpha_\nu I_\nu + j_\nu
\]

• For pure emission, \( I_\nu(s) = I_\nu(0) + \int j_\nu(s') ds' \) \( \text{brightness increases along the l.o.s.} \)

• For pure absorption, \( I_\nu(s) = I_\nu(0) \exp \left[ -\int_0^s \alpha_\nu(s') ds' \right] \equiv I_\nu(0) e^{-\tau_\nu} \) \( \text{brightness decreases exponentially with the optical depth } \tau_\nu, \) defined as

\[
d\tau_\nu = \alpha_\nu ds
\]

Note, \( \tau \) is a more natural «line of sight» unit than \( s \).

When scattering is included, this gets complicated, because photons can be added/removed from the beam, and often numerical solutions are required.

\[
\frac{dI_\nu}{ds} \text{ represents the variation of specific intensity of a beam of solid angle } d\Omega \text{ when crossing a volume } dAd:\Omega:
\]

Power absorbed per unit volume, solid angle, frequency interval: \( \frac{dE}{dA ds d\Omega dt d\nu} = - \alpha_\nu I_\nu \)

Power added per unit volume, solid angle, frequency interval: \( \frac{dE}{dA ds d\Omega dt d\nu} = j_\nu \)
**Optical depth**

The optical depth is a dimensionless quantity, depending on the density of the medium, on the properties of the medium particles, as well as on the traveled length and on the light frequency. It is defined as:

\[ \tau_\nu(s) \equiv \int_0^s \alpha_\nu(s') ds' \sim n \sigma_\nu s \]  

(the last step only holds for an homogeneous medium).

A medium with \( \tau_\nu > 1 \) is called thick or opaque; A medium with \( \tau_\nu < 1 \) is called thin or transparent.

- The probability for a photon to travel a given distance \( s \) in the medium without being absorbed is proportional to \( \exp(-\tau_\nu(s)) \). Thus one usually defines the **mean free path** of a photon as \( \ell_\nu = \frac{1}{n \sigma_\nu} = \frac{1}{\alpha_\nu} \).

**Formal solution to radiative transfer equation**

We rewrite \( \frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \) having defined the **Source function** \( S_\nu \equiv \frac{j_\nu}{\alpha_\nu} \);

a formal solution is then:

\[ I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau'_\nu) e^{-(\tau_\nu - \tau'_\nu)} d\tau' \]

The intensity emerging from the medium is the irradiated intensity attenuated by absorption, plus the integrated source function, also attenuated by absorption. But scattering and stimulated emission have to be also accounted for.
Case of constant source function

If $S_\nu$ is independent on $\tau'_\nu$ (e.g. a blob of uniform composition, $T, n$), then

$$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + S_\nu(1-e^{-\tau_\nu}) = S_\nu + e^{-\tau_\nu} (I_\nu(0) - S_\nu)$$

Interesting limits:

**Optically thick regime:** $\tau_\nu \gg 1$ \hspace{1cm} $I_\nu \rightarrow S_\nu$

(e.g., a rock): the background source if fully absorbed, photons only from the upper layes of the cloud

**Optically thin regime:** $\tau_\nu \rightarrow 0$ \hspace{1cm} $I_\nu \rightarrow I_\nu(0) + S_\nu \tau_\nu$

linear behaviour: unattenuated background source + photons from the cloud.

**Radiation Transfer as a relaxation process**

From the transfer equation, rewritten as

$$\frac{dI_\nu}{ds} = -\alpha_\nu (I_\nu - S_\nu) = \alpha_\nu (S_\nu - I_\nu)$$

we see that:

- if $I_\nu < S_\nu$ then $dI_\nu/ds > 0$: intensity increases along path
- if $I_\nu > S_\nu$, intensity decreases

The equation is "self regulating": $I_\nu$ relaxes to attractor $S_\nu$ and the characteristic length scale for relaxation is mean free path.
Radiative transfer in presence of elastic scattering

In addition to emission and absorption, photons can also be scattered in the medium. The emission of a volume element in the presence of scattering depends on the incident flux. Remember the mono-directional relations: \((ds=c\,dt)\)

Power **absorbed** per unit volume, solid angle, frequency interval: \(\frac{dE}{dA\,ds\,d\Omega\,dt\,dv} = -\alpha_{\nu} I_{\nu} = \frac{dI_{\nu}}{ds}\) negative

Power **added** per unit volume, solid angle, frequency interval: \(\frac{dE}{dA\,ds\,d\Omega\,dt\,dv} = j_{\nu} = \frac{dI_{\nu}}{ds}\) positive

Assume isotropic scattering (efficiency of scattering independent on \(\Omega\)) and **coherent** scattering (energy received per frequency interval \(dv\)= energy scattered in the same frequency interval; also named **elastic** or **monochromatic** scattering). Integrating over \(d\Omega\),

\[
\Delta E_{\nu} (\text{scattered over } 4\pi) = 4\pi j_{\nu}^{Sc} \, dA \, ds \, dt \, dv = [\int \alpha_{\nu}^{Sc} I_{\nu}(\Omega) \, d\Omega] \cdot dA \, ds \, dt \, dv = \Delta E_{\nu} (\text{received from all directions})
\]

Power scattered in all directions per unit volume, time \(dt\), \(dv\)

\[
j_{\nu}^{Sc} = \alpha_{\nu}^{Sc} \frac{1}{4\pi} \int I_{\nu} \, d\Omega = \alpha_{\nu}^{Sc} j_{\nu} \quad \text{(erg s}^{-1} \text{cm}^{-3} \text{Hz}^{-1} \text{ster}^{-1})
\]

Thus the intensity change due to scattered radiation **in a beam** (monodirectional ray of light in solid angle \(d\Omega\)) is

\[
dI_{\nu} = -\alpha_{\nu}^{Sc} (I_{\nu} - j_{\nu}) \, ds
\]

Part of the incoming beam is deviated, subtracting intensity. More intensity is added by rays from other directions, deflected into the direction of our beam.
Combining scattering and absorption, the monodirectional equation for a monochromatic beam is

\[\frac{dI_\nu}{ds} = -\alpha^{sc}_\nu (I_\nu - J_\nu) ds + \alpha^{abs}_\nu (S_\nu - I_\nu) ds \quad \text{where} \quad S_\nu = \frac{j_\nu}{\alpha^{abs}_\nu} \quad \text{(source function for absorption/emission only)}\]

We can define a Source function for the whole extinction process, due to absorption+scattering:

\[S^\text{ext}_\nu = \frac{j_\nu + \alpha^{sc}_\nu j_\nu}{\alpha^{abs}_\nu + \alpha^{sc}_\nu}\]

And the transfer equation becomes

\[\frac{dI_\nu}{ds} = (\alpha^{abs}_\nu + \alpha^{sc}_\nu)(S^\text{ext}_\nu - I_\nu)\]

The mean free path becomes \(\ell = \frac{1}{\alpha^{abs}_\nu + \alpha^{sc}_\nu}\) (path of a photon before being absorbed or scattered).

During the random walk, the probability that a free path end with a true absorption is \(\varepsilon_\nu = \frac{\alpha^{abs}_\nu}{\alpha^{abs}_\nu + \alpha^{sc}_\nu}\) (single-scattering albedo)

The corresponding probability for scattering is \(1 - \varepsilon_\nu = \frac{\alpha^{sc}_\nu}{\alpha^{abs}_\nu + \alpha^{sc}_\nu}\)

A photon has \(N = 1/\varepsilon\) scatterings (number of free paths) before absorption, so the total path, before being absorbed, is

\(\ell_* = N\ell = \frac{\ell}{\sqrt{\varepsilon}} = \frac{1}{\sqrt{\alpha^{abs}_\nu (\alpha^{abs}_\nu + \alpha^{sc}_\nu)}} = \text{thermalization length or diffusion length or effective mean free path} \quad (*)\)

\((*)\) In a random walk, the net displacement of a photon is \(R = r_1 + r_2 + r_3 + \ldots + r_N\). The average of these displacements would be 0. Thus we use the mean square displacement \(\ell_*^2 \equiv \langle R^2 \rangle = N\ell^2\), where \(\ell = \langle r_1^2 \rangle\) is the root mean square of the photon free path.